Two infinite families of polyominoes that tile the plane by translation in two distinct ways[☆]

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Abstract

It has been proved that, among the polyominoes that tile the plane by translation, the so-called squares tile the plane in at most two distinct ways. In this paper, we focus on double squares, that is, the polyominoes that tile the plane in exactly two distinct ways. Our approach is based on solving equations on words, which allows to exhibit properties about their shape. Moreover, we describe two infinite families of double squares. The first one is directly linked to Christoffel words and may be interpreted as segments of thick straight lines. The second one stems from the Fibonacci sequence and reveals some fractal features.

Keywords: Tesselation, Polyomino, Christoffel word, Fibonacci sequence

1. Introduction

During the DGCI 2006 conference held in Szeged [1], E. Andres asked for a description of tesselations of the plane with tiles whose boundary is composed

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of discrete segments. That was the starting point of an investigation that shed new light on connections between discrete geometry, combinatorics on words and also number theory.

The basic object of study is the *polyomino*, ubiquitous in the literature for having applications in numerous fields whose listing is needless for our purpose. There are different types of polyominoes, and by *polyomino* we mean a finite union of unit lattice squares (pixels) in the discrete plane whose boundary is a simple closed path. In particular, a polyomino is simply connected (without holes), and its boundary is simple (does not cross itself). Paths are conveniently encoded by words, called Freeman chain codes, on the alphabet $\mathcal{F} = \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}\}$, representing the elementary grid steps $\{\rightarrow, \uparrow, \leftarrow, \downarrow\}$.

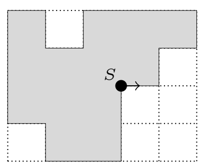


Figure 1: A polyomino and its boundary.

For instance, starting from S the boundary $\mathbf{b}(P)$ of the polyomino P in Figure 1 is coded (counterclockwise) by the word

$$w = \mathbf{0101222321233303001 1}.$$

Observe that we may consider these boundary words as circular, which avoids fixing an origin. Moreover, the *perimeter* of a polyomino P is the length of its boundary words and is an even number.

The problem of deciding if a given polyomino tiles the plane by translation was first considered by Wisjhoff and Van Leeuven [2] who coined the term *exact polyomino* for these. Beauquier and Nivat [3] characterized them by stating that the boundary $b(P)$ of an exact polyomino P satisfies the following (not necessarily in a unique way) factorization

$$\mathbf{b}(P) = A \cdot B \cdot C \cdot \widehat{A} \cdot \widehat{B} \cdot \widehat{C} \tag{1}$$

where at most one of the variables is empty, and where $\widehat{X}$ denotes the path X traveled in the opposite direction. Hereafter, this condition is referred to as the BN-factorization. For example, the polyomino in Figure 2 (left) is exact and its boundary may be factorized as $\mathbf{101} \cdot \mathbf{2} \cdot \mathbf{23212} \cdot \mathbf{323} \cdot \mathbf{0} \cdot \mathbf{03010}$.

Polyominoes having a factorization with A, B and C nonempty are called *pseudo-hexagons*, while *pseudo-squares* designate those for which one of the variable is empty. From now on, we call them *squares* for simplicity. It has been shown in [4] that there exist polyominoes admitting an arbitrary number of distinct non trivial factorizations as pseudo-hexagons. Surprisingly, the situation is different for squares, and it was conjectured in [4] that a polyomino cannot have more than two distinct square factorizations.

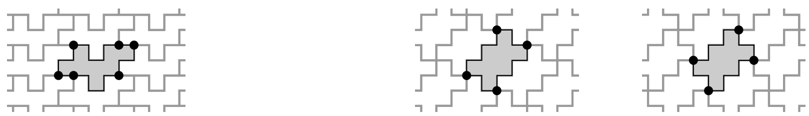


Figure 2: Left: a pseudo-hexagon. Right: A double square and its two tilings.

Polyominoes admitting two distinct square factorizations (Figure 2 right) are called *double squares*. An exhaustive search based on Equation (1) allows to enumerate double squares exhaustively, but since they have very specific structural properties, this leads to a more efficient way to generate them. Moreover, another conjecture on double squares states that the factors of the BN-factorizations are palindromes. For more details on tiling by translation and square tilings see [1, 4, 5].

In this paper we use a combinatorial approach, relying on efficient techniques [1, 6], for constructing two classes of double square polyominoes [7]. These two families are important for the zoology because they describe entirely the table of small double squares available in [4]. The first is composed of Christoffel tiles, those for which the boundary word is composed of crenelated versions of two digitized segments (answering partially to E. Andres' question), for which a characterization is provided (Theorem 6). The second is built on the Fibonacci recurrence: a special family of Fibonacci tiles is completely described (Theorem 11). The palindromicity of the factors in the BN-factorization is proved for both families, and four derived classes of double squares are also presented.

2. Preliminaries

The usual terminology and notation on words is from Lothaire [8]. An *alphabet* $\mathcal{A}$ is a finite set whose elements are called *letters*. A finite word w is a function $w : [1, 2, \ldots, n] \to \mathcal{A}$, where w_i is the i-th letter, $1 \leq i \leq n$. The *length* of w, denoted by $|w|$, is the integer n. The length of the empty word ε is 0. The *free monoid* $\mathcal{A}^*$ is the set of all finite words over $\mathcal{A}$. The *reversal* of $w = w_1 w_2 \cdots w_n$ is the word $\widetilde{w} = w_n w_{n-1} \cdots w_1$. Words p satisfying $p = \widetilde{p}$ are called *palindromes*. A word u is a *factor* of another word w if there exist $x, y \in \mathcal{A}^*$ such that $w = xuy$. We denote by $|w|_u$ the number of occurrences of u in w. Two words u and v are *conjugate*, written $u \equiv v$ or sometimes $u \equiv_{|x|} v$, when x, y are such that $u = xy$ and $v = yx$. Conjugacy is an equivalence relation.

