Combinatorial aspects of Escher tilings

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Introduction

Figure 1: Maurits Cornelis Escher (1898-1972)

In the late 30's, Maurits Cornelis Escher astonished the artistic world by producing some puzzling drawings. In particular, the tesselations of the plane obtained by using a single tile appear to be a major concern in his work, drawing attention from the mathematical community. Since then, tesselations of the plane have been widely studied: see Grünbaum and Shephard (1987) for a general presentation.

Among the many types Escher discovered, the simplest one concerns tilings obtained with translated copies of a single tile: *hexagonal* (right) and *square* (below) tilings appeared in numerous of its drawings and prints. Immediately, two natural questions arise:

1. How can we recognize a tile?
2. How can we generate tiles?

Figure 2: An hexagonal Tiling.

Figure 3: A square Tiling.

Of course, when dealing with boundaries described by continuous functions, it is necessary to represent them conveniently. As quoted on Doron's website :

> *Cauchy ruined mathematics. Let's throw out all that epsilon-delta nonsense.*
>
> —Adriano Garsia,
> talk at the Journées Pierre Leroux, Montreal, Sept. 8, 2006.

So that a good way to deal with tiles is to use polyominoes, whose boundary is conveniently encoded on the four letter alphabet $\Sigma = \{\mathbf{0},\mathbf{1},\mathbf{2},\mathbf{3}\}$.

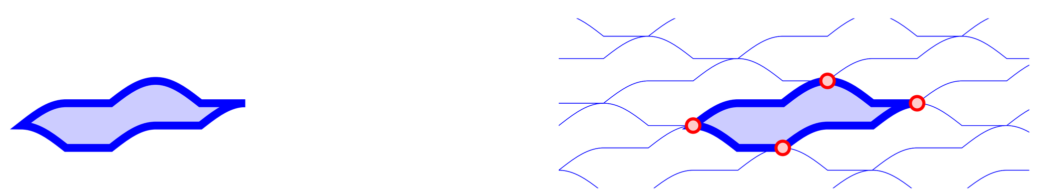


Figure 4: TILINGS OF $\mathbb{R}^2$. The general problem of deciding whether a given region tile the Euclidean plane $\mathbb{R}^2$ is of great interest when studying tilings and cristal networks problems. The region R illustrated above has a continuous boundary that may be factorized in four paths that are pairwise parallel, so that it tiles the plane by translation.

Now by using tools from combinatorics on words, it is possible to tackle these problems. For this purpose, let us quote the following characterization:

Theorem 1 (Beauquier-Nivat,1991)**.** *A polyomino P tiles the plane by translation if and only if there exist $A,B,C \in \Sigma^*$ such that*

$$W \equiv A\cdot B\cdot C\cdot \widehat{A}\cdot \widehat{B}\cdot \widehat{C},$$

where W is some boundary word of P and at most one of the variables A, B, C is empty.

Recognition of tiles Efficient algorithms have been designed:

Square tiles: a linear optimal algorithm (B. and Provençal, 2006).

Hexagonal tiles: if the polyominoes do not have too long square factors then the algorithm is still linear (B. and P., 2008). A general $O(n\log^3(n))$ algorithm also appears in Provençal's thesis (2008).

Conjecture: a linear algorithm exists.

