Every polyomino yields at most two square tilings

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1 Introduction

The problem of designing an efficient algorithm for deciding whether a given polygon tiles the plane becomes more tractable when restricted to polyominoes, that is, subsets of the square lattice $\mathbb{Z}^2$ whose boundary is a non-crossing closed path (see [6] for more on tilings and [3] for related problems). Here, we consider tilings obtained by translation of a single polyomino, called *exact* in [9]. Paths are conveniently described by words on the alphabet $\{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}\}$, representing the elementary grid steps $\{\rightarrow, \uparrow, \leftarrow, \downarrow\}$. Beauquier and Nivat [1] characterized exact polyominoes by showing that the boundary word $b(P)$ of such a polyomino satisfies the equation $b(P) = X \cdot Y \cdot Z \cdot \widehat{X} \cdot \widehat{Y} \cdot \widehat{Z}$, where at most one of the variables is empty and where $\widehat{W}$ is the path W traveled in the opposite direction. Frow now on, this condition is referred as the BN-factorization. An exact polyomino is said to be a *hexagon* if none of the variables X, Y, Z is empty and a *square* if one of them is so. Note that a single polyomino may lead to many tilings

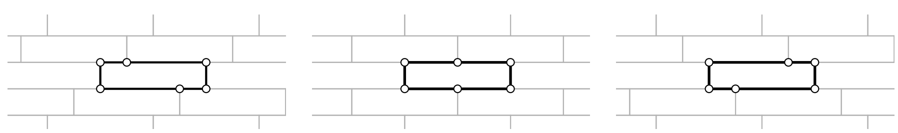

Figure 1: The three hexagonal tilings of the 4×1 rectangle.

of the plane: for instance the $n \times 1$ rectangle does it in $n - 1$ distinct ways as a hexagon (see Figure 1) whereas Christoffel and Fibonacci tiles introduced recently [2] are examples of double squares (see Figure 2). However, it was

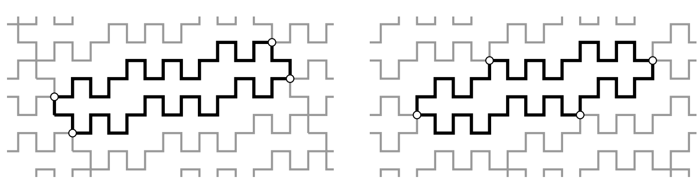

Figure 2: A Christoffel Tile yields two distinct non-symmetric square tilings of the plane.

conjectured by Brlek, Dulucq, Fédou and Provençal in 2007 (see [8] for more details) that an exact polyomino tiles the plane as a square in at most two distinct ways. In this extended abstract, we prove this conjecture, using the symmetries induced by the BN-factorization.

Theorem 1. *Every polyomino yields at most two square tilings.*

2 Preliminaries

The usual terminology and notation on words is from Lothaire [7]. An *alphabet* $\mathcal{A}$ is a finite set whose elements are *letters*. A finite word w is a function $w : [1, 2, \ldots, n] \to \mathcal{A}$, where w_i is the i-th letter, $1 \leq i \leq n$. The *length* of w, denoted by $|w|$, is the integer n. The length of the empty word is 0. The *free monoid* $\mathcal{A}^*$ is the set of all finite words over $\mathcal{A}$. The *reversal* of $w = w_1 w_2 \cdots w_n$ is the word $\widetilde{w} = w_n w_{n-1} \cdots w_1$. A word u is a *factor* of another word w if there exist $x, y \in \mathcal{A}^*$ such that $w = xuy$. We denote by $|w|_u$ the number of times that u appears in w. Two words u and v are *conjugate*, written $u \equiv v$ or sometimes $u \equiv_{|x|} v$, if there exist x, y such that $u = xy$ and $v = yx$. In this paper, the alphabet $\mathcal{F} = \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}\}$ is considered as the additive group of integers mod 4. Basic transformations on $\mathcal{F}$ are rotations $\rho^i : x \mapsto x + i$ and reflections $\sigma_i : x \mapsto i - x$, which extend uniquely to morphisms (w.r.t concatenation) on $\mathcal{F}^*$. Given a nonempty word $w \in \mathcal{F}^*$, the *first differences word* $\Delta w \in \mathcal{F}^*$ of w is