In this paper, the alphabet $\mathcal{F} = \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}\}$ is considered as the additive group of integers mod 4. Basic transformations on $\mathcal{F}$ are rotations $\rho^i : x \mapsto x+i$ and reflections $\sigma_i : x \mapsto i-x$, which extend uniquely to morphisms (w.r.t concatenation) on $\mathcal{F}^*$. Given a nonempty word $w \in \mathcal{F}^*$, the *first differences word* $\Delta(w) \in \mathcal{F}^*$ of w is

$$\Delta(w) = (w_2 - w_1) \cdot (w_3 - w_2) \cdots (w_n - w_{n-1}). \tag{2}$$

Words in $\mathcal{F}^*$ are interpreted as paths in the square grid as usual (see Figure 3), so that we indistinctly talk of any word $w \in \mathcal{F}^*$ as the *path* w.

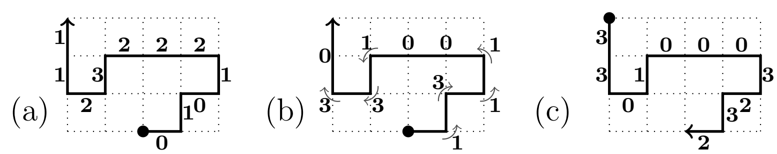


Figure 3: (a) The path $w = \mathbf{01012223211}$. (b) Its first differences word $\Delta(w) = \mathbf{1311001330}$. (c) Its homologous $\widehat{w} = \mathbf{33010003232}$.

Moreover, the word $\widehat{w} := \rho^2(\widetilde{w})$ is *homologous* to w, that is, described in direction opposite to that of w (see Figure 3 (c)).

If at least one letter of w is known, the word w can be recovered completely from $\Delta(w)$ using its sequence of partial sums. Then we define the sequence of partial sums $\Sigma_\alpha(w) \in \mathcal{F}^*$ of a word $w \in \mathcal{F}^*$ starting by the letter $\alpha \in \mathcal{F}$ by

$$\Sigma_\alpha(w) = \alpha \cdot (\alpha + w_1) \cdot (\alpha + w_1 + w_2) \cdots (\alpha + w_1 + w_2 + \cdots + w_n).$$

A word $u \in \mathcal{F}^*$ may contain factors in $\mathcal{C} = \{\mathbf{02}, \mathbf{20}, \mathbf{13}, \mathbf{31}\}$, corresponding to canceling steps on a path. Nevertheless, each word w can be reduced in a unique way to a word w', by sequentially applying the rewriting rules in the set $\{u \mapsto \varepsilon | u \in \mathcal{C}\}$. The *reduced word* w' of w is nothing but a word in $\mathcal{P} = \mathcal{F}^* \setminus \mathcal{F}^*\mathcal{C}\mathcal{F}^*$. Therefore, we define the *turning number*[1] of w by $\mathcal{T}(w) = (|\Delta(w')|_{\mathbf{1}} - |\Delta(w')|_{\mathbf{3}})/4$.

[1]In [6], the authors used the notion of *winding number* of w which is $4\mathcal{T}(w)$

We also introduce two auxiliary length preserving functions on $\mathcal{F}^*$ that are meaningful for closed paths. The first is defined by

$$\overset{\circ}{\Delta}(w) = \Delta(w) \cdot (w_1 - w_n),$$

while the second is defined

$$\overset{\circ}{\Sigma}_\alpha(w) = \alpha \cdot (\alpha + w_1) \cdot (\alpha + w_1 + w_2) \cdots (\alpha + w_1 + w_2 + \cdots + w_{n-1}).$$

We end this section with two results useful for the next section.

Proposition 1 (Provençal [4]). *If a square has two factorizations, then they alternate, i.e. no factor of one factorization is included in a factor of the other one.* □

Lemma 2. *Let W be the boundary word of a square, A and B be words such that $W \equiv AB\widehat{A}\widehat{B}$. Then A and B are palindromes if and only if $W = w\rho^2(w)$ for some word w.*

Proof. If $W = w\rho^2(w)$ then every conjugate of W has this form. Therefore, if $W \equiv AB\widehat{A}\widehat{B}$, we have that $AB = \rho^2(\widehat{A}\widehat{B}) = \widetilde{A}\widetilde{B}$, showing that A and B are palindromes. Conversely, one shows that if A and B are palindromes, then $W \equiv AB\rho^2(AB)$. □

3. Christoffel tiles

Recall that Christoffel words are finite Sturmian words, that is, they are obtained by discretizing a line segment in the plane. Let $(p,q) \in \mathbb{N}^2$ with $\gcd(p,q) = 1$, and let S be the line segment with endpoints $(0,0)$ and (p,q). The word w is a *lower Christoffel word* if the path induced by w is under S and if they both delimit a polygon with no integral interior point. An *upper Christoffel word* is defined similarly. A *Christoffel word* is either a lower Christoffel word or an upper Christoffel word. See the Figure 4 (a).

It is well known that if w and w' are respectively the lower and upper Christoffel words associated to (p,q), then $w' = \widetilde{w}$. Moreover, we have $w = \mathbf{0}m\mathbf{1}$ and $w' = \mathbf{1}m\mathbf{0}$, where m is a palindrome and $\mathbf{0}$, $\mathbf{1}$ are letters. The word m is called *cutting word* . They have been widely studied in the literature (see e.g. [9], where they are also called *central words*).