Multiple tilings

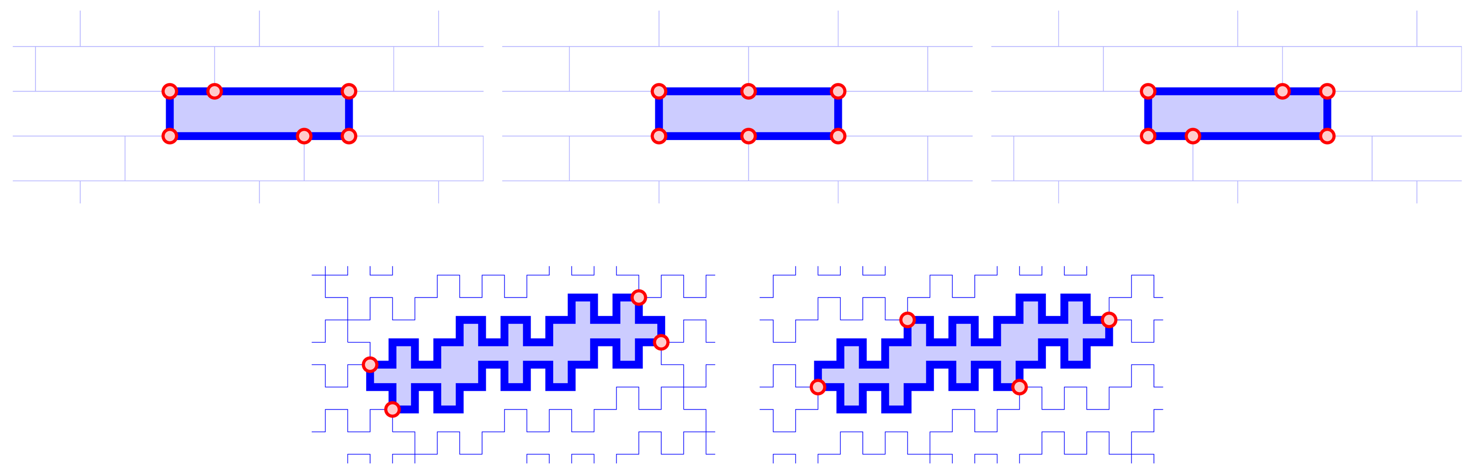


Figure 5: DISTINCT TILINGS. For any positive integer n, there exist polyominoes yielding n distinct hexagonal tilings.

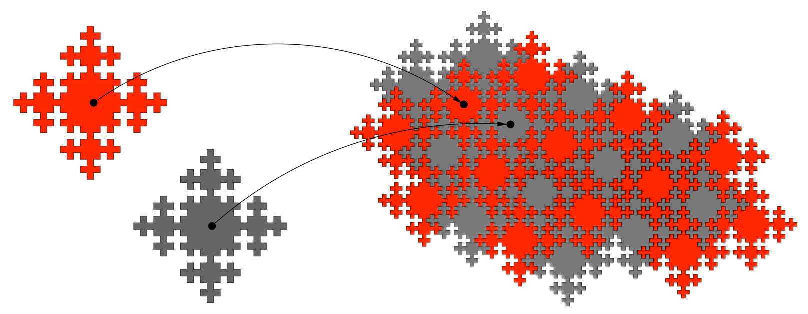
Polyominoes may have both square and hexagonal factorizations. In Figure 5 (top), the 4×1 rectangle has *three* distinct hexagonal tilings. It also has *one* square tiling. More generally, the $(n+1)\times 1$ rectangle yields n hexagonal tilings. Moreover, a hexagonal tile may have at most 1 square tiling.

Figure 5 (bottom) shows a tile admitting two distinct square tilings. And in fact all tiles admit at most two square tilings, that is either 0, 1 or 2 distinct ones:

Theorem 2. *The number of distinct BN-factorizations of a square is at most* 2.

Tiles having exactly two square factorizations define two sets of distinct translations and are called *double squares*.

On the right, one of the two tesselations of a double square. The second one is obtained by taking the (vertical) mirror image.



There are infinite families of such *double squares*, and in particular, two remarkable families of squares are linked to the Christoffel words and to the Fibonacci sequence.

Christoffel tiles

Consider the morphism $\lambda : \{\mathbf{0},\mathbf{1},\mathbf{2},\mathbf{3}\}^* \to \{\mathbf{0},\mathbf{1},\mathbf{2},\mathbf{3}\}^*$ by $\lambda(\mathbf{0}) = \mathbf{0301}$ and $\lambda(\mathbf{1}) = \mathbf{01}$, which can be seen as a "crenelation" of the steps *east* and *north-east*.

