Equations on palindromes and circular words[☆]

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Abstract

In this paper we consider several types of equations on words, motivated by the attempt of characterizing the class of polyominoes that tile the plane by translation in two distinct ways. Words coding the boundary of these polyominoes satisfy an equation whose solutions are in bijection with a subset of the solutions of equations of the form $A\,B\,\widetilde{A}\,\widetilde{B} \equiv X\,Y\,\widetilde{X}\,\widetilde{Y}$. It turns out that the solutions are strongly related to local periodicity involving palindromes and conjugate words.

Keywords: Palindromes, periodicity, Fine and Wilf, polyominoes, tilings

1. Introduction

A large part of combinatorics on words concerns the study of regularities and equations involving them. By describing their combinatorial properties, one is often led to structures that are useful for obtaining faster algorithms for several decidable problems. Their relevance is justified by numerous applications, among which the widely studied pattern matching in strings and more recently the advances in the genoma sequence analysis provided a broad range of algorithms. In addition, the combinatorial properties often provide complexity bounds for both the space representation of objects and the execution time of algorithms. Let us give some examples. The well-known "lemma of three squares" (Lothaire [1], Lemma 8.2.2) states that the equation $x^2 = y^2v = z^2uv$, where z is primitive, implies a condition on the length of z, namely that $|z| < |x|/2$. In other words, it shows that such patterns are rather constrained instead of being freely organized. In turn this result is used for providing upper bounds on the number of squares leading to tighter bounds on square detection algorithms. The problem of overlapping palindromes appears in number theory, when trying to superpose Christoffel words [2] or Beatty sequences [3, 4]. Indeed, a Christoffel word is characterized by a central palindrome. Motivations for the study of palindromic complexity emerge from many areas among which the study

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of Schrödinger operators in physics [5, 6], number theory [7], discrete geometry [8], and combinatorics on words where it appears as a powerful tool for understanding the local structure of words. It has been recently studied in various classes of infinite words, a partial account of which may be found in the survey provided by Allouche et al. [9]. Another example stems from discrete geometry on square lattices, where polyominoes are conveniently represented by words on the 4-letter alphabet $\mathcal{E} = \{a, b, \overline{a}, \overline{b}\}$. A nice application is the detection of digital convexity. Indeed, one needs to combine the computation of the Lyndon factorization with checking if all factors are Christoffel words, that is palindrome words extended by a letter at each end and satisfying an arithmetical condition [8].

In this paper, we study equations that yield periodic words. In the preliminary Section 2 we state some useful lemmas, where the key argument for obtaining periodicity is conjugacy. Then, in Section 3 we consider systems of two equations and show how palindromes propagate in a word. We provide a direct and much simpler proof of a result obtained by Labbé [10], based on a simple but efficient property (Lemma 9). By considering equations involving three palindromes, we obtain results in the spirit of the three squares lemma, which may be seen as extending the results of Paquin [2] for Christoffel words. More precisely, given three fixed palindromes in some overlapping configuration, the relative distances between each pair are constrained in order to apply Fine and Wilf's theorem and obtain periodicity. Finally, we consider equations of type $A\,B\,\widetilde{A}\,\widetilde{B} \equiv X\,Y\,\widetilde{X}\,\widetilde{Y}$ on circular words, which are linked with the representation of a tile yielding tesselations of the plane with translated copies of it. Again, constraints on the way these factors overlap yield periodicity, leading to the discovery of an infinite class of polyominoes (called *double squares* in [11]) that tile the plane by translation in two distinct ways as depicted below.

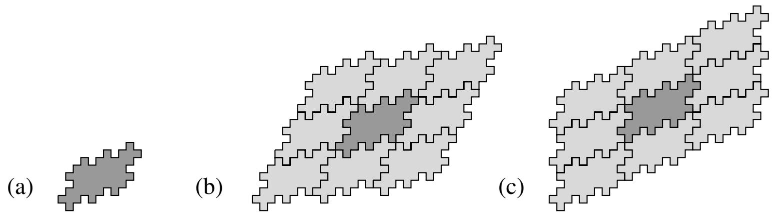


Figure 1: (a) A double square and (b,c) its two associated tilings.

