3-dimensional Continued Fraction Algorithms Cheat Sheets
HD version of `arXiv:1511.08399v1 [math.DS]`

Sébastien Labbé*

Abstract

Multidimensional Continued Fraction Algorithms are generalizations of the Euclid algorithm and find iteratively the gcd of two or more numbers. They are defined as linear applications on some subcone of $\mathbb{R}^d$. We consider multidimensional continued fraction algorithms that acts symmetrically on the positive cone $\mathbb{R}^d_+$ for $d = 3$. We include well-known and old ones (Poincaré, Brun, Selmer, Fully Subtractive) and new ones (Arnoux-Rauzy-Poincaré, Reverse, Cassaigne).

For each algorithm, one page (called cheat sheet) gathers a handful of informations most of them generated with the open source software Sage [9] with the optional Sage package `slabbe-0.2.spkg` [5]. The information includes the n-cylinders, density function of an absolutely continuous invariant measure, domain of the natural extension, lyapunov exponents as well as data regarding combinatorics on words, symbolic dynamics and digital geometry, that is, associated substitutions, generated S-adic systems, factor complexity, discrepancy, dual substitutions and generation of digital planes.

The document ends with a table of comparison of Lyapunov exponents and gives the code allowing to reproduce any of the results or figures appearing in these cheat sheets.

Contents

*Université de Liège, Bât. B37 Institut de Mathématiques, Grande Traverse 12, 4000 Liège, Belgium, `slabbe@ulg.ac.be`.

Brun algorithm

Definition

On $\Lambda = \mathbb{R}^3_+$, the map

$$F(x_1, x_2, x_3) = (x'_1, x'_2, x'_3)$$

is defined by

$$(x'_{\pi 1}, x'_{\pi 2}, x'_{\pi 3}) = (x_{\pi 1}, x_{\pi 2}, x_{\pi 3} - x_{\pi 2})$$

where $\pi \in \mathcal{S}_3$ is the permutation of $\{1, 2, 3\}$ such that $x_{\pi 1} < x_{\pi 2} < x_{\pi 3}$ [3].

Matrix Definition

The partition of the cone is $\Lambda = \cup_{\pi \in \mathcal{S}_3} \Lambda_\pi$ where

$$\Lambda_\pi = \{(x_1, x_2, x_3) \in \Lambda \mid x_{\pi 1} < x_{\pi 2} < x_{\pi 3}\}.$$

The matrices are given by the rule

$$M(\mathbf{x}) = M_\pi \qquad \text{if and only if} \qquad \mathbf{x} \in \Lambda_\pi.$$

The map F on Λ and the projective map f on $\Delta = \{\mathbf{x} \in \Lambda \mid \|\mathbf{x}\|_1 = 1\}$ are:

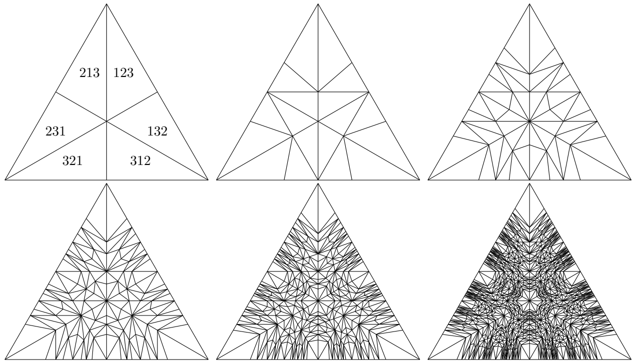
$$F(\mathbf{x}) = M(\mathbf{x})^{-1}\mathbf{x} \qquad \text{and} \qquad f(\mathbf{x}) = \frac{F(\mathbf{x})}{\|F(\mathbf{x})\|_1}.$$

Matrices

$$M_{123} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad M_{132} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad M_{213} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$M_{231} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad M_{312} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad M_{321} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Cylinders



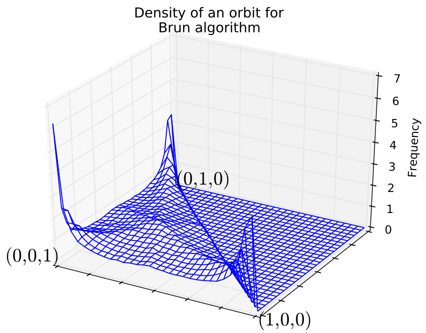
Density function

The density function of the invariant measure of $f : \Delta \to \Delta$ for the Brun algorithm is [1]:

$$\frac{1}{2\, x_{\pi 2}(1 - x_{\pi 2})(1 - x_{\pi 1} - x_{\pi 2})}$$

on the part $\mathbf{x} = (x_1, x_2, x_3) \in \Lambda_\pi \cap \Delta$.

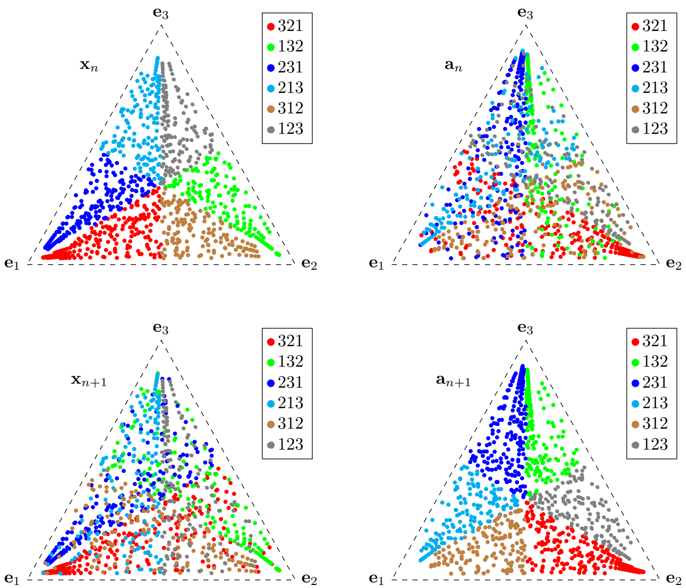
Invariant measure



Natural extension

Two sequences $(\mathbf{x}_{n+1})_{n\geq 0}$ and $(\mathbf{a}_{n+1})_{n\geq 0}$ defined such that

$$\mathbf{x}_{n+1} = M(\mathbf{x}_n)^{-1}\mathbf{x}_n \qquad \text{and} \qquad \mathbf{a}_{n+1} = M(\mathbf{x}_n)^\top \mathbf{a}_n.$$



Lyapunov exponents

(using 30 orbits of 100000000 iterations each)

30 succesfull orbits	min	mean	max	std
θ_1	0.3043	0.3045	0.3047	0.00012
θ_2	−0.11224	−0.11217	−0.11206	0.000048
$1 - \theta_2/\theta_1$	1.36821	1.36834	1.36847	0.000063

Substitutions

$$\sigma_{123} = \begin{cases} 1 \mapsto 1 \\ 2 \mapsto 23 \\ 3 \mapsto 3 \end{cases} \quad \sigma_{132} = \begin{cases} 1 \mapsto 1 \\ 2 \mapsto 2 \\ 3 \mapsto 32 \end{cases} \quad \sigma_{213} = \begin{cases} 1 \mapsto 13 \\ 2 \mapsto 2 \\ 3 \mapsto 3 \end{cases}$$

$$\sigma_{231} = \begin{cases} 1 \mapsto 1 \\ 2 \mapsto 2 \\ 3 \mapsto 31 \end{cases} \quad \sigma_{312} = \begin{cases} 1 \mapsto 12 \\ 2 \mapsto 2 \\ 3 \mapsto 3 \end{cases} \quad \sigma_{321} = \begin{cases} 1 \mapsto 1 \\ 2 \mapsto 21 \\ 3 \mapsto 3 \end{cases}$$

S-adic word example

Using vector $v = (1, e, \pi)$:

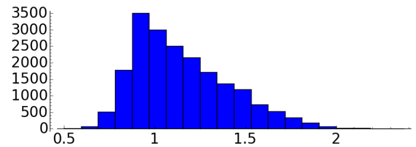
$$\begin{aligned} w &= \sigma_{123}\sigma_{312}\sigma_{312}\sigma_{321}\sigma_{132}\sigma_{123}\sigma_{312}\sigma_{231}\sigma_{231}\sigma_{213}\cdots (1) \\ &= 1232323123233231232332312323123232312323\ldots \end{aligned}$$