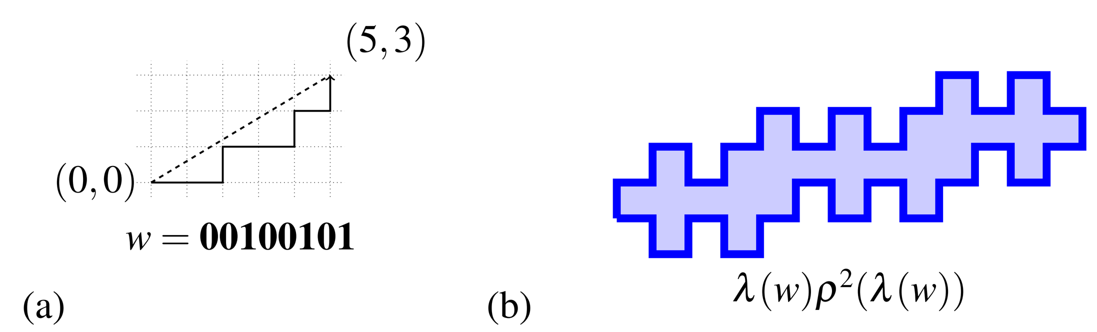


Figure 6: CHRISTOFFEL TILES. An infinite family of double squares are the so-called Christoffel tiles. They are crenellated versions of discrete segments. Indeed, Christoffel tiles are exactly given by $\lambda(w\overline{w})$, where w is any Christoffel word, up to rotations and reflections on $\mathbb{Z}^2$.
(a) The Christoffel word of parameters $(5,3)$. (b) Its associated Christoffel tile.

Fibonacci tiles

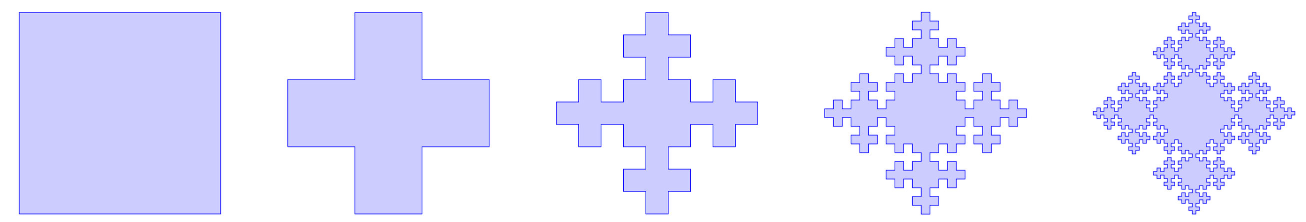


Figure 7: FIBONACCI TILES. Another remarkable family of double-squares are the Fibonacci tiles. Above are listed those of order $n = 0,1,2,3,4$.

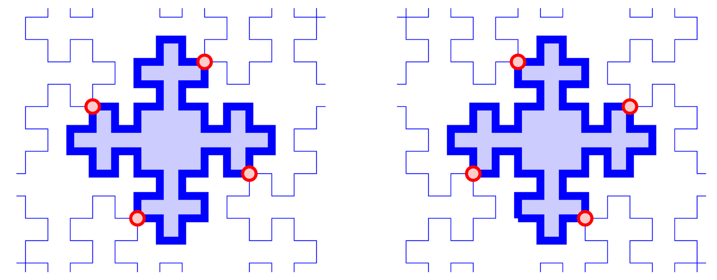


Figure 8: FIBONACCI TILING. Tilings of the Fibonacci Tile of order 2 illustrate that it is a double square tile.

