A d-dimensional extension of Christoffel words

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Abstract

In this article, we extend the definition of Christoffel words to directed subgraphs of the hypercubic lattice in arbitrary dimension that we call Christoffel graphs. Christoffel graphs when $d = 2$ correspond to well-known Christoffel words. We show that Christoffel graphs have similar properties to those of Christoffel words: symmetry of their central part and conjugation with their reversal. Our main result extends Pirillo's theorem (characterization of Christoffel words which asserts that a word amb is a Christoffel word if and only if it is conjugate to bma) in arbitrary dimension. In the generalization, the map $amb \mapsto bma$ is seen as a flip operation on graphs embedded in $\mathbb{Z}^d$ and the conjugation is a translation. We show that a fully periodic subgraph of the hypercubic lattice is a translate of its flip if and only if it is a Christoffel graph.

1 Introduction

This article is a contribution to the study of discrete planes and hyperplanes in any dimension d. We study only rational hyperplanes, that is, those which are defined by an equation with rational coefficients. We extract from such an hyperplane a finite pattern that we call a *Christoffel graph*. We show that they are a generalization of Christoffel words.

Discrete planes were introduced by [21] and further studied [1, 11, 14, 23]. Recognition algorithms were proposed in [15, 20, 22]. See [10] for a complete review about many aspects of digital planarity, such as characterizations in arithmetic geometry, periodicity, connectivity and algorithms. Discrete planes can be seen as an union of square faces. Such stepped surface, introduced in [16, 17] as a way to construct quasiperiodic tilings of the plane, can be generated from multidimensional continued fraction algorithms by introducing substitutions on square faces [2, 4].

While discrete planes are a satisfactory generalization of Sturmian words, it is still unclear what is the equivalent notion of Christoffel words in higher dimension. In [13, Fig. 6.6 and 6.7], fundamental domain of rational discrete planes are constructed from the iteration of generalized substitutions on the unit cube. Recently [12] generalized central words to arbitrary dimension using palindromic closure. In both cases the representation is nonconvex and has a boundary like a fractal.

We propose to extend the definition of Christoffel words to directed subgraphs of the hypercubic lattice in arbitrary dimension that we call Christoffel graphs. A similar construction, called *roundwalk*, but serving a different purpose was given in [6] producing multi-dimensional words

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that are closely related to k-dimensional Sturmian words. Christoffel graphs when $d = 2$ correspond to Christoffel words. Due to its periods, the d-dimensional Christoffel graph can be embedded in a $(d-1)$-torus and when $d = 3$, the torus is a parallelogram. This extension is motivated by Pirillo's theorem which asserts that a word amb is a Christoffel word if and only if it is conjugate to bma. In the generalization, the map $amb \mapsto bma$ is seen as a flip operation on graphs embedded in $\mathbb{Z}^d$ and the conjugation is replaced by some translation. When $d = 3$, our flip corresponds to a flip in a rhombus tiling [3,8,9]. We show that these Christoffel graphs have similar properties to those of Christoffel words: symmetry of their central part (Lemma 10) and conjugation with their reversal (Corollary 13). Our main result is Theorem 14 which extends Pirillo's theorem in arbitrary dimension.

This is an extended abstract to the preprint [18], the latter containing the proofs to each of the result presented below and more.

2 Christoffel words and discrete planes

2.1 Christoffel words

Recall that Christoffel words are obtained by discretizing a line segment in the plane as follows: let $(p, q) \in \mathbb{N}^2$ with $\gcd(p, q) = 1$, and let S be the line segment with endpoints $(0, 0)$ and (p, q). The word $w \in \{a, b\}^*$ is a *lower Christoffel word* if the path induced by w starting at the origin

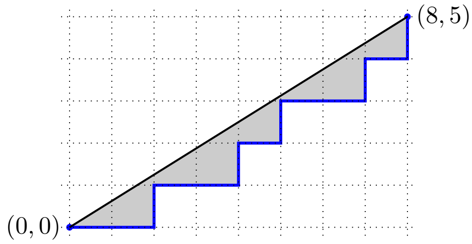

Figure 1: The lower Christoffel word $w = aabaababaabab$.