The next theorem gives a very useful characterization of Christoffel words.

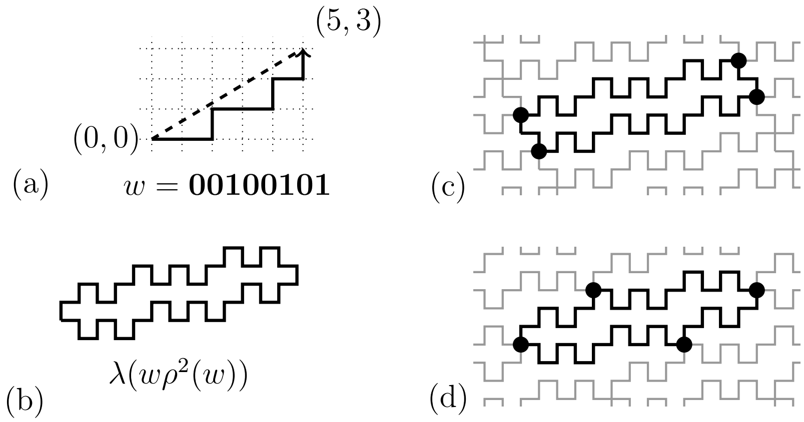


Figure 4: (a) The lower Christoffel word $w = \mathbf{00100101}$. (b) The Christoffel Tile $\lambda(w)\rho^2(\lambda(w))$ is a double square tile, with distinct tilings (c) and (d).

Theorem 3 (Pirillo [10]). *A word m on the two-letter alphabet $\{\mathbf{0}, \mathbf{1}\} \subset \mathcal{F}$ is a cutting word if and only if $\mathbf{0}m\mathbf{1}$ and $\mathbf{1}m\mathbf{0}$ are conjugate.* $\square$

Another useful result is the following.

Proposition 4 (Borel and Reutenauer [11]). *The lower and upper Christoffel words w and w' are conjugate by palindromes.* $\square$

Consider the morphism $\lambda : \mathcal{F}^* \to \mathcal{F}^*$ defined by $\mathbf{0} \mapsto \mathbf{0301}$, $\mathbf{1} \mapsto \mathbf{01}$, $\mathbf{2} \mapsto \mathbf{2123}$ and $\mathbf{3} \mapsto \mathbf{23}$, which can be seen as a "crenelation" of the four canonical steps. Two useful properties of λ are used through the rest of this section and are easy to establish.

Lemma 5. *Let $v, v' \in \mathcal{F}^*$. Then*

(i) *$\mathbf{1}\lambda(v)$ is a palindrome if and only if v is a palindrome.*

(ii) *$\lambda(v) \equiv \lambda(v')$ if and only if $v \equiv v'$.*

Theorem 6. *Let $w = \mathbf{0}v\mathbf{1} \in \{\mathbf{0}, \mathbf{1}\}^* \subset \mathcal{F}^*$.*

(i) *If v is a palindrome, then $\lambda(w\rho^2(w))$ is a square tile.*

(ii) *$\lambda(w\rho^2(w))$ is a double square if and only if w is a Christoffel word.*

PROOF. (i) First, we have the square factorization

$$\lambda(w\rho^2(w)) = \mathbf{0301}\lambda(v)\mathbf{01212}\rho^2(\mathbf{1}\lambda(v))\mathbf{23} \equiv \mathbf{303}\cdot\mathbf{01}\lambda(v)\mathbf{0}\cdot\widehat{\mathbf{303}}\cdot\widehat{\mathbf{01}\lambda(v)}\mathbf{0}.$$

Now we show that $\lambda(w\rho^2(w))$ is simple. Clearly, $\lambda(w)$ and $\lambda(\rho^2(w))$ are simple since they both contain three letters and no word of the form $\alpha\overline{\alpha}$. Moreover, if P and Q denote respectively the starting and ending points of $\lambda(w)$, then the path $\lambda(w)$ is below the line PQ while $\lambda(\rho^2(w))$ is above.

(ii) ($\Rightarrow$) Assume that W is a double square. Let $W = \lambda(w\rho^2(w))$ be its boundary word, where $w = \mathbf{0}v\mathbf{1} \in \mathbf{0}\mathrm{Pal}(\mathcal{F}^*)\mathbf{1}$. Since W factorizes as

$$W = \mathbf{303}\cdot\mathbf{01}\lambda(v)\mathbf{0}\cdot\widehat{\mathbf{303}}\cdot\widehat{\mathbf{01}\lambda(v)}\mathbf{0}, \tag{3}$$

and since the factorizations must alternate (Proposition 1), the second factorization starts with the second or the third letter of W. Let W', W'' be such that $W \equiv_1 W'$ and $W \equiv_2 W''$ and V' and V'' be respectively the first half of W' and W''. Then, by Lemma 2, either V' or V'' is a product of two palindromes x and y. First, assume that the other factorization is obtained from V'. Then $V' = \lambda(\mathbf{0}v\mathbf{1}) = \mathbf{0301}\lambda(v)\mathbf{01} = xy$. Taking the reversal on both sides, we get $\widetilde{\lambda(\mathbf{0}v\mathbf{1})} = yx$, that is $\lambda(\mathbf{0}v\mathbf{1}) \equiv \widetilde{\lambda(\mathbf{0}v\mathbf{1})}$. But