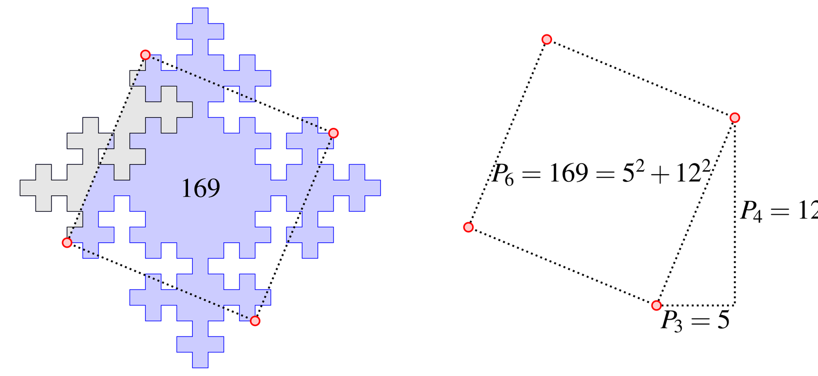


Figure 9: A LINK WITH THE PELL NUMBERS. Fibonacci tiles are related both to the Fibonacci sequence and the Pell numbers, i.e. they contain both the golden and silver ratios.

More precisely, the area of the Fibonacci tiles is described by the subsequence of odd indexed Pell numbers $0,1,2,5,12,29,70,169,408,985,2378,5741,13860,33461,80782,\ldots$, defined by $P_0 = 0$, $P_1 = 1$, $P_n = 2P_{n-1}+P_{n-2}$, for $n > 1$. They are known to satisfy the identity $P_n^2 + P_{n+1}^2 = P_{2n+1}$.

Enumeration and generation of double squares

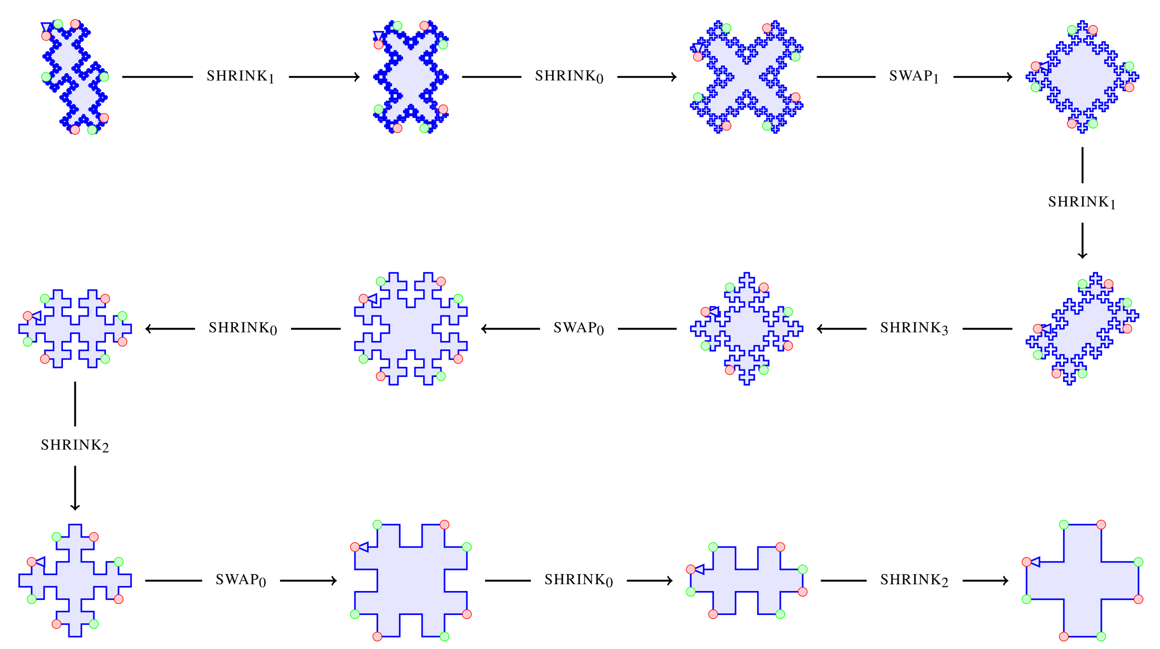


Figure 10: DOUBLE SQUARES. The problem of characterizing the double squares has been studied in Blondin Massé et al. (2010), where it is shown that every double square reduces to a morphic pentamino cross by mean of three operators of reduction acting on possibly self-intersecting double squares.