Factor Complexity of w is $(p_w(n))_{0\leq n\leq 20} =$

$$(1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 22, 24, 26, 28, 30, 32, 34, 36, 38, 40, 42)$$

Discrepancy

Discrepancy [10] for all 19701 S-adic words with directions $v \in \mathbb{N}^3_{>0}$ such that $v_1 + v_2 + v_3 = 200$:



Dual substitutions

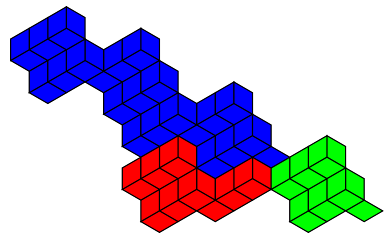
$$\sigma^*_{123} = \begin{cases} 1 \mapsto 1 \\ 2 \mapsto 2 \\ 3 \mapsto 32 \end{cases} \quad \sigma^*_{132} = \begin{cases} 1 \mapsto 1 \\ 2 \mapsto 23 \\ 3 \mapsto 3 \end{cases} \quad \sigma^*_{213} = \begin{cases} 1 \mapsto 1 \\ 2 \mapsto 2 \\ 3 \mapsto 31 \end{cases}$$

$$\sigma^*_{231} = \begin{cases} 1 \mapsto 13 \\ 2 \mapsto 2 \\ 3 \mapsto 3 \end{cases} \quad \sigma^*_{312} = \begin{cases} 1 \mapsto 1 \\ 2 \mapsto 21 \\ 3 \mapsto 3 \end{cases} \quad \sigma^*_{321} = \begin{cases} 1 \mapsto 12 \\ 2 \mapsto 2 \\ 3 \mapsto 3 \end{cases}$$

E one star

Using vector $v = (1, e, \pi)$, the 9-th iteration on the unit cube is:

$$E_1^*(\sigma^*_{123})E_1^*(\sigma^*_{312})E_1^*(\sigma^*_{312})E_1^*(\sigma^*_{321})E_1^*(\sigma^*_{132})\cdots(\ ) =$$



Selmer algorithm

Definition

On $\Lambda = \mathbb{R}^3_+$, the map

$$F(x_1, x_2, x_3) = (x'_1, x'_2, x'_3)$$

is defined by

$$(x'_{\pi 1}, x'_{\pi 2}, x'_{\pi 3}) = (x_{\pi 1}, x_{\pi 2}, x_{\pi 3} - x_{\pi 1})$$

where $\pi \in \mathcal{S}_3$ is the permutation of $\{1, 2, 3\}$ such that $x_{\pi 1} < x_{\pi 2} < x_{\pi 3}$ [8].

Matrix Definition

The partition of the cone is $\Lambda = \cup_{\pi \in \mathcal{S}_3} \Lambda_\pi$ where

$$\Lambda_\pi = \{(x_1, x_2, x_3) \in \Lambda \mid x_{\pi 1} < x_{\pi 2} < x_{\pi 3}\}.$$

The matrices are given by the rule

$$M(\mathbf{x}) = M_\pi \qquad \text{if and only if} \qquad \mathbf{x} \in \Lambda_\pi.$$

The map F on Λ and the projective map f on $\Delta = \{\mathbf{x} \in \Lambda \mid \|\mathbf{x}\|_1 = 1\}$ are:

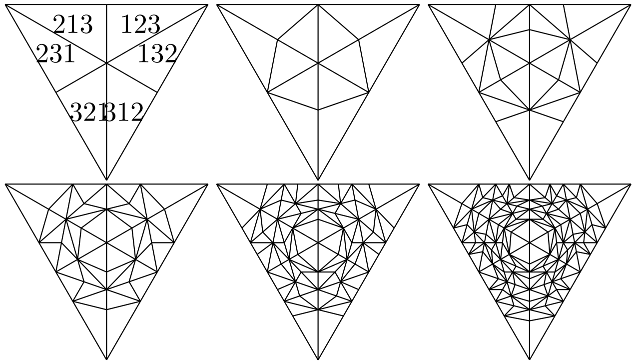
$$F(\mathbf{x}) = M(\mathbf{x})^{-1}\mathbf{x} \qquad \text{and} \qquad f(\mathbf{x}) = \frac{F(\mathbf{x})}{\|F(\mathbf{x})\|_1}.$$

Matrices

$$M_{123} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad M_{132} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad M_{213} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$$M_{231} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad M_{312} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad M_{321} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

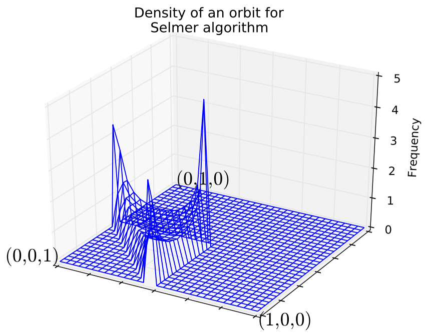
Cylinders



Density function

The sorted version of f admits a σ-finite invariant measure which is absolutely continuous with respect to Lebesgue measure on the central part and its density is known [7].

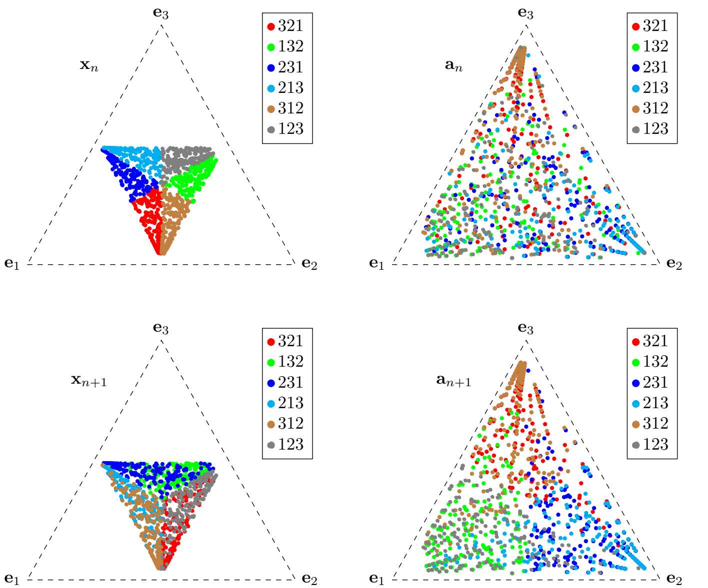
Invariant measure



Natural extension

Two sequences $(\mathbf{x}_{n+1})_{n \geq 0}$ and $(\mathbf{a}_{n+1})_{n \geq 0}$ defined such that

$$\mathbf{x}_{n+1} = M(\mathbf{x}_n)^{-1}\mathbf{x}_n \qquad \text{and} \qquad \mathbf{a}_{n+1} = M(\mathbf{x}_n)^\top \mathbf{a}_n.$$



Lyapunov exponents

(using 30 orbits of 100000000 iterations each)

30 succesfull orbits	min	mean	max	std
θ_1	0.1824	0.1827	0.1829	0.00012
θ_2	-0.07083	-0.07072	-0.07061	0.000059
$1 - \theta_2/\theta_1$	1.38695	1.38712	1.38731	0.000096

Substitutions

$$\sigma_{123} = \begin{cases} 1 \mapsto 13 \\ 2 \mapsto 2 \\ 3 \mapsto 3 \end{cases} \quad \sigma_{132} = \begin{cases} 1 \mapsto 12 \\ 2 \mapsto 2 \\ 3 \mapsto 3 \end{cases} \quad \sigma_{213} = \begin{cases} 1 \mapsto 1 \\ 2 \mapsto 23 \\ 3 \mapsto 3 \end{cases}$$

$$\sigma_{231} = \begin{cases} 1 \mapsto 1 \\ 2 \mapsto 21 \\ 3 \mapsto 3 \end{cases} \quad \sigma_{312} = \begin{cases} 1 \mapsto 1 \\ 2 \mapsto 2 \\ 3 \mapsto 32 \end{cases} \quad \sigma_{321} = \begin{cases} 1 \mapsto 1 \\ 2 \mapsto 2 \\ 3 \mapsto 31 \end{cases}$$