$$\begin{aligned}\widetilde{\lambda(\mathbf{0}v\mathbf{1})} &= \mathbf{10}\widetilde{\lambda(v)}\mathbf{1030} = \mathbf{101}\widetilde{\lambda(v)}\mathbf{030} = \mathbf{101}\lambda(v)\mathbf{030}\\ &\equiv \mathbf{01}\lambda(v)\mathbf{0301} = \lambda(\mathbf{1}v\mathbf{0}),\end{aligned}$$

which means that $\lambda(\mathbf{0}v\mathbf{1}) \equiv \lambda(\mathbf{1}v\mathbf{0})$. Thus, by Lemma 5, it follows that $\mathbf{0}v\mathbf{1} \equiv \mathbf{1}v\mathbf{0}$. Hence, by Theorem 3, v is a cutting word so that $w = \mathbf{0}v\mathbf{1}$ is a lower Christoffel word. It remains to consider the case where the second factorization is obtained from V''. We could then write $V'' = \mathbf{301}\lambda(v)\mathbf{012} = xy$. But such palindromes x and y cannot exist since $\mathbf{2}$ appears only at the end of V''.

($\Leftarrow$) Assume that $w = \mathbf{0}v\mathbf{1}$ is a lower Christoffel word. It is well known that v is a palindrome. Then from (i), $\lambda(w\rho^2(w))$ is the boundary of a square tile. We know from Proposition 4 that $w = \mathbf{0}m\mathbf{01}m'\mathbf{1}$ for some other palindromes m and m'. Therefore,

$$\begin{aligned}\lambda(w\rho^2(w)) &= \lambda(\mathbf{0}m\mathbf{01}m'\mathbf{1})\rho^2(\lambda(\mathbf{0}m\mathbf{01}m'\mathbf{1}))\\ &= \mathbf{0301}\lambda(m)\mathbf{030}\cdot\mathbf{101}\lambda(m')\mathbf{01}\cdot\mathbf{2123}\rho^2(\lambda(m))\mathbf{212}\\ &\quad\cdot\mathbf{323}\rho^2(\lambda(m'))\mathbf{23},\end{aligned}$$

showing that P admits a second square factorization. $\square$

We say that a crenelated tile $\lambda(w\rho^2(w))$ obtained from a lower Christoffel word w is a *basic Christoffel tile* while a *Christoffel tile* is a polyomino isometric to a basic Christoffel tile under some rotations ρ and symmetries σ_i. Observe that, in view of Lemma 2, if $W \equiv AB\widehat{A}\widehat{B}$ is the boundary word of a Christoffel tile, then the factors A and B are palindromes, a result compatible with the conjecture of Provençal and Vuillon [4]. To conclude this section, we extract interesting statistics of the Christoffel tiles.

Proposition 7. *Let T be a Christoffel tile obtained from the (p, q)-Christoffel word, where p and q are relatively prime. Then the perimeter and the area of T are given respectively by $\mathbf{P}(T) = 8p + 4q$ and $\mathbf{A}(T) = 4p + 3q - 2$.*

Proof. Let $w = \mathbf{0}v\mathbf{1}$ be the (p, q)-Christoffel word. First, we have

$$\mathbf{P}(T) = |\lambda(w)\rho^2(\lambda(w))| = 2|\lambda(w)| = 2(4|w|_\mathbf{0} + 2|w|_\mathbf{1}) = 8p + 4q.$$

On the other hand, it follows from Equation (3) in the proof of Theorem 6 that the area of a T is exactly the area of the parallelogram determined by the vectors $\overrightarrow{A} = \overrightarrow{\mathbf{303}} = (1, -2)$ and

$$\overrightarrow{B} = \overrightarrow{\mathbf{01}\lambda(v)\mathbf{0}} = (2, 1) + (2|v|_\mathbf{0} + |v|_\mathbf{1}, |v|_\mathbf{1}) = (2p + q - 1, q).$$

Hence, $\mathbf{A}(T) = |\overrightarrow{A} \times \overrightarrow{B}| = 4p + 3q - 2$. □

4. Fibonacci tiles

Among Sturmian words, the most famous one is the Fibonacci word

$$abaababaabaababaababa \cdots .$$

defined as the limit of the sequence satisfying $f_{-1} = b$, $f_0 = a$, and, for $n \geq 1$, $f_n = f_{n-1}f_{n-2}$. Equivalently, one may prove that $\mathbf{f}$ is the fixed point of the morphism $\varphi : \{a, b\} \to \{a, b\}$ defined by $\varphi(a) = ab$ and $\varphi(b) = a$.

From $\mathbf{f}$, it is possible to derive naturally a path on $\mathcal{F}$ having remarkable properties. The construction is obtained as follows. First, rewrite the Fibonacci word on the alphabet $\{\mathbf{2}, \mathbf{0}\} \subset \mathcal{F}$ instead of $\{a, b\}$. Then