2. Preliminaries

All the basic terminology about words is taken from M. Lothaire [12]. In what follows, Σ is a finite *alphabet* whose elements are called *letters*. By *word* we mean a finite sequence of letters $w : [0..n-1] \longrightarrow \Sigma$, where $n \in \mathbb{N}$. The length of w is $|w| = n$ and $w[i]$ or w_i denote its i-th letter. The set of n-length words over Σ is denoted Σ^n. By convention, the *empty* word is denoted ε and its length is 0. The free monoid generated by Σ is defined by $\Sigma^* = \bigcup_{n\geq 0} \Sigma^n$. The *kth* power of w is defined recursively by $w^0 = \varepsilon$ and $w^k = w^{k-1}w$. A word is said to be *primitive* if it is not a power of another word. Given a word $w \in \Sigma^*$, a *factor* u of w is a word $u \in \Sigma^*$ such that $w = xuy$, with $x \in \Sigma^*$ and $y \in \Sigma^*$. If $x = \varepsilon$ (resp. $y = \varepsilon$) then u is called *prefix* (resp. *suffix*). The set of all factors of w is denoted by $\mathrm{Fact}(w)$, and $\mathrm{Pref}(w)$ is the set of all its prefixes. An *antimorphism* is a map $\varphi : \Sigma^* \to \Sigma^*$ such that $\varphi(uv) = \varphi(v)\varphi(u)$ for any word $u, v \in \Sigma^*$. A useful example is the *reversal* of $w \in \Sigma^n$ defined by $\widetilde{w} = w_{n-1}w_{n-2}\cdots w_0$. A *palindrome* is a word w such that $w = \widetilde{w}$.

A *period* of a word w is an integer k such that $w[i] = w[i + k]$, for all $i < |w| - k$. In particular, every $k \geq |w|$ is a period of w. An important result about periods is due to Fine and Wilf.

Theorem 1 (Fine and Wilf). *Let w be a word having k and ℓ for periods. If $|w| \geq k+\ell-\gcd(k, \ell)$, then $\gcd(k, \ell)$ is also a period of w.*

Two words u and v are *conjugate*, written $u \equiv v$, when there are words x, y such that $u = xy$ and $v = yx$. Clearly, it is an equivalence relation, and the conjugacy class of a word w is also called its *circular word*. Moreover, we say that *u is conjugate to v with delay d*, denoted by $u \equiv_d v$, if $d = |x|$. Note that $\equiv_d$ is not a symmetric relation. We allow the delay to be negative by setting $v \equiv_{-d} u$ if and only if $u \equiv_d v$. The reader may verify easily the following fact.

Lemma 2. *If $u \equiv_{d_1} v$ and $v \equiv_{d_2} w$, then $u \equiv_{d_1+d_2 \bmod |u|} w$.*

Moreover, periodicity is related to conjugacy as the next propositions show. The first one comes from Lothaire.

Proposition 3 (Lothaire [12] Prop.1.3.2). *Let x and y be two words such that $xy = yx$. Then there exist a word p and two nonnegative integers i, j such that $x = p^i$ and $y = p^j$.*

As a consequence, we obtain the following fact.

Proposition 4. *If $w = ABC = CBA$, then $|AB|$, $|CB|$ and $\gcd(|AB|, |CB|)$ are periods of w.*

Proof. By hypothesis, we have

$$AB \cdot CB = ABC \cdot B = CBA \cdot B = CB \cdot AB$$

which is of the form $xy = yx$, and therefore there exist a word p, and two integers i and j such that $x = p^i$ et $y = p^j$ (Proposition 3). Hence $AB = p^i$ and $BC = p^j$. It follows that $|p|$ divides both $|AB|$ and $|BC|$, so that $|p|$ also divides $\gcd(|AB|, |CB|)$. □

Observe that if $A = \varepsilon$, then w is conjugate to itself with delay $d = |B|$, so that $\gcd(d, |w|)$ is a period of w. Moreover, the statement above may be refined to take into account the fact that $|p|$ is a period of w. In fact, a more general result holds. Indeed, let $\sigma : \{A, B, C\} \longrightarrow \{A, B, C\}$ be a bijection (or permutation). Then, periodicity appears whenever the factors A, B and C have two occurrences in $w = ABC$.

Proposition 5. *Let $w = ABC = \sigma(A) \cdot \sigma(B) \cdot \sigma(C)$, where $\sigma \neq \mathrm{Id}$. If A, B and $C \neq \varepsilon$, then the following conditions hold*

(i) *if σ has no fixed point then there exist $p \in \Sigma^*$ and an integer $k > 1$ such that $ABC = p^k$;*

(ii) *otherwise, either ABC or AB or BC is periodic.*

Proof. (i) In this case we have either $ABC = BCA$ or $ABC = CAB$. In both cases, the equation is of the form $xy = yx$, so that Proposition 4 applies.