S-adic word example

Using vector $v = (1, e, \pi)$:

$$\begin{aligned} w &= \sigma_{123}\sigma_{132}\sigma_{123}\sigma_{132}\sigma_{213}\sigma_{321}\sigma_{312}\sigma_{231}\sigma_{123}\sigma_{312}\cdots(1) \\ &= 132323132322323132323132322323132321323 2... \end{aligned}$$

Factor Complexity of w is $(p_w(n))_{0 \leq n \leq 20} =$

$$(1, 3, 7, 11, 16, 20, 24, 28, 32, 36, 40, 44, 48, 52, 56, 60, 64, 68, 72, 76, 80)$$

Discrepancy

ValueError: On input=[198, 1, 1], algorithm Selmer loops on (1.0, 1.0, 0.0)

Dual substitutions

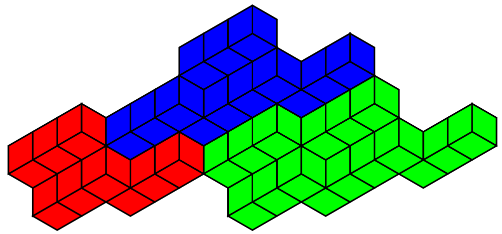
$$\sigma^*_{123} = \begin{cases} 1 \mapsto 1 \\ 2 \mapsto 2 \\ 3 \mapsto 31 \end{cases} \quad \sigma^*_{132} = \begin{cases} 1 \mapsto 1 \\ 2 \mapsto 21 \\ 3 \mapsto 3 \end{cases} \quad \sigma^*_{213} = \begin{cases} 1 \mapsto 1 \\ 2 \mapsto 2 \\ 3 \mapsto 32 \end{cases}$$

$$\sigma^*_{231} = \begin{cases} 1 \mapsto 12 \\ 2 \mapsto 2 \\ 3 \mapsto 3 \end{cases} \quad \sigma^*_{312} = \begin{cases} 1 \mapsto 1 \\ 2 \mapsto 23 \\ 3 \mapsto 3 \end{cases} \quad \sigma^*_{321} = \begin{cases} 1 \mapsto 13 \\ 2 \mapsto 2 \\ 3 \mapsto 3 \end{cases}$$

E one star

Using vector $v = (1, e, \pi)$, the 13-th iteration on the unit cube is:

$$E_1^*(\sigma^*_{123})E_1^*(\sigma^*_{132})E_1^*(\sigma^*_{123})E_1^*(\sigma^*_{132})E_1^*(\sigma^*_{213})\cdots(\ ) =$$



Poincaré algorithm

Definition

On $\Lambda = \mathbb{R}^3_+$, the map

$$F(x_1, x_2, x_3) = (x'_1, x'_2, x'_3)$$

is defined by

$$(x'_{\pi 1}, x'_{\pi 2}, x'_{\pi 3}) = (x_{\pi 1}, x_{\pi 2} - x_{\pi 1}, x_{\pi 3} - x_{\pi 2})$$

where $\pi \in \mathcal{S}_3$ is the permutation of $\{1, 2, 3\}$ such that $x_{\pi 1} < x_{\pi 2} < x_{\pi 3}$ [6].

Matrix Definition

The partition of the cone is $\Lambda = \cup_{\pi \in \mathcal{S}_3} \Lambda_\pi$ where

$$\Lambda_\pi = \{(x_1, x_2, x_3) \in \Lambda \mid x_{\pi 1} < x_{\pi 2} < x_{\pi 3}\}.$$

The matrices are given by the rule

$$M(\mathbf{x}) = M_\pi \qquad \text{if and only if} \qquad \mathbf{x} \in \Lambda_\pi.$$

The map F on Λ and the projective map f on $\Delta = \{\mathbf{x} \in \Lambda \mid \|\mathbf{x}\|_1 = 1\}$ are:

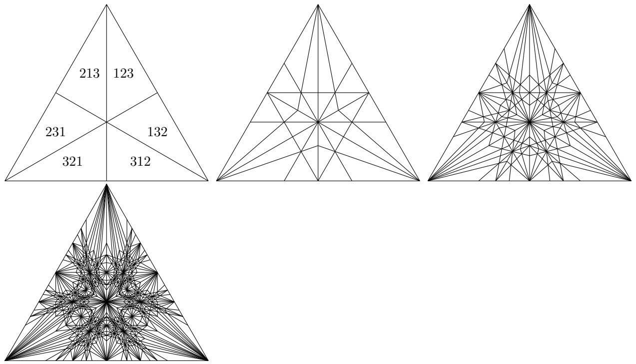
$$F(\mathbf{x}) = M(\mathbf{x})^{-1}\mathbf{x} \qquad \text{and} \qquad f(\mathbf{x}) = \frac{F(\mathbf{x})}{\|F(\mathbf{x})\|_1}.$$

Matrices

$$M_{123} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \quad M_{132} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \quad M_{213} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

$$M_{231} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad M_{312} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad M_{321} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

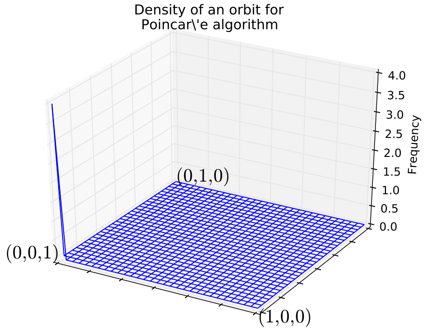
Cylinders



Density function

The sorted version of f admits a σ-finite invariant measure which is absolutely continuous with respect to Lebesgue measure and its density is known [6, 7].

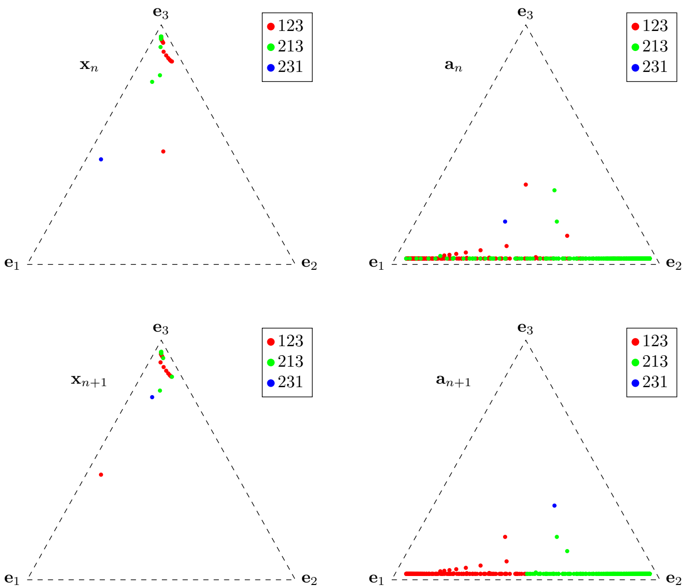
Invariant measure



Natural extension

Two sequences $(\mathbf{x}_{n+1})_{n \geq 0}$ and $(\mathbf{a}_{n+1})_{n \geq 0}$ defined such that

$$\mathbf{x}_{n+1} = M(\mathbf{x}_n)^{-1}\mathbf{x}_n \qquad \text{and} \qquad \mathbf{a}_{n+1} = M(\mathbf{x}_n)^\top \mathbf{a}_n.$$



Lyapunov exponents

(using 30 orbits of 100000000 iterations each)

24 succesfull orbits	min	mean	max	std
θ_1	4.2×10^{-9}	7.5×10^{-8}	2.0×10^{-7}	5.8×10^{-8}
θ_2	-3.7×10^{-8}	8.5×10^{-7}	9.2×10^{-7}	1.8×10^{-7}
$1 - \theta_2/\theta_1$	$-200.$	$-30.$	1.5	$42.$

Substitutions

$$\sigma_{123} = \begin{cases} 1 \mapsto 123 \\ 2 \mapsto 23 \\ 3 \mapsto 3 \end{cases} \quad \sigma_{132} = \begin{cases} 1 \mapsto 132 \\ 2 \mapsto 2 \\ 3 \mapsto 32 \end{cases} \quad \sigma_{213} = \begin{cases} 1 \mapsto 13 \\ 2 \mapsto 213 \\ 3 \mapsto 3 \end{cases}$$

$$\sigma_{231} = \begin{cases} 1 \mapsto 1 \\ 2 \mapsto 231 \\ 3 \mapsto 31 \end{cases} \quad \sigma_{312} = \begin{cases} 1 \mapsto 12 \\ 2 \mapsto 2 \\ 3 \mapsto 312 \end{cases} \quad \sigma_{321} = \begin{cases} 1 \mapsto 1 \\ 2 \mapsto 21 \\ 3 \mapsto 321 \end{cases}$$