$$\mathbf{f} = \mathbf{202202022022020220202} \cdots .$$

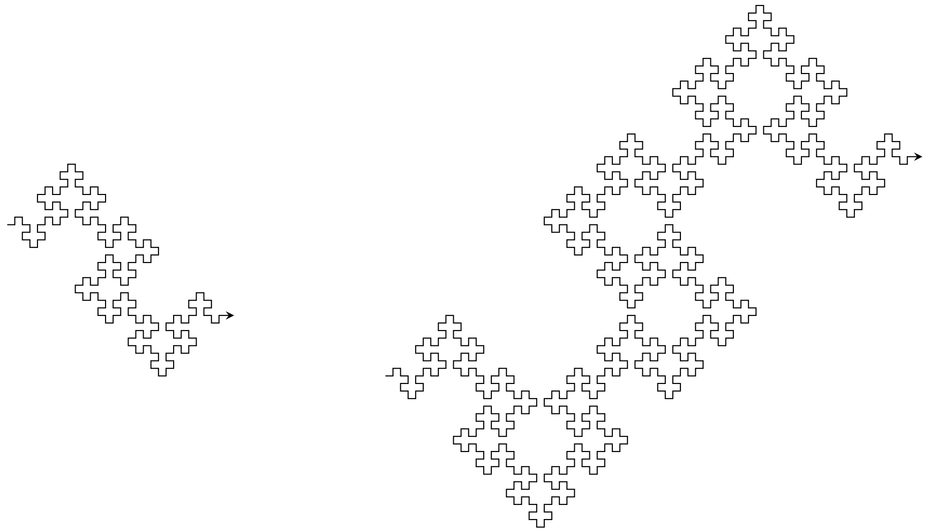


Figure 5: The prefixes of length 233 (left) and 987 (right) of the path $\mathbf{p}$.

Next, apply the operator $\Sigma_\mathbf{1}$ followed by the operator $\Sigma_\mathbf{0}$. This yields the word

$$\mathbf{p} = \Sigma_\mathbf{0}\Sigma_\mathbf{1}\mathbf{f} = \mathbf{0103032303010121010301012123 2}\cdots .$$

which is as an infinite path on the square grid (see Figure 5). It has been discovered independently in [12], where the construction is equivalent but slightly different.

It is convenient to express the path $\mathbf{p}$ by means of the turns *right* and *left* (encoded respectively by $\mathbf{3}$ and $\mathbf{1}$) instead of the four elementary steps $\mathbf{0}$, $\mathbf{1}$, $\mathbf{2}$ and $\mathbf{3}$. Thus, the sequence of turns of the path $\mathbf{p}$ is given by $\Sigma_\mathbf{1}\mathbf{f}$ or, equivalently, by $\Delta\mathbf{p}$. We shall denote the sequence of turns by $\mathbf{q}$.

Hereafter, in order to simplify the notation, we define $\overline{w} = \sigma_0(w)$ for all $w \in \mathcal{F}^*$. Hence, $\overline{\mathbf{0}} = \mathbf{0}$, $\overline{\mathbf{1}} = \mathbf{3}$, $\overline{\mathbf{2}} = \mathbf{2}$ and $\overline{\mathbf{3}} = \mathbf{1}$ and the words w of $\mathcal{F}^*$ satisfying $\widetilde{w} = \overline{w}$ are called *σ_0-palindromes.*

Consider the sequence $(q_n)_{n\in\mathbb{N}}$ in $\mathcal{F}^*$ defined by $q_0 = \varepsilon$, $q_1 = \mathbf{3}$ and

$$q_n = \begin{cases} q_{n-1}q_{n-2} & \text{if } n \equiv 2 \mod 3, \\ q_{n-1}\overline{q_{n-2}} & \text{if } n \equiv 0, 1 \mod 3. \end{cases}$$

whenever $n \geq 2$. The first terms of $(q_n)_{n\in\mathbb{N}}$ are

$$
\begin{array}{lll}
q_0 = \varepsilon & q_3 = \mathbf{31} & q_6 = \mathbf{31131133} \\
q_1 = \mathbf{3} & q_4 = \mathbf{311} & q_7 = \mathbf{3113113313313} \\
q_2 = \mathbf{3} & q_5 = \mathbf{31131} & q_8 = \mathbf{311311331331331131133}
\end{array}
$$

Moreover, $|q_n| = F_n$ is the n-th Fibonacci number.

Proposition 8 (Blondin Massé and Paquin [13]). *The sequence* $\mathbf{q}$ *of turns is the limit of the sequence* $(q_n)_{n\in\mathbb{N}}$.

Proof. Since $\Delta(\mathbf{q}) = \mathbf{f}$, it suffices to show that $\Delta(q_n)\alpha_n = f_{n-1}$ for all integers $n \geq 3$, where $\alpha_n = \sigma_2^n(\mathbf{2})$. The proof is done by induction on n. First, we have $\Delta(q_3)\sigma_2^3(\mathbf{2}) = \Delta(\mathbf{31})\mathbf{0} = \mathbf{20} = f_2$, $\Delta(q_4)\sigma_2^4(\mathbf{2}) = \Delta(\mathbf{311})\mathbf{2} = \mathbf{202} = f_3$ and $\Delta(q_5)\sigma_2^5(\mathbf{2}) = \Delta(\mathbf{31131})\mathbf{0} = \mathbf{20220} = f_4$. Now, assume that the result holds for all integers m such that $3 \leq m < n$ and let us show that it also holds for n. We only prove the case $n \equiv 2 \bmod 3$ since the argument is similar for the cases $n \equiv i \bmod 3$, $i \in \{0, 1\}$. Let $n = 3k + 2$ for some integer k. Then

$$
\begin{aligned}
\Delta(q_{3k+2})\alpha_{3k+2} &= \Delta(q_{3k+1}q_{3k})\alpha_{3k+2} \\
&= \Delta(q_{3k+1})\Delta(\sigma_0^k(\mathbf{3})\mathbf{3})\Delta(q_{3k})\alpha_{3k+2} \\
&= \Delta(q_{3k+1})\sigma_2^k(\mathbf{0})\Delta(q_{3k})\alpha_{3k+2} \\
&= \Delta(q_{3k+1})\alpha_{3k+1}\Delta(q_{3k})\alpha_{3k} \\
&= f_{3k+1}f_{3k} = f_{3k+2},
\end{aligned}
$$

and the result follows. □

Given $\alpha \in \mathcal{F}$, the path $\Sigma_\alpha q_n$ exhibits interesting symmetry properties.