(ii) $ABC = CBA$ implies that ABC has period $\gcd(|AB|, |CB|)$, while $AB \cdot C = BA \cdot C$ implies that AB has period $\gcd(|A|, |B|)$ and $A \cdot BC = A \cdot CB$ implies that BC has period $\gcd(|B|, |C|)$. □

For a language $L \subseteq \Sigma^*$, the set of palindromic elements is denoted by Pal(L). Every word contains palindromes, the letters and ε being necessarily part of them. This justifies the introduction of the function LPS(w) which associates to any word w its longest palindromic suffix. We recall from [13] a useful combinatorial property (see also [14]).

Proposition 6 (Blondin Massé et al. [13]). *Assume that $w = xy = yz$ with $|y| \neq 0$. Then for some u, v, and some $i \geq 0$ we have from Lothaire* [12]

$$x = uv, y = (uv)^i u, z = vu; \tag{1}$$

and the following conditions are equivalent :

(i) $x = \widetilde{z}$;

(ii) *u and v are palindromes;*

(iii) *w is a palindrome;*

(iv) *xyz is a palindrome.*

Moreover, if one of the equivalent conditions above holds then

(v) *y is a palindrome.*

An interesting consequence is the following result.

Corollary 7 (Blondin Massé et al. [15]). *Assume that $w = xp = qz$ where p and q are palindromes such that $|q| > |x|$. Then w has period $|x| + |z|$, and $x\widetilde{z}$ is a product of two palindromes.*

This basic property is best possible. Indeed, it says that when two distinct palindromes overlap, even for one letter, then a period appears in the word.

Finally, the following technical lemma is useful for analyzing equations involving circular words. It is derived from the fundamental result of Lothaire already mentioned in Proposition 6.

Lemma 8. *Let S, T, p, q, g and h be six non-empty words such that*

$$pS = Tq \quad \text{and} \quad gT = Sh.$$

Then for some u, v, and some $i \geq 0$, $pg = uv$, $T = (uv)^i u$ and $qh = vu$.

Proof. The following relations hold : $pgT = pSh = Tqh$. Setting $x = pg$, $y = T$ and $z = qh$, this equation has the form $w = xy = yz$. Since $|y| \neq 0$, Equation (1) in Proposition 6 ensures that for some u, v, and some $i \geq 0$, $x = uv$, $y = (uv)^i u$, $z = vu$. □

3. Equations involving palindromes

A very special case of this problem already appears in the literature. Indeed, Christoffel words (finite Sturmian words) have the property that the central word is a palindrome, and their superposition is possible under some arithmetic constraints. For more details, see for instance Simpson [3, 4] or more recently Paquin [2]. The following results are excerpts of the last author Master thesis [10]. The presentation given here is much simpler and is based on the following very useful lemma, which may be considered as a special case of Proposition 6. It is however independent since it does not rely on periodicity.

Lemma 9. *Let $w = \widetilde{y}p = qy$ where $p, q \in \mathrm{Pal}(\Sigma^*)$. Then $w \in \mathrm{Pal}(\Sigma^*)$.*

Proof. Consider the sequence of equalities $\widetilde{y}\,\widetilde{y}\widetilde{w} = \widetilde{y}\,\widetilde{y}py = \widetilde{y}qyy = \widetilde{w}yy$. Since the equality $\widetilde{y}\,\widetilde{y}\cdot\widetilde{w} = \widetilde{w}\cdot yy$ has the form $ab = bc$ with $a = \widetilde{c}$ and $b = \widetilde{w}$, it follows from Proposition 6 that w is a palindrome. □

Proposition 10. *Let $x, y, p, s \in \Sigma^*$ with $|p| + |s| > 0$, and let $v, w \in \mathrm{Pal}(\Sigma^*)$. Then the following properties hold :*

(i) *if $xs = pv$ and $\widetilde{x}s = pw$, then x is a palindrome;*

(ii) *if $xs = py$ and $\widetilde{x}s = p\widetilde{y}$, then x and y are palindromes;*

(iii) *if $sx = vp$ and $s\widetilde{x} = wp$, then x is a palindrome.*

Proof. (i) We write $\widetilde{s}xs = \widetilde{s}\cdot pv = w\widetilde{p}\cdot s$. Then Lemma 9 applies with $y = \widetilde{p}s$ so that $\widetilde{s}xs$ is a palindrome, and x as well.