S-adic word example

Using vector $v = (1, e, \pi)$:

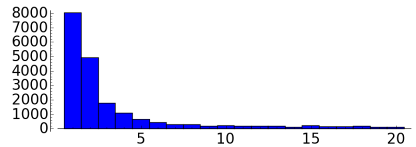
$$\begin{aligned} w &= \sigma_{123}\sigma_{312}\sigma_{312}\sigma_{213}\sigma_{123}\sigma_{132}\sigma_{213}\sigma_{213}\sigma_{213}\sigma_{213} \cdots (1) \\ &= 1232323312323123232323123232331232312323\ldots \end{aligned}$$

Factor Complexity of w is $(p_w(n))_{0 \leq n \leq 20} =$

$$(1, 3, 5, 7, 9, 11, 14, 17, 19, 21, 23, 25, 27, 29, 31, 33, 35, 37, 39, 41, 43)$$

Discrepancy

Discrepancy [10] for all 19701 S-adic words with directions $v \in \mathbb{N}^3_{>0}$ such that $v_1 + v_2 + v_3 = 200$:



Dual substitutions

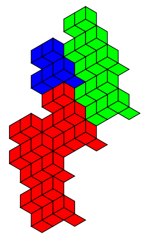
$$\sigma^*_{123} = \begin{cases} 1 \mapsto 1 \\ 2 \mapsto 21 \\ 3 \mapsto 321 \end{cases} \quad \sigma^*_{132} = \begin{cases} 1 \mapsto 1 \\ 2 \mapsto 231 \\ 3 \mapsto 31 \end{cases} \quad \sigma^*_{213} = \begin{cases} 1 \mapsto 12 \\ 2 \mapsto 2 \\ 3 \mapsto 312 \end{cases}$$

$$\sigma^*_{231} = \begin{cases} 1 \mapsto 132 \\ 2 \mapsto 2 \\ 3 \mapsto 32 \end{cases} \quad \sigma^*_{312} = \begin{cases} 1 \mapsto 13 \\ 2 \mapsto 213 \\ 3 \mapsto 3 \end{cases} \quad \sigma^*_{321} = \begin{cases} 1 \mapsto 123 \\ 2 \mapsto 23 \\ 3 \mapsto 3 \end{cases}$$

E one star

Using vector $v = (1, e, \pi)$, the 5-th iteration on the unit cube is:

$$E_1^*(\sigma^*_{123})E_1^*(\sigma^*_{312})E_1^*(\sigma^*_{312})E_1^*(\sigma^*_{213})E_1^*(\sigma^*_{123})($$  $$) =$$



Fully Subtractive algorithm

Definition

On $\Lambda = \mathbb{R}^3_+$, the map

$$F(x_1, x_2, x_3) = (x'_1, x'_2, x'_3)$$

is defined by

$$(x'_{\pi 1}, x'_{\pi 2}, x'_{\pi 3}) = (x_{\pi 1}, x_{\pi 2} - x_{\pi 1}, x_{\pi 3} - x_{\pi 1})$$

where $\pi \in \mathcal{S}_3$ is the permutation of $\{1,2,3\}$ such that $x_{\pi 1} < x_{\pi 2} < x_{\pi 3}$ [7].

Matrix Definition

The partition of the cone is $\Lambda = \cup_{i\in\{1,2,3\}} \Lambda_i$ where

$$\Lambda_i = \{(x_1, x_2, x_3) \in \Lambda \mid x_i = \min\{x_1, x_2, x_3\}\}.$$

The matrices are given by the rule

$$M(\mathbf{x}) = M_i \qquad \text{if and only if} \qquad \mathbf{x} \in \Lambda_i.$$

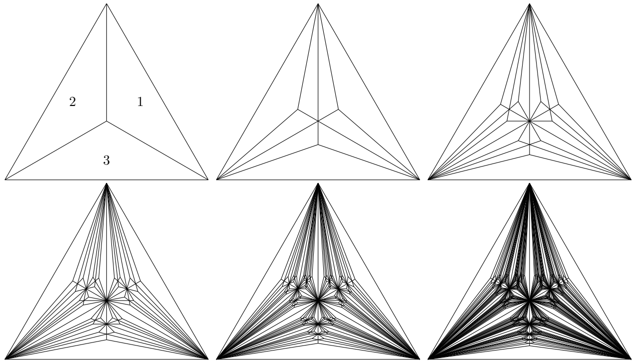
The map F on Λ and the projective map f on $\Delta = \{\mathbf{x} \in \Lambda \mid \|\mathbf{x}\|_1 = 1\}$ are:

$$F(\mathbf{x}) = M(\mathbf{x})^{-1}\mathbf{x} \qquad \text{and} \qquad f(\mathbf{x}) = \frac{F(\mathbf{x})}{\|F(\mathbf{x})\|_1}.$$

Matrices

$$M_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad M_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad M_3 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

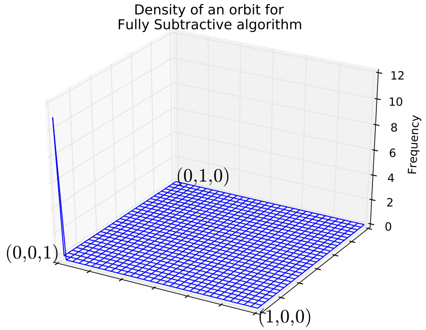
Cylinders



Density function

The sorted version of f admits a σ-finite invariant measure which is absolutely continuous with respect to Lebesgue measure and its density is known [7].

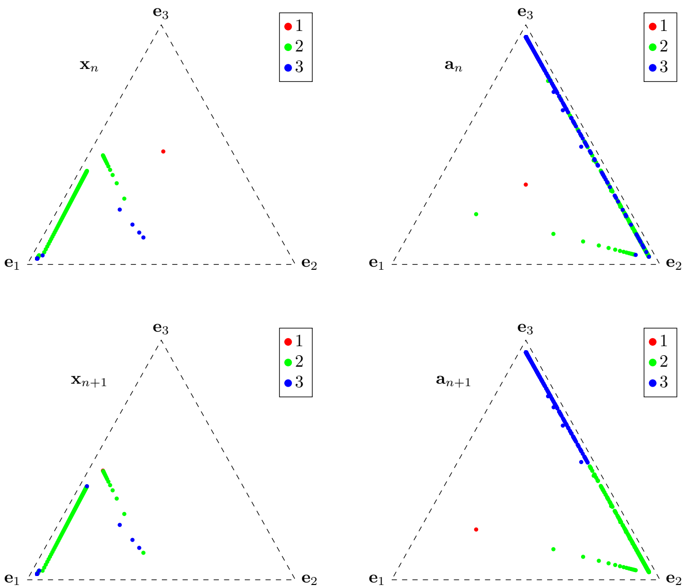
Invariant measure



Natural extension

Two sequences $(\mathbf{x}_{n+1})_{n\geq 0}$ and $(\mathbf{a}_{n+1})_{n\geq 0}$ defined such that

$$\mathbf{x}_{n+1} = M(\mathbf{x}_n)^{-1}\mathbf{x}_n \qquad \text{and} \qquad \mathbf{a}_{n+1} = M(\mathbf{x}_n)^\top \mathbf{a}_n.$$



Lyapunov exponents

(using 30 orbits of 100000000 iterations each)

21 succesfull orbits	min	mean	max	std
θ_1	2.6×10^{-9}	2.6×10^{-8}	1.2×10^{-7}	2.5×10^{-8}
θ_2	8.7×10^{-7}	9.0×10^{-7}	9.3×10^{-7}	1.8×10^{-8}
$1 - \theta_2/\theta_1$	$-360.$	$-77.$	-6.5	$83.$

Substitutions

$$\sigma_1 = \begin{cases} 1 \mapsto 123 \\ 2 \mapsto 2 \\ 3 \mapsto 3 \end{cases} \qquad \sigma_2 = \begin{cases} 1 \mapsto 1 \\ 2 \mapsto 231 \\ 3 \mapsto 3 \end{cases} \qquad \sigma_3 = \begin{cases} 1 \mapsto 1 \\ 2 \mapsto 2 \\ 3 \mapsto 312 \end{cases}$$