ends at (p, q), is under S and the path and the segment S delimit a polygon with no integral interior point. An *upper Christoffel word* is defined similarly, by taking the path which is above the segment. A *Christoffel word* is a lower Christoffel word. See Figure 1 and reference [5]. An astonishing result about Christoffel words is the following characteristic property given by Pirillo [19]. Recall that two words w and w' are *conjugate* if there exist two words u and v such that $w = uv$ and $w' = vu$.

Theorem 1 (Pirillo)**.** *A word $w = amb \in \{a, b\}^*$ is a Christoffel word if and only if amb and bma are conjugate.*

In this article, we generalize Theorem 1 to dimension $d \geq 3$.

3 Discrete hyperplane graphs

Let $a_1, \ldots, a_d$ be relatively prime positive integers and $s = \|\mathbf{a}\|_1 = \sum a_i$ be their sum. We denote $\mathbf{a} = (a_1, a_2, \ldots, a_d) \in \mathbb{N}^d$. We define the mapping $\mathcal{F}_{\mathbf{a}} : \mathbb{Z}^d \to \mathbb{Z}/s\mathbb{Z}$ sending each integral vector $(x_1, \ldots, x_d)$ onto $\sum_i a_i x_i \bmod s$. We identify $\mathbb{Z}/s\mathbb{Z}$ and $\{0, 1, \ldots, s-1\}$. A total order on $\mathbb{Z}/s\mathbb{Z}$ is defined correspondingly; it is this order that is used in the definition of $\mathcal{H}_{\mathbf{a}}$ below. The map $\mathcal{F}_{\mathbf{a}}$

induces a $\mathbb{Z}^d$-action $\mathbf{x} \cdot g = g + \mathcal{F}_{\mathbf{a}}(\mathbf{x})$ on the cyclic group $\mathbb{Z}/s\mathbb{Z}$, so that it is a rational case of the $\mathbb{Z}^2$-action on the torus as studied in [4,7]. We consider $\mathbb{E}_d = \{(\mathbf{u}, \mathbf{u}+\mathbf{e}_i) : \mathbf{u} \in \mathbb{Z}^d \text{ and } 1 \leq i \leq d\}$, the set of oriented edges of the hypercubic lattice. Note that the set $\mathbb{E}_d$ also corresponds to the Cayley graph of $\mathbb{Z}^d$ with generators $\mathbf{e}_i$ for all i with $1 \leq i \leq d$.

3.1 The Christoffel graph $\mathcal{H}_{\mathbf{a}}$

The *Christoffel graph* $\mathcal{H}_{\mathbf{a}}$ of normal vector $\mathbf{a}$ is the subset of edges of $\mathbb{E}_d$ increasing for the function $\mathcal{F}_{\mathbf{a}}$:

$$\mathcal{H}_{\mathbf{a}} = \{(\mathbf{u}, \mathbf{u}+\mathbf{e}_i) \in \mathbb{E}_d : \mathcal{F}_{\mathbf{a}}(\mathbf{u}) < \mathcal{F}_{\mathbf{a}}(\mathbf{u}+\mathbf{e}_i)\}.$$

An example of the graph $\mathcal{H}_{\mathbf{a}}$ when $d = 2$ and $\mathbf{a} = (a_1, a_2) = (2, 5)$ is shown in Figure 2 where the edges are represented in blue and a small red circle surrounds the origin. A first observation

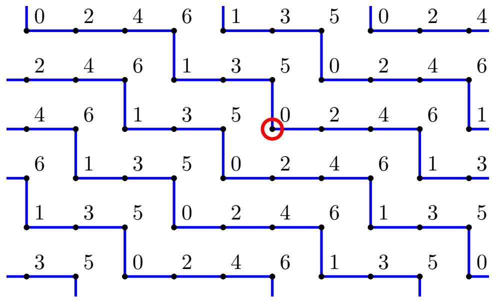

Figure 2: The graph $\mathcal{H}_{\mathbf{a}}$ with $\mathbf{a} = (2, 5)$.

is stated in the next lemma.