Lemma 9 (Blondin Massé et al. [14]). *Let* $n \in \mathbb{N}$ *and* $\alpha = \sigma_0^n(\mathbf{3})$. *Then* $q_{3n+1} = p\alpha$, $q_{3n+2} = r\alpha$ *and* $q_{3n+3} = s\overline{\alpha}$ *for some* σ_0*-palindrome* p *and some palindromes* r *and* s.

Proof. By induction on n. For $n = 0$, we have indeed $q_1 = \varepsilon \cdot \mathbf{3}$, $q_2 = \varepsilon \cdot \mathbf{3}$ and $q_3 = \mathbf{3} \cdot \mathbf{1}$. Now, assume that $q_{3n+1} = p\alpha$, $q_{3n+2} = r\alpha$ and $q_{3n+3} = s\overline{\alpha}$ for some σ_0-palindrome p, some palindromes r, s and $\alpha = \sigma_0^n(\mathbf{3})$. Then

$$
\begin{aligned}
q_{3n+4} &= q_{3n+3}\overline{q_{3n+2}} = q_{3n+2}\overline{q_{3n+1}q_{3n+2}} = r\alpha\overline{p\alpha r} \cdot \sigma_0^{n+1}(\mathbf{3}) \\
q_{3n+5} &= q_{3n+4}q_{3n+3} = q_{3n+3}\overline{q_{3n+2}}q_{3n+3} = s\overline{\alpha r \alpha}s \cdot \sigma_0^{n+1}(\mathbf{3}) \\
q_{3n+6} &= q_{3n+5}\overline{q_{3n+4}} = q_{3n+4}q_{3n+3}\overline{q_{3n+4}} = r\alpha\overline{p\alpha r\alpha}s\overline{\alpha r\alpha}p\alpha r \cdot \sigma^{n+2}(\mathbf{3})
\end{aligned}
$$

Since $r\alpha\overline{p\alpha r}$ is a σ_0-palindrome and $s\overline{\alpha r\alpha}s$, $r\alpha\overline{p\alpha r\alpha}s\overline{\alpha r\alpha}p\alpha r$ are palindromes, the result follows. □

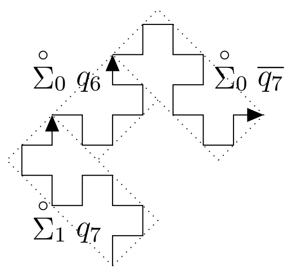
Lemma 10 (Blondin Massé et al. [14]). *Let $n \in \mathbb{N}$ and $\alpha \in \mathcal{F}$.*

(i) *The path $\Sigma_\alpha q_n$ is simple.*

(ii) *The path $\mathring{\Sigma}_\alpha\, (q_{3n+1})^4$ is the boundary word of a polyomino.* □

Proof. Since the proof is rather technical, we only describe the basic ideas.

(i) By induction on n. This is clearly verified for $n = 1, 2, 3$. Now, assume that the result holds for all integers m such that $1 \leq m < n$ and let us show that this is also true for n. The idea is to divide the path $\mathring{\Sigma}_\alpha\, q_n$ into three smaller parts as follows.



Applying the induction hypothesis, one deduces that the paths $\mathring{\Sigma}_\beta\ q_{n-3}$ and $\mathring{\Sigma}_\gamma\ q_{n-2}$ are simple as well. It only remains to show that the three smaller paths are contained in disjoint boxes and the result follows.

(ii) It is sufficient to show that $\mathring{\Sigma}_\alpha\, (q_{3n+1})^3$ is simple. First, notice that $\overline{q_{3n+5}} = \overline{q_{3n+4}q_{3n+3}} = \overline{q_{3n+3}}q_{3n+2}\overline{q_{3n+2}}q_{3n+1} = \overline{q_{3n+2}}q_{3n+1}q_{3n+1}q_{3n}\overline{q_{3n+2}}q_{3n+1}$. But q_{3n+1} is a prefix of $q_{3n}\overline{q_{3n+2}}$, so that q_{3n+1}^3 is a factor of $\overline{q_{3n+5}}$. From (i), we conclude that $\mathring{\Sigma}_\alpha\, (q_{3n+1})^3$ is simple as well. □

A Fibonacci tile of order n is a polyomino having $\mathring{\Sigma}_\alpha\, (q_{3n+1})^4$ as a boundary word, where $n \in \mathbb{N}$. The first Fibonacci tiles are illustrated in Figure 6.

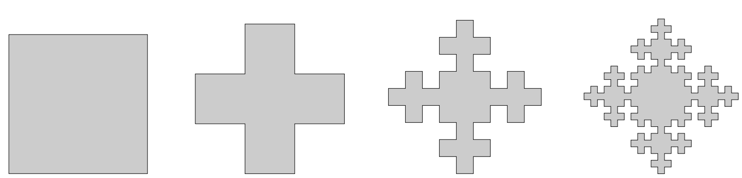


Figure 6: Fibonacci tiles of order $n = 0, 1, 2, 3, 4$.

Fibonacci tiles were considered in [14] from a number theory point of view, and presented slightly differently.

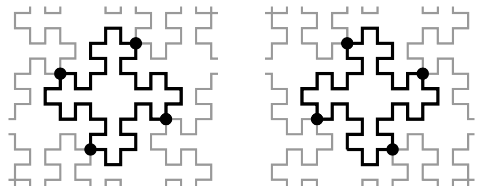


Figure 7: Tilings of the Fibonacci Tile of order 2 illustrate that it is a double square tile.