S-adic word example

Using vector $v = (1, e, \pi)$:

$$\begin{aligned} w &= \sigma_1\sigma_1\sigma_2\sigma_1\sigma_3\sigma_1\sigma_3\sigma_3\sigma_3\sigma_3 \cdots (1) \\ &= 1232323123233231232331232323123233231232... \end{aligned}$$

Factor Complexity of w is $(p_w(n))_{0\leq n\leq 20} =$

$$(1, 3, 5, 8, 11, 14, 16, 18, 19, 20, 21, 21, 21, 21, 21, 21, 21, 21, 21, 21, 21)$$

Discrepancy

ValueError: On input=[198, 1, 1], algorithm Fully Subtractive loops on (197.0, 1.0, 0.0)

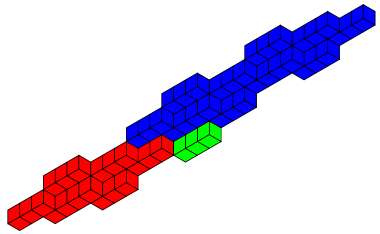
Dual substitutions

$$\sigma_1^* = \begin{cases} 1 \mapsto 1 \\ 2 \mapsto 21 \\ 3 \mapsto 31 \end{cases} \qquad \sigma_2^* = \begin{cases} 1 \mapsto 12 \\ 2 \mapsto 2 \\ 3 \mapsto 32 \end{cases} \qquad \sigma_3^* = \begin{cases} 1 \mapsto 13 \\ 2 \mapsto 23 \\ 3 \mapsto 3 \end{cases}$$

E one star

Using vector $v = (1, e, \pi)$, the 7-th iteration on the unit cube is:

$$E_1^*(\sigma_1^*)E_1^*(\sigma_1^*)E_1^*(\sigma_2^*)E_1^*(\sigma_1^*)E_1^*(\sigma_3^*)\cdots(\quad) =$$



Arnoux-Rauzy-Poincaré algorithm

Definition

On $\Lambda = \mathbb{R}^3_+$, the map

$$F(x_1, x_2, x_3) = (x'_1, x'_2, x'_3)$$

is defined by

$$(x'_{\pi 1}, x'_{\pi 2}, x'_{\pi 3}) = \begin{cases} (x_{\pi 1}, x_{\pi 2}, x_{\pi 3} - x_{\pi 1} - x_{\pi 2}) & \text{if } x_{\pi 3} > x_{\pi 1} + x_{\pi 2} \\ (x_{\pi 1}, x_{\pi 2} - x_{\pi 1}, x_{\pi 3} - x_{\pi 2}) & \text{otherwise.} \end{cases}$$

where $\pi \in \mathcal{S}_3$ is the permutation of $\{1, 2, 3\}$ such that $x_{\pi 1} < x_{\pi 2} < x_{\pi 3}$ [2].

Matrix Definition

The subcones are

$$\Lambda_i = \{(x_1, x_2, x_3) \in \Lambda \mid 2x_i > x_1 + x_2 + x_3\}, \qquad i \in \{1, 2, 3\},$$
$$\Lambda_\pi = \{(x_1, x_2, x_3) \in \Lambda \mid x_{\pi 1} < x_{\pi 2} < x_{\pi 3}\}, \qquad \pi \in \mathcal{S}_3.$$

The matrices are given by the rule

$$M(\mathbf{x}) = \begin{cases} M_i & \text{if } \mathbf{x} \in \Lambda_i, \\ M_\pi & \text{else if } \mathbf{x} \in \Lambda_\pi. \end{cases}$$

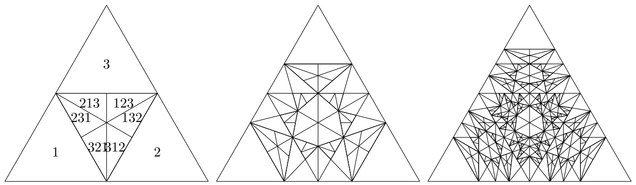
The map F on Λ and the projective map f on $\Delta = \{\mathbf{x} \in \Lambda \mid \|\mathbf{x}\|_1 = 1\}$ are:

$$F(\mathbf{x}) = M(\mathbf{x})^{-1}\mathbf{x} \qquad \text{and} \qquad f(\mathbf{x}) = \frac{F(\mathbf{x})}{\|F(\mathbf{x})\|_1}.$$

Matrices

$$M_1 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad M_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$
$$M_{123} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \quad M_{132} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \quad M_{213} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$
$$M_{231} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad M_{312} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad M_{321} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

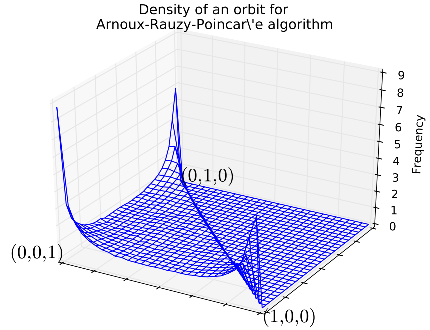
Cylinders



Density function

The density of the absolutely continuous invariant measure is unknown [1].

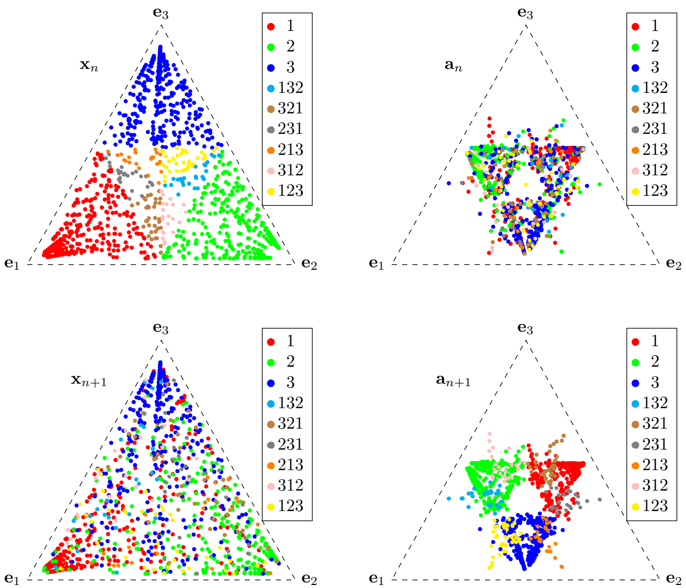
Invariant measure



Natural extension

Two sequences $(\mathbf{x}_{n+1})_{n\geq 0}$ and $(\mathbf{a}_{n+1})_{n\geq 0}$ defined such that

$$\mathbf{x}_{n+1} = M(\mathbf{x}_n)^{-1}\mathbf{x}_n \qquad \text{and} \qquad \mathbf{a}_{n+1} = M(\mathbf{x}_n)^\top \mathbf{a}_n.$$



Lyapunov exponents

(using 30 orbits of 100000000 iterations each)

30 succesfull orbits	min	mean	max	std
θ_1	0.4427	0.4430	0.4433	0.00014
θ_2	-0.17235	-0.17222	-0.17210	0.000062
$1 - \theta_2/\theta_1$	1.38869	1.38879	1.38888	0.000051

Substitutions

$$\sigma_1 = \begin{cases} 1 \mapsto 1 \\ 2 \mapsto 21 \\ 3 \mapsto 31 \end{cases} \quad \sigma_2 = \begin{cases} 1 \mapsto 12 \\ 2 \mapsto 2 \\ 3 \mapsto 32 \end{cases} \quad \sigma_3 = \begin{cases} 1 \mapsto 13 \\ 2 \mapsto 23 \\ 3 \mapsto 3 \end{cases}$$
$$\sigma_{123} = \begin{cases} 1 \mapsto 123 \\ 2 \mapsto 23 \\ 3 \mapsto 3 \end{cases} \quad \sigma_{132} = \begin{cases} 1 \mapsto 132 \\ 2 \mapsto 2 \\ 3 \mapsto 32 \end{cases} \quad \sigma_{213} = \begin{cases} 1 \mapsto 13 \\ 2 \mapsto 213 \\ 3 \mapsto 3 \end{cases}$$
$$\sigma_{231} = \begin{cases} 1 \mapsto 1 \\ 2 \mapsto 231 \\ 3 \mapsto 31 \end{cases} \quad \sigma_{312} = \begin{cases} 1 \mapsto 12 \\ 2 \mapsto 2 \\ 3 \mapsto 312 \end{cases} \quad \sigma_{321} = \begin{cases} 1 \mapsto 1 \\ 2 \mapsto 21 \\ 3 \mapsto 321 \end{cases}$$