Lemma 2. *The graph $\mathcal{H}_{\mathbf{a}}$ is invariant under the translation by the vector $\sum_{i=1}^{d} \mathbf{e}_i = (1, 1, \ldots, 1)$.*

Definition 3 (Image)**.** *Let $f : \mathbb{Z}^d \to S$ be an homomorphism of $\mathbb{Z}$-module. For some subset of edges $X \subseteq \mathbb{E}_d$, we define the image by f of the edges X by*

$$f(X) = \{(f(\mathbf{u}), f(\mathbf{v})) \mid (\mathbf{u}, \mathbf{v}) \in X\}.$$

This definition allows to define the graphs $I_{\mathbf{a}}$ and $\mathcal{G}_{\mathbf{a}}$ as projections of $\mathcal{H}_{\mathbf{a}}$.

3.2 The graph $I_{\mathbf{a}}$

Let π be the orthogonal projection from $\mathbb{R}^d$ onto the hyperplane $\mathcal{D}$ of equation $\sum x_i = 0$. We consider the directed graph whose set of edges is $I_{\mathbf{a}} = \pi(\mathcal{H}_{\mathbf{a}})$. The graphs $I_{\mathbf{a}}$ for $\mathbf{a} = (a_1, a_2) = (2, 5)$ and $\mathbf{a} = (a_1, a_2, a_3) = (2, 3, 5)$ are shown in Figure 3. Note that the orientation of an edge is redundant when $d = 3$, since each edge is oriented as one of the vector $\mathbf{h}_i$.

Proposition 4. *The graph $I_{\mathbf{a}}$ produces a tiling of $\mathcal{D}$ by d types of $(d-1)$-dimensional parallelotopes.*

3.3 The graph $\mathcal{G}_{\mathbf{a}}$

Let $d \geq 2$ be an integer and $\mathbf{a}$ as before. The graph $\mathcal{G}_{\mathbf{a}}$ of normal vector $\mathbf{a} \in \mathbb{Z}^d$ is the directed graph $\mathcal{G}_{\mathbf{a}} = \mathcal{F}_{\mathbf{a}}(\mathcal{H}_{\mathbf{a}})$. It is also equal to

$$\mathcal{G}_{\mathbf{a}} = \{(k, k + a_i) \mid k \in \mathbb{Z}/s\mathbb{Z}, 1 \leq i \leq d \text{ and } k < k + a_i\}.$$

Examples are shown at Figure 4.

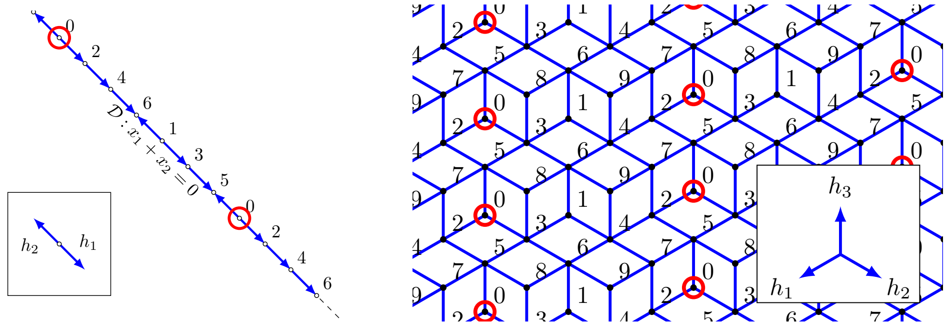

Figure 3: Left: the graph $I_{\mathbf{a}}$ when $\mathbf{a} = (2, 5)$. Right: the graph $I_{\mathbf{a}}$ when $\mathbf{a} = (2, 3, 5)$. The label at each vertex is its image under $\mathcal{F}_{\mathbf{a}}$.