Theorem 11. *Fibonacci tiles of order $n > 0$ are double squares.*

Proof. We know from Lemma 9 that $q_{3n+1} = px$ for some σ_0-palindrome p and some letter $x \in \{\mathbf{1}, \mathbf{3}\}$. If $x = \mathbf{3}$, we consider the reversal of the path, i.e. $\sigma_0((\widetilde{q_{3n+1}})^4)$, so that we may suppose that $x = \mathbf{1}$. Therefore, on one hand we obtain

$$\mathring{\Sigma}_\alpha\ (q_{3n+1})^4 = \Sigma_\alpha\big(p\mathbf{1} \cdot p\mathbf{1} \cdot \sigma_0(\widetilde{p})\mathbf{1} \cdot \sigma_0(\widetilde{p})\big) = \Sigma_\alpha p \cdot \Sigma_{\rho(\alpha)} p \cdot \widehat{\Sigma_\alpha p} \cdot \widehat{\Sigma_{\rho(\alpha)} p},$$

because $\mathcal{T}(p) = 0$. On the other hand, the conjugate $q'_{3n+1} = \overline{q_{3n-1}} q_{3n}$ of q_{3n+1} corresponds to another boundary word of the same tile. Using again Lemma 9, we may write $q_{3n} = r\mathbf{1}$ and $q_{3n-1} = q\mathbf{3}$, for some palindromes q and r. Therefore, $p\mathbf{1} = q_{3n+1} = q_{3n}\overline{q_{3n-1}} = r\mathbf{1}\overline{q}\mathbf{1}$ so that $p = r\mathbf{1}\overline{q}$. But p is an σ_0-palindrome, which means that $q'_{3n+1} = \overline{q_{3n-1}} q_{3n} = \overline{q}\mathbf{1}r\mathbf{1} = \widetilde{p}\mathbf{1} = \overline{p}\mathbf{1}$. Hence, since $\overline{p}$ is an σ_0-palindrome as well, we find

$$\mathring{\Sigma}_\alpha\ (q'_{3n+1})^4 = \Sigma_\alpha\big(\overline{p}\mathbf{1} \cdot \overline{p}\mathbf{1} \cdot \widetilde{p}\mathbf{1} \cdot \widetilde{p}\big) = \Sigma_\alpha \overline{p} \cdot \Sigma_{\rho(\alpha)} \overline{p} \cdot \widehat{\Sigma_\alpha \overline{p}} \cdot \widehat{\Sigma_{\rho(\alpha)} \overline{p}}. \quad \square$$

As for Christoffel tiles, Fibonacci tiles also suggest that the conjecture of Provençal and Vuillon for palindromes in double squares [4] holds.

Corollary 12. *If $AB\widehat{A}\widehat{B}$ is a BN-factorization of a Fibonacci tile, then A and B are palindromes.*

Proof. The conclusion follows from Theorem 11. Indeed, since p is a σ_0-palindrome, then $\Sigma_\alpha p$ is a palindrome. The same argument applies for the second factorization. $\square$

Moreover, the sequence of areas of the Fibonacci tiles

$$\mathbf{A}(n) = 1, 5, 29, 169, 985, 5741, 33461, \cdots$$

is precisely the odd index subsequence of the well-known Pell numbers

$$P(n) = 0, 1, 2, 5, 12, 29, 70, 169, 408, 985, 2378, 5741, 13860, 33461, \cdots .$$

The identity $\mathbf{A}(n) = P_{2n+1}$ reveals a link with number theory where Pell numbers play an important role, as established in [14].

4.1. Fibonacci tiles variants

We conclude this section by introducing four variants of the Fibonacci tiles. Consider the sequence $(r_{d,m,n})_{(d,m,n)\in\mathbb{N}^3}$ satisfying the following recurrence, for $d \geq 2$,

$$r_{d,m,n} = \begin{cases} r_{d-1,n,m}\overline{r_{d-2,n,m}} & \text{if } d \equiv 0 \bmod 3 \\ r_{d-1,n,m}r_{d-2,n,m} & \text{if } d \equiv 1 \bmod 3 \\ r_{d-1,m,n}\overline{r_{d-2,m,n}} & \text{if } d \equiv 2 \bmod 3 \end{cases}$$

Using similar arguments as in the Fibonacci tiles case, one shows that both families obtained respectively with seed values

$$r_{0,m,n} = (\mathbf{3113})^m\mathbf{313}, \qquad r_{1,m,n} = (\mathbf{3113})^n\mathbf{3}, \tag{4a}$$
$$r_{0,m,n} = (\mathbf{31})^m\mathbf{313}, \qquad r_{1,m,n} = (\mathbf{31})^n\mathbf{31} \tag{4b}$$

are such that $\overset{\circ}{\Sigma}_\alpha\ (r_{3d,m,n}r_{3d,n,m})^2$ is a boundary word whose associated polyomino is a double square (see Figure 8), where $\alpha \in \mathcal{F}$. Intuitively, the parameters m and n measure the thickness of the tiles (in orthogonal directions), while the parameter d measures their level of fractality.

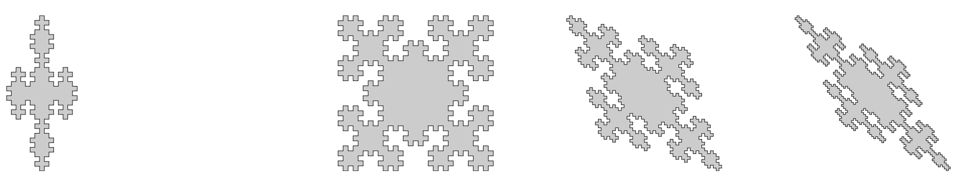


Figure 8: The tile $\overset{\circ}{\Sigma}_0\ (r_{6,0,1}r_{6,1,0})^2$ with seeds (4a); tiles $\overset{\circ}{\Sigma}_0\ (r_{9,m,0}r_{9,0,m})^2$ with seeds of type (4b) for $m = 0, 1, 2$.