$$\Delta w = (w_2 - w_1) \cdot (w_3 - w_2) \cdots (w_n - w_{n-1}).$$

One may verify that if $z \in \mathcal{F}^*$, then $\Delta wz = \Delta w \Delta(w_n z_1) \Delta z$. We introduce another function well-defined on conjugacy classes and circular words:

$$\overset{\circ}{\Delta} w = (w_2 - w_1) \cdot (w_3 - w_2) \cdots (w_n - w_{n-1}) \cdot (w_1 - w_n) = \Delta w \cdot (w_1 - w_n).$$

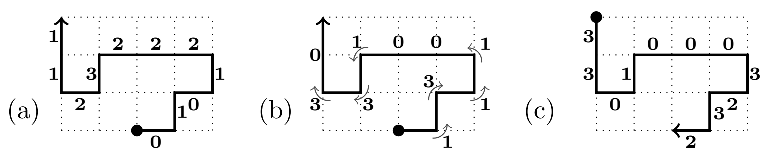

Figure 3: (a) The path $w = \mathbf{01012223211}$. (b) Its first differences word $\Delta w = \mathbf{1311001330}$. (c) Its reversal $\widehat{w} = \mathbf{33010003232}$.

Let M be a $\mathbb{Z}$-module and $\mu : \mathcal{F}^* \to M$ be a morphism, i.e., $\mu(wz) = \mu(w) + \mu(z)$ for all $w, z \in \mathcal{F}^*$. The *μ-path* of a word $w \in \mathcal{F}^*$ is the finite sequence of partial sums $\mathcal{P}w = \left(\sum_{i=1}^{k} \mu(w_i)\right)_{0 \leq k \leq n}$. In what follows, we consider the square lattice $M = \mathbb{Z} \times \mathbb{Z}$ and

$$\mu : \quad \mathbf{0} \mapsto (1, 0), \quad \mathbf{1} \mapsto (0, 1), \quad \mathbf{2} \mapsto (-1, 0), \quad \mathbf{3} \mapsto (0, -1).$$

which corresponds to the Freeman chain code [5], we write *path* as a shorthand for μ-path and we say that w describes the path $\mathcal{P}w$. Furthermore, we say that a path $\mathcal{P}w$ is *closed* if $\mu(w) = (0, 0)$ or equivalently if it satisfies $|w|_\mathbf{0} = |w|_\mathbf{2}$ and

$|w|_{\mathbf{1}} = |w|_{\mathbf{3}}$. A path $\mathcal{P}w$ is *simple* if no proper factor of w describes a closed path. A *boundary word* describes a simple and closed path. A *polyomino* is a subset of $\mathbb{Z}^2$ contained in some boundary word. Finally, the word $\widehat{w} := \rho^2(\widetilde{w})$ describes the same path as w traveled in the opposite direction (see Figure 3). The *turning number* of a closed path $\mathcal{P}w$ is $\mathcal{T}(\overset{\circ}{\Delta}\, w) = \left(|\overset{\circ}{\Delta}\, w|_{\mathbf{1}} - |\overset{\circ}{\Delta}\, w|_{\mathbf{3}}\right)/4$ and corresponds to its total curvature divided by 2π. A closed path is *positively oriented* if its turning number is positive.

Lemma 1. *If $XY\widehat{X}\widehat{Y}$ is the positively oriented boundary word of a square, then*

$$\overset{\circ}{\Delta}\, XY\widehat{X}\widehat{Y} = \Delta X \cdot \mathbf{1} \cdot \Delta Y \cdot \mathbf{1} \cdot \Delta\widehat{X} \cdot \mathbf{1} \cdot \Delta\widehat{Y} \cdot \mathbf{1}.$$

Proof. The equation $\mathcal{T}(\Delta w) = -\mathcal{T}(\Delta\widehat{w})$ holds for all $w \in \mathcal{F}^*$ and the turning number of a positively oriented boundary word is 1. □

3 Proof of Theorem 1

In this section, we suppose that there exists a polyomino that tiles the plane as a square in three ways, i.e., its positively oriented boundary word has three distinct square factorizations given by

$$UV\widehat{U}\widehat{V} \equiv_{d_1} XY\widehat{X}\widehat{Y} \equiv_{d_2} WZ\widehat{W}\widehat{Z}. \tag{1}$$

Lemma 2. [4, 8] *If an exact polyomino satisfies $UV\widehat{U}\widehat{V} \equiv_{d_1} XY\widehat{X}\widehat{Y}$, then the factorization must alternate, i.e., $0 < d_1 < |U| < d_1 + |X|$.*

Hence, we must have the situation depicted in Figure 4 (a).