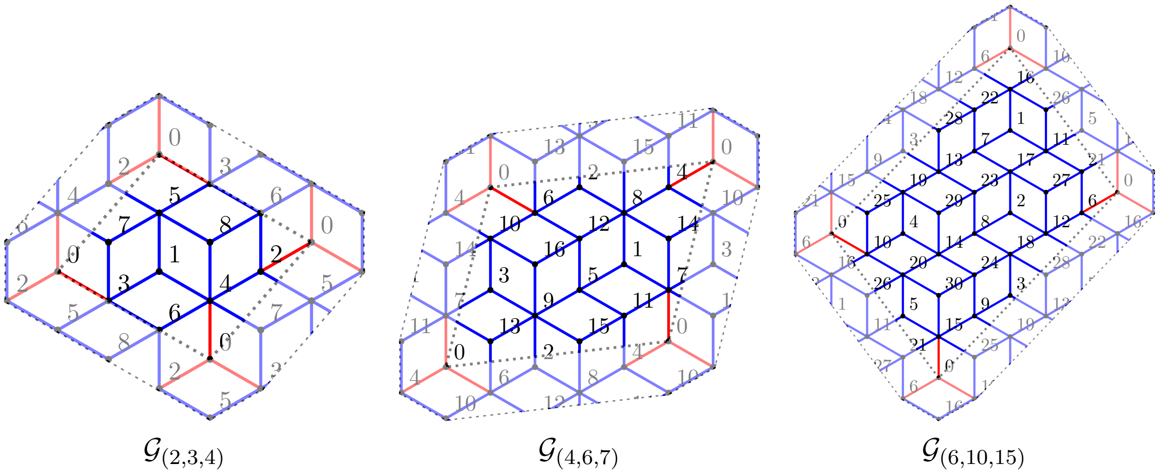

Figure 4: Some Christoffel graphs in dimension $d = 3$. Legs are the edges of the Christoffel graphs incident to zero. Every other edges constitute the body of the Christoffel graph.

4 Flip, reversal and translation

In this short section, we define the flip, reversal and translate of set of edges $X \subseteq \mathbb{E}_d$.

Definition 5 (edges of $\mathbb{E}_d$ incident to zero)**.** *Let $d \geq 2$ be an integer and $\mathbf{a} \in \mathbb{Z}^d$ be a vector of relatively prime positive integers. The set of* edges of $\mathbb{E}_d$ incident to zero *is*

$$\mathcal{Q} = \{(\mathbf{u}, \mathbf{v}) \in \mathbb{E}_d : \mathcal{F}_{\mathbf{a}}(\mathbf{u}) = 0 \text{ or } \mathcal{F}_{\mathbf{a}}(\mathbf{v}) = 0\}.$$

Definition 6 (body, legs)**.** *Let $X \subseteq \mathbb{E}_d$. The set $X \setminus \mathcal{Q}$ is the* body *and the edges of $X \cap \mathcal{Q}$ are the* legs *of X.*

See Figure 4 where the legs of graphs $\mathcal{G}_{\mathbf{a}}$ are represented in red, and the body in black.

Definition 7 (FLIP)**.** *For a subset of edges $X \subseteq \mathbb{E}_d$, we define the* FLIP *operation which exchanges edges incident to zero:*

$$\text{FLIP} : X \mapsto (X \setminus \mathcal{Q}) \cup (\mathcal{Q} \setminus X).$$

We see that FLIP(X) exchanges the legs of X and keeps the body of X invariant. The FLIP is an operation which generalizes the function $amb \mapsto bma$ defined for Christoffel words. While we define the flip on graphs, it can also be seen as a flip in a rhombus tiling when $d = 3$ [3,8,9].