Similarly, let $(s_{d,m,n})_{(d,m,n)\in\mathbb{N}^3}$ be a sequence satisfying for $d \geq 2$ the recurrence

$$s_{d,m,n} = \begin{cases} s_{d-1,n,m}\overline{s_{d-2,n,m}} & \text{if } d \equiv 0, 2 \bmod 3, \\ s_{d-1,m,n}s_{d-2,m,n} & \text{if } d \equiv 1 \bmod 3. \end{cases}$$

Then the families obtained with seed values

$$s_{0,m,n} = (\mathbf{3113})^m\mathbf{313}, \qquad s_{1,m,n} = \mathbf{31}, \tag{5a}$$
$$s_{0,m,n} = (\mathbf{31})^m\mathbf{313}, \qquad s_{1,m,n} = \mathbf{3} \tag{5b}$$

yield double squares $\overset{\circ}{\Sigma}_\alpha\ (s_{3d,m,n}s_{3d,n,m})^2$ as well (see Figure 9). One may verify that $r_{d,0,0} = s_{d,0,0}$ for any $d \in \mathbb{N}$ if the seed values are respectively (4a) and (5b) or respectively (4b) and (5a).

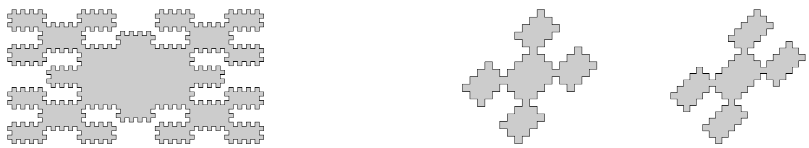


Figure 9: Tile $\overset{\circ}{\Sigma}_0\ (s_{9,2,0}s_{9,0,2})^2$ with seeds (5a); tiles $\overset{\circ}{\Sigma}_0\ (s_{6,0,n}s_{6,n,0})^2$ with seeds (5b) for $n = 1, 2$.

5. Concluding remarks

The study of double squares suggests interesting and challenging problems. For instance, it is appealing to conjecture that a double square is either of Christoffel type or of Fibonacci type. However, that is not the case, as illustrated in Figure 10. This begs for a thorough study in order to exhibit a complete zoology of such tilings. Another problem is to prove that Christoffel and Fibonacci tiles are prime, that is, they are not obtained by composition of smaller square tiles (see Figure 11). On the other hand, there is a conjecture of [4] stating that if $AB\widehat{A}\widehat{B}$ is the BN-factorization of a prime double square, then A and B are palindromes, for which no counter-example has been provided. This leads to a number of questions on the "arithmetics" of tilings, such as the unique decomposition, distribution of prime tiles, and their enumeration. Moreover, it would be interesting to verify if the following statements hold:

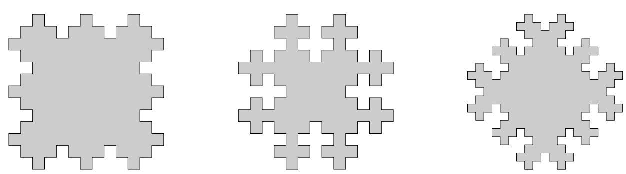


Figure 10: Three double squares not in the Christoffel and Fibonacci tiles families.

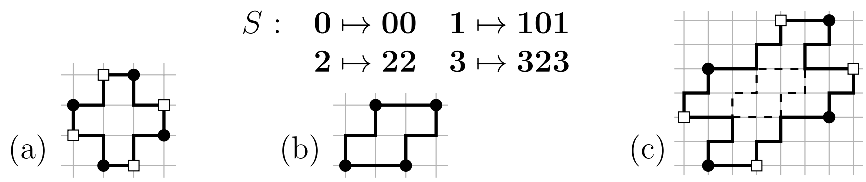


Figure 11: (a) A prime double square D. (b) A square tile S. (c) The tile $S(D)$, which is obtained by replacing each unit square of D by S, is a double square tile. It is not prime.

- Let $m, n \in \mathbb{N}$ be fixed. The sequence of areas indexed by $d \in \mathbb{N}$ of the four variants of Fibonacci tiles satisfy, for $d \geq 2$, the recurrence

$$\mathbf{A}(d) = 6\mathbf{A}(d-1) - \mathbf{A}(d-2).$$

- The first differences sequence of $\lim_{n\to\infty} q_n$ is the Fibonacci word. Describe the first differences sequence of $\lim_{d\to\infty} r_{d,m,n}$ and $\lim_{d\to\infty} s_{d,m,n}$.
- If $\alpha\alpha$ appears in the boundary word of a double square tile D, where $\alpha \in \mathcal{F}$, then D is not prime.

Finally, the Fibonacci tiles have fractal characteristics suggesting that Lindemayer systems (L-systems) might be used for their generation. The formal grammars that describe them have been widely studied, and their impact is significant in biology, computer graphics [15] and modeling of plants [16]. Snowflakes (Fibonacci tiles) are one of the numerous designs that are included in this category.

Acknowledgements

The results found in that paper were partly driven by computer exploration using the open-source mathematical software `Sage` [17]. This paper is an extended version of the communication [7] presented at DGCI'09. Finally, we are grateful to the anonymous referees for their careful reading and helpful suggestions that simplified and improved the presentation.

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