S-adic word example

Using vector $v = (1, e, \pi)$:

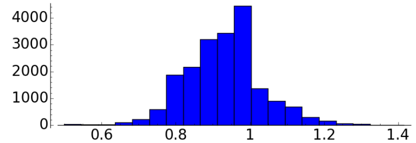
$$w = \sigma_{123}\sigma_2\sigma_1\sigma_{123}\sigma_1\sigma_{231}\sigma_3\sigma_3\sigma_3\sigma_3 \cdots (1)$$
$$= 1232323123233231232332312323123232312323\ldots$$

Factor Complexity of w is $(p_w(n))_{0\leq n\leq 20} =$

$$(1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 22, 24, 26, 28, 30, 32, 34, 36, 38, 40, 42)$$

Discrepancy

Discrepancy [10] for all 19701 S-adic words with directions $v \in \mathbb{N}^3_{>0}$ such that $v_1 + v_2 + v_3 = 200$:



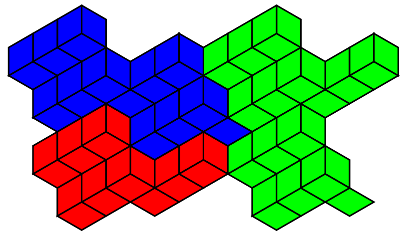
Dual substitutions

$$\sigma^*_1 = \begin{cases} 1 \mapsto 123 \\ 2 \mapsto 2 \\ 3 \mapsto 3 \end{cases} \quad \sigma^*_2 = \begin{cases} 1 \mapsto 1 \\ 2 \mapsto 231 \\ 3 \mapsto 3 \end{cases} \quad \sigma^*_3 = \begin{cases} 1 \mapsto 1 \\ 2 \mapsto 2 \\ 3 \mapsto 312 \end{cases}$$
$$\sigma^*_{123} = \begin{cases} 1 \mapsto 1 \\ 2 \mapsto 21 \\ 3 \mapsto 321 \end{cases} \quad \sigma^*_{132} = \begin{cases} 1 \mapsto 1 \\ 2 \mapsto 231 \\ 3 \mapsto 31 \end{cases} \quad \sigma^*_{213} = \begin{cases} 1 \mapsto 12 \\ 2 \mapsto 2 \\ 3 \mapsto 312 \end{cases}$$
$$\sigma^*_{231} = \begin{cases} 1 \mapsto 132 \\ 2 \mapsto 2 \\ 3 \mapsto 32 \end{cases} \quad \sigma^*_{312} = \begin{cases} 1 \mapsto 13 \\ 2 \mapsto 213 \\ 3 \mapsto 3 \end{cases} \quad \sigma^*_{321} = \begin{cases} 1 \mapsto 123 \\ 2 \mapsto 23 \\ 3 \mapsto 3 \end{cases}$$

E one star

Using vector $v = (1, e, \pi)$, the 5-th iteration on the unit cube is:

$$E^*_1(\sigma^*_{123})E^*_1(\sigma^*_2)E^*_1(\sigma^*_1)E^*_1(\sigma^*_{123})E^*_1(\sigma^*_1)(\ ) =$$



Reverse algorithm

Definition

On $\Lambda = \mathbb{R}^3_+$, the map

$$F(x_1, x_2, x_3) = (x'_1, x'_2, x'_3)$$

is defined by

$$\begin{pmatrix} x'_{\pi 1} \\ x'_{\pi 2} \\ x'_{\pi 3} \end{pmatrix} = \begin{cases} \begin{pmatrix} x_{\pi 1} \\ x_{\pi 2} \\ x_{\pi 3} - x_{\pi 1} - x_{\pi 2} \end{pmatrix} & \text{if } x_{\pi 3} > x_{\pi 1} + x_{\pi 2} \\ \frac{1}{2}\begin{pmatrix} -x_{\pi 1} + x_{\pi 2} + x_{\pi 3} \\ x_{\pi 1} - x_{\pi 2} + x_{\pi 3} \\ x_{\pi 1} + x_{\pi 2} - x_{\pi 3} \end{pmatrix} & \text{otherwise.} \end{cases}$$

where $\pi \in \mathcal{S}_3$ is the permutation of $\{1, 2, 3\}$ such that $x_{\pi 1} < x_{\pi 2} < x_{\pi 3}$ [1].

Matrix Definition

The subcones are

$$\Lambda_i = \{(x_1, x_2, x_3) \in \Lambda \mid 2x_i > x_1 + x_2 + x_3\}, \qquad i \in \{1, 2, 3\},$$
$$\Lambda_4 = \Lambda \setminus (\Lambda_1 \cup \Lambda_2 \cup \Lambda_3)$$

The matrices are given by the rule

$$M(\mathbf{x}) = M_i \qquad \text{if and only if} \qquad \mathbf{x} \in \Lambda_i.$$

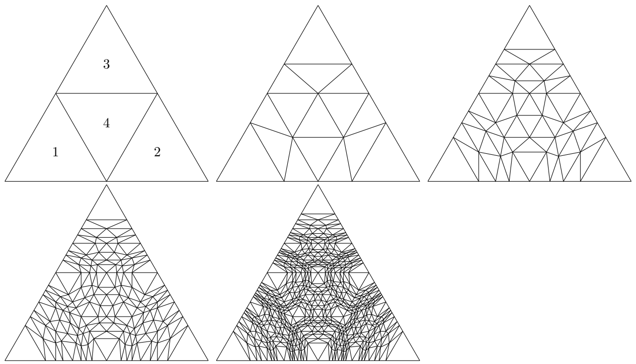
The map F on Λ and the projective map f on $\Delta = \{\mathbf{x} \in \Lambda \mid \|\mathbf{x}\|_1 = 1\}$ are:

$$F(\mathbf{x}) = M(\mathbf{x})^{-1}\mathbf{x} \qquad \text{and} \qquad f(\mathbf{x}) = \frac{F(\mathbf{x})}{\|F(\mathbf{x})\|_1}.$$

Matrices

$$M_1 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad M_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$
$$M_4 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Cylinders

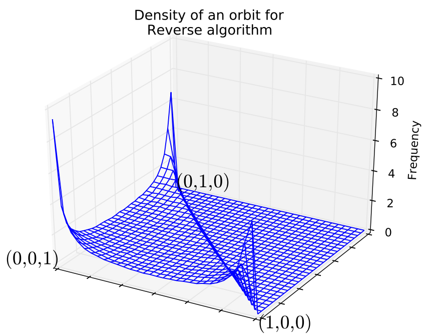


Density function

The density function of the invariant measure of $f : \Delta \to \Delta$ for the Reverse algorithm is [1]:

$$\frac{1}{(1 - x_1)(1 - x_2)(1 - x_3)}.$$

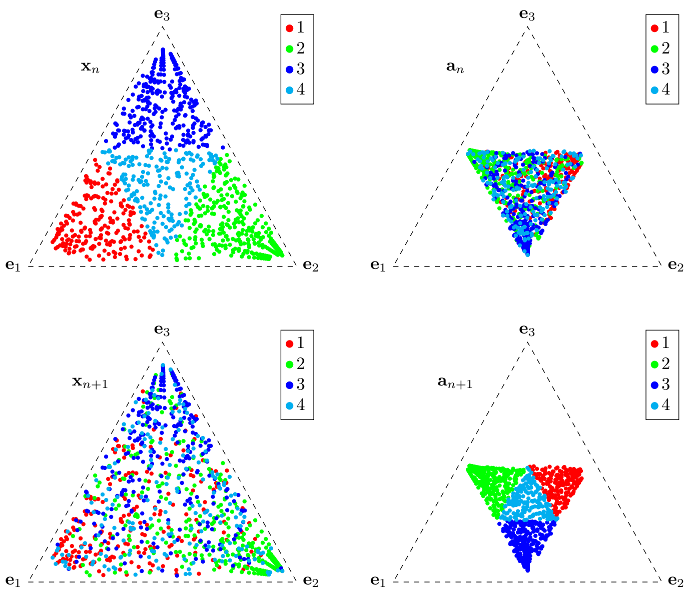
Invariant measure



Natural extension

Two sequences $(\mathbf{x}_{n+1})_{n \geq 0}$ and $(\mathbf{a}_{n+1})_{n \geq 0}$ defined such that

$$\mathbf{x}_{n+1} = M(\mathbf{x}_n)^{-1}\mathbf{x}_n \qquad \text{and} \qquad \mathbf{a}_{n+1} = M(\mathbf{x}_n)^{\top}\mathbf{a}_n.$$