Definition 8 (Reversal, Translate)**.** *Let $X \subseteq \mathbb{E}_d$ be a subset of edges. We define the* reversal *$-X$ of X and the* translate *$X + \mathbf{t}$, for some $\mathbf{t} \in \mathbb{Z}^d$, of X as*

$$-X = \{(-\mathbf{v}, -\mathbf{u}) \mid (\mathbf{u}, \mathbf{v}) \in X\} \qquad \textit{and} \qquad X + \mathbf{t} = \{(\mathbf{u} + \mathbf{t}, \mathbf{v} + \mathbf{t}) \mid (\mathbf{u}, \mathbf{v}) \in X\}.$$

5 Flipping is translating

In this section, we show that the flip of the Christoffel graph $\mathcal{H}_\mathbf{a}$ is a translate of $\mathcal{H}_\mathbf{a}$; this is a generalization of one implication of Theorem 1 (it generalizes the fact that a Christoffel word amb is conjugate to to bma). We also show that the body of $\mathcal{H}_\mathbf{a}$ is symmetric (generalizes the fact that central words are palindromes), the reversal of $\mathcal{H}_\mathbf{a}$ is equal to its flip (generalizes the fact that the reversal $\widetilde{amb}$ of a Christoffel word is equal to bma) and as a consequence we obtain that a Christoffel graph is a translate of its reversal (a Christoffel word is conjugate to its reversal).

Lemma 9 (Legs of $\mathcal{H}_\mathbf{a}$)**.** *An edge $(\mathbf{u}, \mathbf{v})$ is a leg of $\mathcal{H}_\mathbf{a}$ if and only if $\mathcal{F}_\mathbf{a}(\mathbf{u}) = 0$.*

Lemma 10. *The body of $\mathcal{H}_\mathbf{a}$ is symmetric, i.e., $-(\mathcal{H}_\mathbf{a} \setminus \mathcal{Q}) = \mathcal{H}_\mathbf{a} \setminus \mathcal{Q}$.*

Lemma 11. *The reversal of $\mathcal{H}_\mathbf{a}$ is equal to its flip, i.e., $-\mathcal{H}_\mathbf{a} = \textsc{flip}(\mathcal{H}_\mathbf{a})$.*

Proposition 12. *Let $\mathbf{t} \in \mathbb{Z}^d$ be such that $\mathcal{F}_\mathbf{a}(\mathbf{t}) = 1$. The translate by $\mathbf{t}$ of $\mathcal{H}_\mathbf{a}$ is equal to its flip, i.e., $\mathcal{H}_\mathbf{a} + \mathbf{t} = \textsc{flip}(\mathcal{H}_\mathbf{a})$.*

Proposition 12 is illustrated in Figure 5 and Figure 6.

Corollary 13. *Let $\mathbf{t} \in \mathbb{Z}^d$ be such that $\mathcal{F}_\mathbf{a}(\mathbf{t}) = 1$. Then $-\mathcal{H}_\mathbf{a} = \mathcal{H}_\mathbf{a} + \mathbf{t}$.*

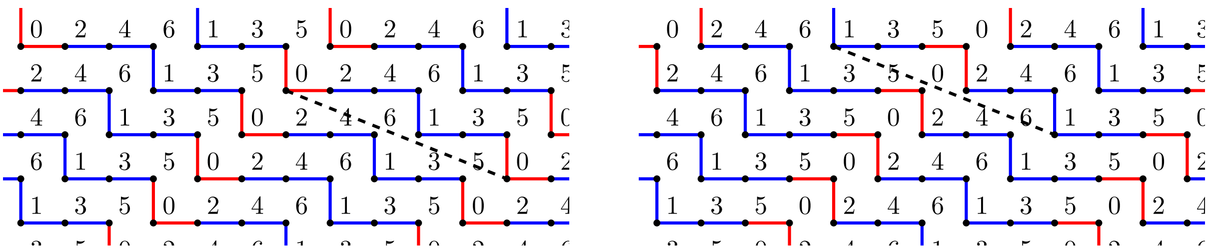

Figure 5: Left: the graph $\mathcal{H}_\mathbf{a}$ with $\mathbf{a} = (2, 5)$. Right: $\textsc{flip}(\mathcal{H}_\mathbf{a})$.