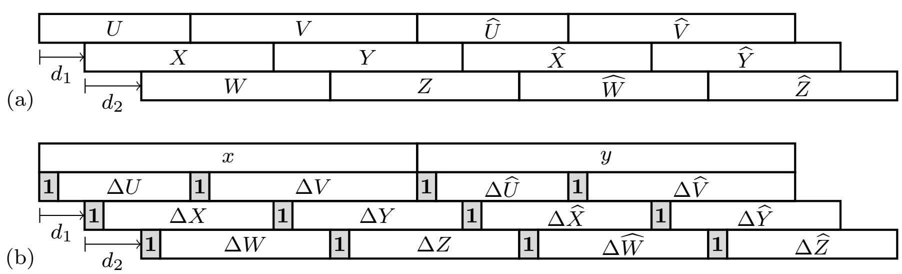

Figure 4: (a) Three distinct square factorizations of a tile. Note that $0 < d_1 < d_1 + d_2 < |U| < d_1 + |X| < d_1 + d_2 + |W|$. (b) One has $x_i = y_i = \mathbf{1}$ for all $i \in I$.

Let $I = \{0, d_1, d_1+d_2, |U|, d_1+|X|, d_1+d_2+|W|\}$. It follows from Lemma 2 that all these positions are distinct, that is $|I| = 6$. Furthermore, it is convenient to consider the first differences word of the boundary word as two parts

$$\begin{aligned} x &= x_0x_1x_2\cdots x_{n-1} = \mathbf{1}\cdot\Delta U\cdot\mathbf{1}\cdot\Delta V, \\ y &= y_0y_1y_2\cdots y_{n-1} = \mathbf{1}\cdot\Delta\widehat{U}\cdot\mathbf{1}\cdot\Delta\widehat{V}, \end{aligned}$$

where $n = |x| = |y|$ is the half-perimeter. Note that **1** occurs in both x and y for each position $i \in I$ (see Figure 4 (b)). Three reflections on $\mathbb{Z}_n$ are useful:

$$\begin{aligned} s_1 : i &\mapsto (|U| - i) \mod n, \\ s_2 : i &\mapsto (|X| + 2d_1 - i) \mod n, \\ s_3 : i &\mapsto (|W| + 2(d_1 + d_2) - i) \mod n. \end{aligned}$$

From Lemma 2, the reflections s_1, s_2 and s_3 are pairwise distinct. We say that the application s_1 is *admissible* on i if $i \notin \{0, |U|\}$ and similarly for the application s_2 if $i \notin \{d_1, |X| + d_1\}$ and for s_3 if $i \notin \{d_1 + d_2, |W| + d_1 + d_2\}$. Below we denote $\overline{\alpha} := \sigma_0(\alpha)$ so that $\overline{\mathbf{0}} = \mathbf{0}$, $\overline{\mathbf{1}} = \mathbf{3}$, $\overline{\mathbf{2}} = \mathbf{2}$ and $\overline{\mathbf{3}} = \mathbf{1}$. The fact that $(\Delta w)_i = \overline{(\Delta \widehat{w})_{|w|-i}}$ for all $w \in \{U, V, X, Y, W, Z\}$ and $1 \leq i \leq |w| - 1$ then translates nicely in terms of x, y and reflections s_1, s_2 and s_3.

Lemma 3. *Let $i \in \mathbb{Z}_n$ and $j \in \{1, 2, 3\}$ such that s_j is admissible on i. Then*

(i) $y_i = \overline{x_{s_j(i)}}$ *and* $x_i = \overline{y_{s_j(i)}}$.