Lyapunov exponents

(using 30 orbits of 100000000 iterations each)

30 succesfull orbits	min	mean	max	std
θ_1	0.4045	0.4049	0.4052	0.00017
θ_2	-0.10328	-0.10320	-0.10309	0.000048
$1 - \theta_2/\theta_1$	1.25479	1.25488	1.25496	0.000045

Substitutions

$$\sigma_1 = \begin{cases} 1 \mapsto 1 \\ 2 \mapsto 21 \\ 3 \mapsto 31 \end{cases} \quad \sigma_2 = \begin{cases} 1 \mapsto 12 \\ 2 \mapsto 2 \\ 3 \mapsto 32 \end{cases} \quad \sigma_3 = \begin{cases} 1 \mapsto 13 \\ 2 \mapsto 23 \\ 3 \mapsto 3 \end{cases}$$
$$\sigma_4 = \begin{cases} 1 \mapsto 23 \\ 2 \mapsto 31 \\ 3 \mapsto 12 \end{cases}$$

S-adic word example

Using vector $v = (1, e, \pi)$:

$$\begin{aligned} w &= \sigma_4\sigma_1\sigma_1\sigma_4\sigma_3\sigma_1\sigma_1\sigma_3\sigma_3\sigma_3 \cdots (1) \\ &= 23312323312323122323233123233123231223 23\ldots \end{aligned}$$

Factor Complexity of w is $(p_w(n))_{0 \leq n \leq 20} =$

$$(1, 3, 6, 9, 12, 14, 17, 20, 23, 26, 29, 32, 35, 38, 41, 44, 47, 50, 53, 56, 58)$$

Discrepancy

ValueError: On input=[197, 2, 1], algorithm Reverse reaches non integer entries (0.5, 0.5, 1.5)

Dual substitutions

$$\sigma_1^* = \begin{cases} 1 \mapsto 123 \\ 2 \mapsto 2 \\ 3 \mapsto 3 \end{cases} \quad \sigma_2^* = \begin{cases} 1 \mapsto 1 \\ 2 \mapsto 231 \\ 3 \mapsto 3 \end{cases} \quad \sigma_3^* = \begin{cases} 1 \mapsto 1 \\ 2 \mapsto 2 \\ 3 \mapsto 312 \end{cases}$$
$$\sigma_4^* = \begin{cases} 1 \mapsto 23 \\ 2 \mapsto 13 \\ 3 \mapsto 12 \end{cases}$$

E one star

ValueError: The substitution (1->23, 2->1233, 3->1232) must be unimodular.

Cassaigne algorithm

Definition

On $\Lambda = \mathbb{R}^3_+$, the map is [4]

$$F(x_1, x_2, x_3) = \begin{cases} (x_1 - x_3, x_3, x_2) & \text{if } x_1 > x_3 \\ (x_2, x_1, x_3 - x_1) & \text{if } x_1 < x_3. \end{cases}$$

Matrix Definition

The partition of the cone is $\Lambda = \cup_{\pi \in \mathcal{S}_3} \Lambda_\pi$ where

$$\Lambda_1 = \{(x_1, x_2, x_3) \in \Lambda \mid x_1 > x_3\},$$
$$\Lambda_2 = \{(x_1, x_2, x_3) \in \Lambda \mid x_1 < x_3\}.$$

The matrices are given by the rule

$$M(\mathbf{x}) = M_i \qquad \text{if and only if} \qquad \mathbf{x} \in \Lambda_i.$$

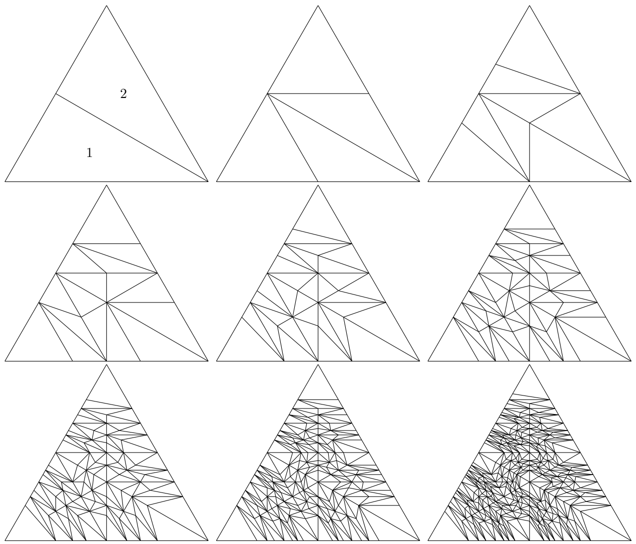
The map F on Λ and the projective map f on $\Delta = \{\mathbf{x} \in \Lambda \mid \|\mathbf{x}\|_1 = 1\}$ are:

$$F(\mathbf{x}) = M(\mathbf{x})^{-1}\mathbf{x} \qquad \text{and} \qquad f(\mathbf{x}) = \frac{F(\mathbf{x})}{\|F(\mathbf{x})\|_1}.$$

Matrices

$$M_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad M_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

Cylinders

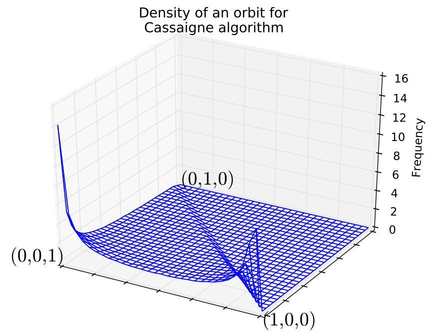


Density function

The density function of the invariant measure of $f : \Delta \to \Delta$ for the Cassaigne algorithm is [1]

$$\frac{1}{(1 - x_1)(1 - x_3)}.$$

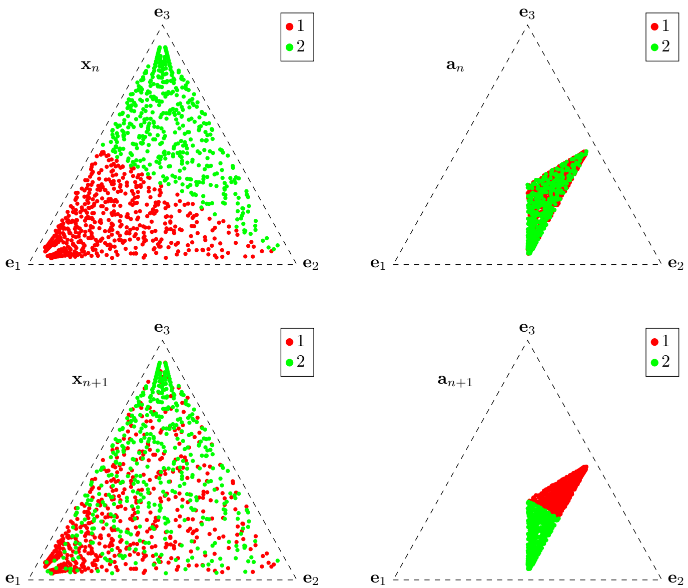
Invariant measure



Natural extension

Two sequences $(\mathbf{x}_{n+1})_{n\geq 0}$ and $(\mathbf{a}_{n+1})_{n\geq 0}$ defined such that

$$\mathbf{x}_{n+1} = M(\mathbf{x}_n)^{-1}\mathbf{x}_n \qquad \text{and} \qquad \mathbf{a}_{n+1} = M(\mathbf{x}_n)^\top \mathbf{a}_n.$$



Lyapunov exponents

(using 30 orbits of 100000000 iterations each)

30 succesfull orbits	min	mean	max	std
θ_1	0.1825	0.1827	0.1829	0.00012
θ_2	-0.07081	-0.07074	-0.07064	0.000048
$1 - \theta_2/\theta_1$	1.38697	1.38712	1.38729	0.000065