6 Higher-dimensional Pirillo's theorem

This section considers the converse of Proposition 12 and contains the main result of this article which generalizes Pirillo's theorem (Theorem 1) to arbitrary dimension: a graph $M \subseteq \mathbb{E}_d$ is a translate of its flip if and only if it is a Christoffel graph. In order to do so, we need to extend the definition of the Christoffel graph $\mathcal{H}_{\mathbf{a},\omega}$ for a vector $\mathbf{a} \in \mathbb{Z}^d$ and width ω.

6.1 The graph $\mathcal{H}_{\mathbf{a},\omega}$

Again let $\mathbf{a} \in \mathbb{N}^d$ be a vector of relatively prime positive integers and $s = \|\mathbf{a}\|_1 = \sum a_i$. Let $\omega \in \mathbb{N}$ be some *width* such that s/ω is a positive integer strictly smaller than the dimension d: $0 < s/\omega < d$. We define the mapping $\mathcal{F}_{\mathbf{a},\omega} : \mathbb{Z}^d \to \mathbb{Z}/\omega\mathbb{Z}$ sending each integral vector $(x_1, \ldots, x_d)$

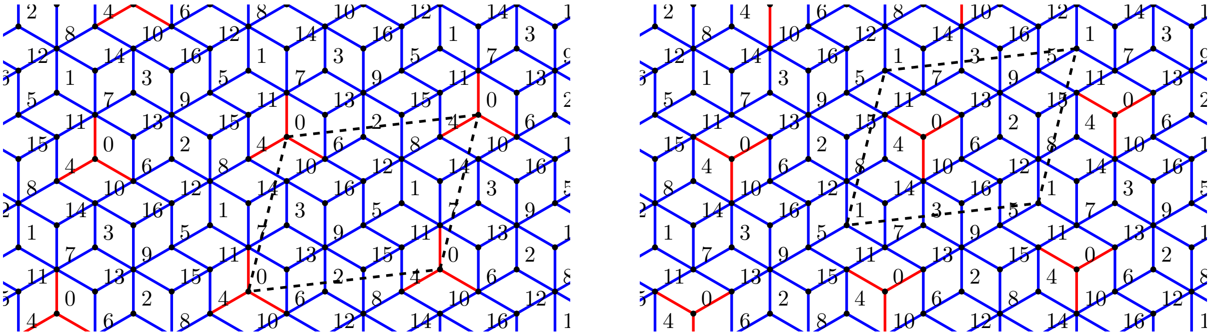

Figure 6: Left: the graph $I_{\mathbf{a}}$ with $\mathbf{a} = (4, 6, 7)$. Right: $\textsc{flip}(I_{\mathbf{a}})$. Consider the Christoffel parallelogram P with vertices labeled by 0 embedded in $I_{\mathbf{a}}$. The parallelogram P also appears in the graph $\textsc{flip}(I_{\mathbf{a}})$ with vertices labeled by 1.

onto $\sum_i a_i x_i \bmod \omega$. We identify $\mathbb{Z}/\omega\mathbb{Z}$ and $\{0, 1, \cdots, \omega - 1\}$. A total order on $\mathbb{Z}/\omega\mathbb{Z}$ is defined correspondingly. The *Christoffel graph of normal vector* $\mathbf{a} \in \mathbb{N}^d$ *of width* ω is the subset of edges $\mathcal{H}_{\mathbf{a},\omega} \subseteq \mathbb{E}_d$ defined by

$$\mathcal{H}_{\mathbf{a},\omega} = \{(\mathbf{u}, \mathbf{v}) \in \mathbb{E}_d \mid \mathcal{F}_{\mathbf{a},\omega}(\mathbf{u}) < \mathcal{F}_{\mathbf{a},\omega}(\mathbf{v})\}.$$