(ii) We write $\widetilde{s}xs = \widetilde{s}\cdot py = y\widetilde{p}\cdot s$. If $|y| > 0$, since this equality has the form $uy = y\widetilde{u}$ with $u = \widetilde{s}p$, by Proposition 6 (i) and (iii) $\widetilde{s}xs$ is a palindrome and so is x. If $|y| = 0$, then $xs = p = \widetilde{x}s$ and x is a palindrome.

(iii) By applying the reversal operator on both sides of equalities, the result follows by (i). □

Corollary 11. *Let $x, y, p, s \in \Sigma^*$, and $v, w \in \mathrm{Pal}(\Sigma^*)$. Then the following properties hold.*

(i) *If $sx = pv$ and $s\widetilde{x} = pw$, then x is a palindrome;*

(ii) *Let $|p| \neq |s|$. If $sx = py$ and $s\widetilde{x} = p\widetilde{y}$, then x and y are palindromes.*

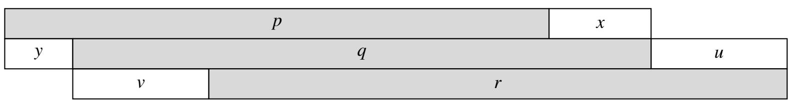
Proof. (i) If $|p| > |s|$, then there exists $p' \neq \varepsilon$ such that $x = p'v$ and $\widetilde{x} = p'w$. From Proposition 10 (i), we conclude that x is a palindrome. If $|p| < |s|$, then there exists $s' \neq \varepsilon$ such that $s'x = v$ and $s'\widetilde{x} = w$ and x is a palindrome from Proposition 10 (iii).

(ii) Without loss of generality consider $|p| > |s|$. Then there exists $p' \neq \varepsilon$ such that $x = p'y$ and $\widetilde{x} = p'\widetilde{y}$. The result follows from Proposition 10 (ii). □

We consider now the relative positions of three palindromes in a word, and start with a property inducing periodicity in overlapping palindromes.

Proposition 12. *Let $w = pxu = yqu = yvr$ where p, q and r are palindromes with $|q| > |x|$ and $|q| > |v|$. Let $a = |x| + |y|$ and $b = |u| + |v|$. If $|q| \geq a + b - \gcd(a, b)$, then $\gcd(a, b)$ is a period of w.*

Proof. The relative positions of p, q and r are as follows.



Applying Corollary 7 to the equations $px = yq$ and $qu = vr$, we obtain that yq has period $a = |x| + |y|$ and period $b = |u| + |v|$. In particular, q has both periods. Since $|q| \geq a + b - \gcd(a, b)$, then the Fine and Wilf's theorem applies so that q has period $\gcd(a, b)$ as well. Finally, since $|q| \geq a + b - \gcd(a, b)$ implies that $|q| \geq a$ and $|q| \geq b$, then yq and qu both have period $\gcd(a, b)$ and $w = yqu$ has period $\gcd(a, b)$. □

As a direct consequence of this proposition, we know a little bit more when q and r are the longest palindromic suffixes. Using the same notation, we have the following result.

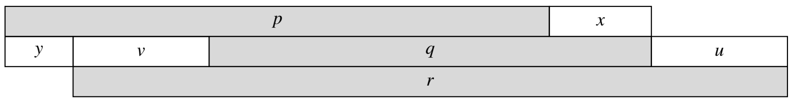
Corollary 13. *Assume that* $q = \mathrm{LPS}(px)$ *or* $r = \mathrm{LPS}(qu)$*, where* $|x| > 0$ *and* $|v| > 0$*. Then,* $|q| \geq a + b - 1$ *implies that* $\gcd(a, b) \neq 1$.

Proof. If $\gcd(a, b) = 1$, then $w = \alpha^{|w|}$ by Proposition 12. Therefore $\mathrm{LPS}(qu) = vr \neq r$ since $|v| > 0$, a contradiction. □

The next proposition is similar but deals with another configuration.