Substitutions

$$\sigma_1 = \begin{cases} 1 \mapsto 1 \\ 2 \mapsto 13 \\ 3 \mapsto 2 \end{cases} \qquad \sigma_2 = \begin{cases} 1 \mapsto 2 \\ 2 \mapsto 13 \\ 3 \mapsto 3 \end{cases}$$

S-adic word example

Using vector $v = (1, e, \pi)$:

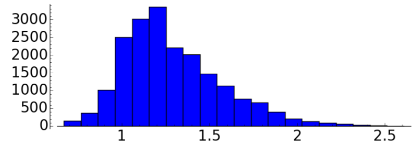
$$\begin{aligned} w &= \sigma_2\sigma_1\sigma_2\sigma_1\sigma_1\sigma_1\sigma_1\sigma_2\sigma_1\sigma_1 \cdots (1) \\ &= 2323213232323132323213232321323231323232\ldots \end{aligned}$$

Factor Complexity of w is $(p_w(n))_{0\leq n\leq 20} =$

$$(1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33, 35, 37, 39, 41)$$

Discrepancy

Discrepancy [10] for all 19701 S-adic words with directions $v \in \mathbb{N}^3_{>0}$ such that $v_1 + v_2 + v_3 = 200$:



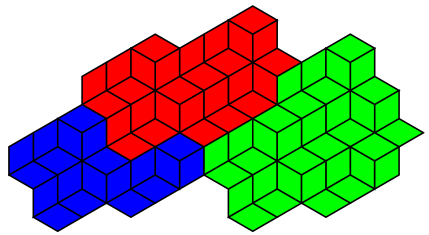
Dual substitutions

$$\sigma_1^* = \begin{cases} 1 \mapsto 12 \\ 2 \mapsto 3 \\ 3 \mapsto 2 \end{cases} \qquad \sigma_2^* = \begin{cases} 1 \mapsto 2 \\ 2 \mapsto 1 \\ 3 \mapsto 23 \end{cases}$$

E one star

Using vector $v = (1, e, \pi)$, the 13-th iteration on the unit cube is:

$$E_1^*(\sigma_2^*)E_1^*(\sigma_1^*)E_1^*(\sigma_2^*)E_1^*(\sigma_1^*)E_1^*(\sigma_1^*)\cdots($$$$) =$$



Comparison of Lyapunov exponents

(30 orbits of 1000000000 iterations each)

Algorithm	#Orbits	θ_1 (std)	θ_2 (std)	$1 - \theta_2/\theta_1$ (std)
Arnoux-Rauzy-Poincaré	30	0.44290 (0.000083)	-0.17219 (0.000035)	1.38879 (0.000017)
Selmer	30	0.18269 (0.000032)	-0.07072 (0.000013)	1.38710 (0.000021)
Cassaigne	30	0.18268 (0.000041)	-0.07072 (0.000017)	1.38709 (0.000028)
Brun	30	0.30449 (0.000049)	-0.11216 (0.000019)	1.36833 (0.000015)
Reverse	30	0.40489 (0.000057)	-0.10320 (0.000015)	1.25489 (0.000016)
Fully Subtractive	26	2.5e-9 (1.6e-9)	9.3e-8 (1.5e-9)	-69. (87.)
Poincaré	22	6.9e-9 (4.8e-9)	7.8e-8 (3.2e-8)	-24. (40.)

Sage Code

This section shows how to reproduce any of the results in these Cheat Sheets.

Requirements

The image and experimental results in these cheat sheets were created with the following version of Sage [9]

```
$ sage -v
SageMath Version 6.10.beta3, Release Date: 2015-11-05
```

and my optional Sage package [5] which can be installed with:

```
$ sage -p http://www.slabbe.org/Sage/slabbe-0.2.spkg
```

Definition

Define a Multidimensional Continued Fraction algorithm:

```
sage: from slabbe.mult_cont_frac import Brun
sage: algo = Brun()
```

You may replace `Brun` above by any of the following:

```
Brun, Poincare, Selmer, FullySubtractive,
ARP, Reverse, Cassaigne
```

Matrices

```
sage: cocycle = algo.matrix_cocycle()
sage: cocycle.gens()
```

Cylinders

```
sage: cocycle = algo.matrix_cocycle()
sage: t = cocycle.tikz_n_cylinders(3, scale=3)
sage: t.pdf()
```

Density function

This section is hand written.

Invariant measure

```
sage: fig = algo.invariant_measure_wireframe_plot(
....:        n_iterations=10^6, ndivs=30, norm='1')
sage: fig.savefig('a.pdf')
```

Natural extension

```
sage: t = algo.natural_extension_tikz(n_iterations=1200,
....:           marksize=.8, group_size="2 by 2")
sage: t.png()
```

Lyapunov exponents

The algorithm that computes Lyapunov exponents was provided to me by Vincent Delecroix, in June 2013. I translated his C code into cython.

```
sage: from slabbe.lyapunov import lyapunov_table
sage: lyapunov_table(algo, n_orbits=30, n_iterations=10^7)
```

Substitutions

```
sage: algo.substitutions()
```

S-adic word example

```
sage: v = (1,e,pi)
sage: it = algo.coding_iterator(v)
sage: [next(it) for _ in range(10)]
sage: algo.s_adic_word(v)
sage: map(w[:10000].number_of_factors, range(21))
```

Discrepancy

```
sage: D = algo.discrepancy_statistics(length=20)
sage: histogram(D.values())
```

Dual substitutions

```
sage: algo.dual_substitutions()
```

E one star

```
sage: from slabbe import TikzPicture
sage: P = algo.e_one_star_patch(v=(1,e,pi), n=8)
sage: s = P.plot_tikz()
sage: TikzPicture(s).pdf()
```

Comparison of Lyapunov exponents

```
sage: import slabbe.mult_cont_frac as mcf
sage: from slabbe.lyapunov import lyapunov_comparison_table
sage: algos = [mcf.Brun(), mcf.Selmer(), mcf.ARP(),
....:           mcf.Reverse(), mcf.Cassaigne()]
sage: lyapunov_comparison_table(algos, n_orbits=30,
....:                             n_iterations=10^7)
```

Acknowledgments

This work is part of the project “Dynamique des algorithmes du pgcd : une approche Algorithmique, Analytique, Arithmétique et Symbolique (Dyna3S)” (ANR-13-BS02-0003) supported by the Agence Nationale de la Recherche. The author is supported by a postdoctoral Marie Curie fellowship (BeIPD-COFUND) cofunded by the European Commission. I wish to thank Valérie Berthé, Pierre Arnoux, Vincent Delecroix and Thierry Monteil for many discussions on the experimental aspects of MCF algorithms.

References

[1] Pierre Arnoux and Sébastien Labbé. On some symmetric multidimensional continued fraction algorithms. *arXiv:1508.07814*, August 2015.

[2] V. Berthé and S. Labbé. Factor complexity of S-adic words generated by the arnoux–rauzy–poincaré algorithm. *Advances in Applied Mathematics*, 63(0):90 – 130, 2015.

[3] Viggo Brun. Algorithmes euclidiens pour trois et quatre nombres. In *Treizième congrès des mathèmaticiens scandinaves, tenu à Helsinki 18-23 août 1957*, pages 45–64. Mercators Tryckeri, Helsinki, 1958.

[4] Julien Cassaigne. Un algorithme de fractions continues de complexité linéaire. October 2015. DynA3S meeting, LIAFA, Paris, October 12th, 2015.

[5] Sébastien Labbé. Sébastien labbé research code v0.2, `slabbe-0.2.spkg`. `http://www.slabbe.org/Sage`, 2015.

[6] A. Nogueira. The three-dimensional Poincaré continued fraction algorithm. *Israel J. Math.*, 90(1-3):373–401, 1995.

[7] F. Schweiger. *Multidimensional Continued Fraction*. Oxford Univ. Press, New York, 2000.

[8] Ernst S. Selmer. Continued fractions in several dimensions. *Nordisk Nat. Tidskr.*, 9:37–43, 95, 1961.

[9] William A. Stein et al. *Sage Mathematics Software (Version 6.9)*. The Sage Development Team, 2015.

[10] R. Tijdeman. The chairman assignment problem. *Discrete Math.*, 32(3):323–330, 1980.