(ii) *If $x_i = y_i$, then $x_{s_j(i)} = y_{s_j(i)}$.*

We say that the application of a product of reflections $s_{j_m} s_{j_{m-1}} \cdots s_{j_2} s_{j_1}$ is *admissible* on i if each application of s_{j_k} is admissible on $s_{j_{k-1}} \cdots s_{j_2} s_{j_1}(i)$. Finally we say that $i \in \mathbb{Z}_n$ is *reachable* if there exist an admissible product of reflections S and $i' \in I$ such that $i = S(i')$. Roughly speaking, an index is reachable if one of the six initial **1** letters propagate to that position.

Lemma 4. *Let $i \in \mathbb{Z}_n$ be reachable and S be an admissible product of reflections on i. Then $x_i = y_i$ and*

$$x_i = \begin{cases} x_{S(i)} & \text{if } S \text{ is a rotation,} \\ \overline{x_{S(i)}} & \text{if } S \text{ is a reflection.} \end{cases}$$

We are now ready to show the main result.

Proof of Theorem 1. Arguing by contradiction, assume that a polyomino satisfying Equation (1) exists, and that the formalism and lemmas above apply. We have $s_1 = s_2 s_3 s_1 s_2 s_3$. If $s_2 s_3 s_1 s_2 s_3$ is admissible on 0, then

$$\mathbf{1} = x_0 = \overline{x_{s_2 s_3 s_1 s_2 s_3(0)}} = \overline{x_{s_1(0)}} = \overline{x_{|U|}} = \overline{\mathbf{1}} = \mathbf{3}$$

which is a contradiction. Thus $s_2 s_3 s_1 s_2 s_3$ is not admissible on 0. Having s_3 not admissible on 0 is impossible since s_3 is admissible on everything but $d_1 + d_2$ and $|W| + d_1 + d_2$. Having s_2 not admissible on $s_3(0)$ is also impossible since this implies that

$$\mathbf{1} = x_0 = \overline{x_{s_3(0)}} = \overline{\mathbf{1}} = \mathbf{3}.$$

Similar arguments show that supposing s_2 not admissible on $s_3 s_1 s_2 s_3(0)$ or s_3 not admissible on $s_1 s_2 s_3(0)$ leads to a contradiction. Hence, we must have that s_1 is not admissible on $s_2 s_3(0)$. Again there are two cases: either $s_2 s_3(0) = 0$

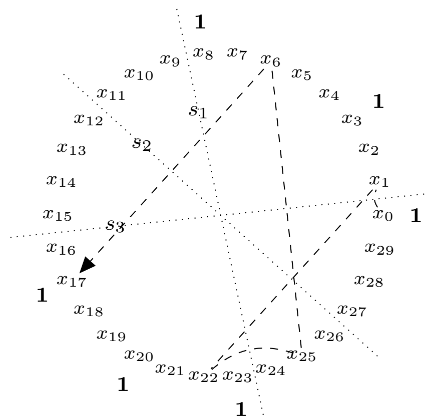

Figure 5: Reflections in action on x where $n = 30$, $d_1 = 3$, $d_2 = 5$, $|U| = 17$, $|X| = 17$ and $|W| = 15$. The product of reflections $s_3s_2s_1s_3s_2$ is admissible on 0 so that $\mathbf{1} = x_0 = \overline{x_{s_3s_2s_1s_3s_2(0)}} = \overline{x_{17}} = \overline{\mathbf{1}} = \mathbf{3}$ a contradiction.

or $s_2s_3(0) = |U|$. In the first case, we must have $s_2 = s_3$ which is a contradiction. Hence, there is only one possibility : $s_2s_3(0) = |U|$. We also have $s_1 = s_3s_2s_1s_3s_2$ and using exactly the same argument, we also conclude that $s_3s_2(0) = |U|$. But then, $s_3s_2 = s_2s_3$ so that s_2 and s_3 must be perpendicular since they are not equal. We also have $s_2 = s_1s_3s_2s_1s_3$ and $s_2 = s_3s_1s_2s_3s_1$ so that for the same reason as above, we deduce that s_1 and s_3 are perpendicular. But we already know that s_2 and s_3 are perpendicular. This implies $s_1 = s_2$ which is impossible, and the proof is complete. □

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