Proposition 14. *Let* $w = pxu = yvqu = yr$ *where* p, q *and* r *are palindromes such that* $|q| > |x|$ *and* $|r| > |xu|$*. Let* $a = |x| + |y| + |u|$ *and* $b = |y| + |v| + |x|$*. If* $|yvq| > a + b - \gcd(a, b)$*, then* $\gcd(a, b)$ *is a period of* w*.*

Proof. The relative positions of p, q and r are depicted below.



Applying Corollary 7 to the equations $w = pxu = yr$ and $px = yvq$, we obtain that w has period a and yvq has period b. If $|yvq| > a + b - \gcd(a, b)$, then by Fine and Wilf's theorem, $\gcd(a, b)$ is a period of yvq. Finally, since $|yvq| > a$, the period $\gcd(a, b)$ extends to whole w. □

4. Equations on circular words

Circular words are convenient for coding discrete objects like polyominoes, and therefore, it is natural to study the structure of equations involving them. In this section, we consider equations of type $A\,B\,\widetilde{A}\,\widetilde{B} \equiv X\,Y\,\widetilde{X}\,\widetilde{Y}$, or equivalently, words $w \in \Sigma^*$ satisfying the following properties:

(i) $w = A\,B\,\widetilde{A}\,\widetilde{B}$, with $A, B \in \Sigma^*$

(ii) $wp = pX\,Y\,\widetilde{X}\,\widetilde{Y}$ or $sw = X\,Y\,\widetilde{X}\,\widetilde{Y}s$

where p and s are respectively a prefix and a suffix of w. Without loss of generality, the case $sw = X\,Y\,\widetilde{X}\,\widetilde{Y}s$ may be dropped since it amounts to a renaming of the variables. An example of the situation $wp = pX\,Y\,\widetilde{X}\,\widetilde{Y}$ is depicted in Figure 2.

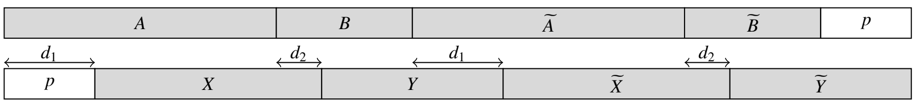


Figure 2: Equation $A\,B\,\widetilde{A}\,\widetilde{B} \equiv X\,Y\,\widetilde{X}\,\widetilde{Y}$.

Let d_1 be the delay between A and X, and d_2 the one between B and Y. Clearly, the overlap between $A\,B\,\widetilde{A}\,\widetilde{B}$ and $X\,Y\,\widetilde{X}\,\widetilde{Y}$ is completely determined by A, B, d_1 and d_2. Indeed, by construction, $|X| = |A| - d_1 + d_2$ and $|Y| = |B| - d_2 + d_1$. We assume in addition that $|A| > |d_1| > 0$

and $|B| > |d_2| > 0$. We are interested in the particular cases where none of the factors A, B, X, Y is included in some other one, that is, the case $d_1 d_2 > 0$. The reason for this restriction will become clear later. Without loss of generality, we may restrict our study to the case $d_1, d_2 > 0$, since the case $d_1, d_2 < 0$ is obtained from the first one by applying the reversal operator on both sides of the equation $A\,B\,\widetilde{A}\,\widetilde{B} \equiv X\,Y\,\widetilde{X}\,\widetilde{Y}$. Therefore, we define an *admissible configuration* as a 4-tuple $(A, B, d_1, d_2) \in \Sigma^* \times \Sigma^* \times \mathbb{N}_{>0} \times \mathbb{N}_{>0}$ such that there exist two words X and Y satisfying $A\,B\,\widetilde{A}\,\widetilde{B} \equiv X\,Y\,\widetilde{X}\,\widetilde{Y}$ with delays d_1 and d_2. We finally set $d = d_1 + d_2$.

Let A_1, A_2 be respectively the prefix and the suffix of A of length d_1, and B_1, B_2 those of B of length d_2, that is

$$A = A_1 S = T A_2 \quad \text{and} \quad B = B_1 V = W B_2.$$

Observe that with this notation, we can write X and Y as

$$X = B_2 T = S B_1 \quad \text{and} \quad Y = \widetilde{A_1} W = V \widetilde{A_2},$$

as shown in Figure 3.