This graph is related but does not correspond exactly to discrete plane of width ω. In fact, $\mathcal{H}_{\mathbf{a},\omega}$ can be obtained by the superposition of s/ω discrete plane of width ω. The definition of $\mathcal{H}_{\mathbf{a},\omega}$ is motivated by Pirillo's theorem, because this is what allows to generalize Pirillo's theorem in arbitrary dimension (see Theorem 14). Of course if $\omega = s$, then $\mathcal{H}_{\mathbf{a},\omega} = \mathcal{H}_{\mathbf{a}}$ is the Christoffel graph of normal vector $\mathbf{a}$. Also, if $d = 2$ then $s = \omega$. If $d = 3$, then either $\omega = s$ or $\omega = s/2$.

Theorem 14 (**d-dimensional Pirillo's theorem**). *Let K be a subgroup of finite index of $\mathbb{Z}^d$ such that $\sum_{i=1}^{d} \mathbf{e}_i \in K$. Let $M \subseteq \mathbb{E}_d$ be a subset of edges invariant for the group of translations K such that the edges of M incident to zero mod K are $\mathcal{Q} \cap M = \{(\mathbf{0}, \mathbf{e}_i) \mid 1 \leq i \leq d\} + K$. There exists $\mathbf{t} \in \mathbb{Z}^d$ such that $M = \textsc{flip}(M + \mathbf{t})$ if and only if $M = \mathcal{H}_{\mathbf{a},\omega}$ is the Christoffel graph of normal vector $\mathbf{a}$ and width ω.*

The result is illustrated in Figure 7 by an example when $d = 3$.

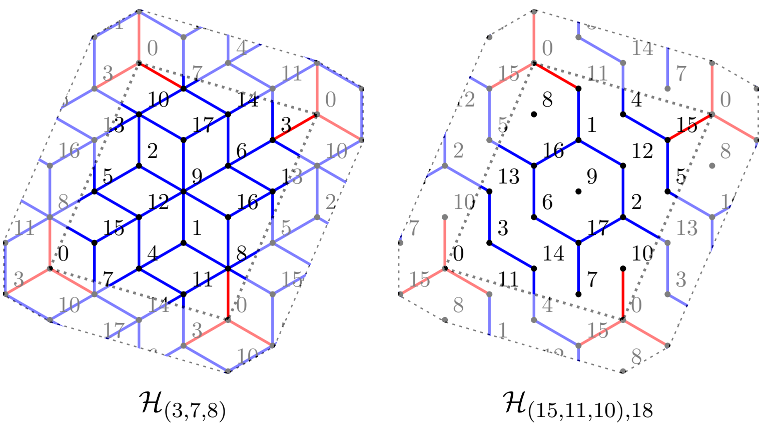

Figure 7: Left: the Christoffel graph $\mathcal{H}_{\mathbf{a}}$ for the vector $\mathbf{a} = (3, 7, 8)$. It satisfies the equation $M = \textsc{flip}(M + \mathbf{t})$ for the translation vector $\mathbf{t} = \mathbf{e}_3 - \mathbf{e}_2$. Right: the complement of the reversal of the Christoffel graph for the vector $\mathbf{b} = (3, 7, 8)$, i.e. $\mathbb{E}_d \setminus -\mathcal{H}_{\mathbf{b}}$. It corresponds to the Christoffel graph $\mathcal{H}_{\mathbf{a},\omega}$ for the vector $\mathbf{a} = (15, 11, 10)$ and width $\omega = 18$. It satisfies the equation $M = \textsc{flip}(M + \mathbf{t})$ for the translation vector $\mathbf{t} = \mathbf{e}_2 - \mathbf{e}_3$. They represent the only two possibilities for a pattern M satisfying $M = \textsc{flip}(M + \mathbf{t})$ when $d = 3$ and K is the subgroup of $\mathbb{Z}^3$ given by $\langle (0, 4, 1), (-2, 0, 3), (1, 1, 1) \rangle$.

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