<table>
<tr><td colspan="2">A</td><td colspan="2">B</td><td colspan="2">$\widetilde{A}$</td><td colspan="2">$\widetilde{B}$</td><td>p</td></tr>
<tr><td>A_1</td><td>S</td><td>B_1</td><td>V</td><td>$\widetilde{A_2}$</td><td>$\widetilde{T}$</td><td>$\widetilde{B_2}$</td><td>$\widetilde{W}$</td><td>A_1</td></tr>
<tr><td>p</td><td colspan="2">X</td><td colspan="2">Y</td><td colspan="2">$\widetilde{X}$</td><td colspan="2">$\widetilde{Y}$</td></tr>
</table>

Figure 3: Finer factorization of $A\,B\,\widetilde{A}\,\widetilde{B}$.

It is convenient to work with this decomposition that leads to a finer one, as shown in the next proposition.

Proposition 15. *With the notation and configuration introduced above, there exist 8 words u, v, r, s, g, h, m, n and $i, j \in \mathbb{N}$ such that*

(i) $A_1 B_2 = uv$, $T = (uv)^i u$, $A_2 B_1 = vu$

(ii) $B_2 A_1 = rs$, $S = (rs)^i r$, $B_1 A_2 = sr$

(iii) $B_1 \widetilde{A_1} = gh$, $W = (gh)^j g$, $B_2 \widetilde{A_2} = hg$

(iv) $\widetilde{A_1} B_1 = mn$, $V = (mn)^j m$, $\widetilde{A_2} B_2 = nm$

with $|u| = |r|$ and $|g| = |m|$.

Proof. First, consider case (i). We have that $A_1 S = T A_2$ and $B_2 T = S B_1$. Since both equations satisfy the hypothesis of Lemma 8, we conclude that for some u, v and some $i \geq 0$,

$$A_1 B_2 = uv, T = (uv)^i u, A_2 B_1 = vu.$$

The argument is the same for proving (ii), (iii) and (iv), using respectively the pairs of equations $(B_2 T = S B_1\,; A_1 S = T A_2)$, $(B_1 V = W B_2\,; \widetilde{A_1} W = V A_2)$ and $(\widetilde{A_1} W = V \widetilde{A_2}\,; B_1 V = W B_2)$. Finally, since $|S| = |T|$ and $|V| = |W|$, the result follows. □

As a consequence we obtain more information on the periodic structure of factors in the circular equation.

Corollary 16. *A, B, X and Y have period d.*

Proof. Proposition 15 ensures that the following equalities hold

$$AB_1 = TA_2B_1 = (uv)^i uA_2B_1 = (uv)^i uvu = (uv)^{i+1}u.$$

Since $|uv| = |A_1B_2| = d$, AB_1 has the period d and so does A. The argument is exactly the same for B, X and Y. □

Corollary 17. *If $u = \varepsilon$ or $g = \varepsilon$, then $A_1 = A_2$ and $B_1 = B_2$.*

Proof. If $u = \varepsilon$, then by Proposition 15 (i), $A_1B_2 = \varepsilon \cdot v = v \cdot \varepsilon = A_2B_1$ and the result follows. The proof is similar for the case $g = \varepsilon$. □

Corollary 18. *If $u = \varepsilon$ (resp. $g = \varepsilon$) and $|g| = k$ (resp. $|u| = k$), then B and Y (resp. A and X) have periods k and $\gcd(k, d)$.*

Proof. By Corollary 17, the hypothesis $u = \varepsilon$ implies that $A_1 = A_2$, $B_1 = B_2$ and so $gh = hg$. Let $w = (gh)^N$ such that $|w| > |B\widetilde{A_1}|$. Clearly, $|gh| = d$ is a period of w. Moreover, the equalities $w \cdot g = (gh)^N g = g(hg)^N = g(gh)^N = g \cdot w$ imply that $k = |g|$ is also a period of w. Since N can be chosen large enough so that the theorem of Fine and Wilf applies, w has period $\gcd(k, d)$, and so is the case for B and Y since $B\widetilde{A_1} = B_1Y$ is a prefix of w. □

Proposition 15 enables us to build a set of equalities (Lemma 19) revealing the local periodicity of factors in the circular equation (Corollary 20). It also provides a tool for deriving new admissible configurations from the initial one by adding (Corollary 21) or removing (Corollary 22) periods in A and B.

Lemma 19. *The following equalities hold:*

(i) *$A_1r = uA_2$ and $sB_2 = B_1v$,*

(ii) *$B_2u = rB_1$ and $vA_1 = A_2s$,*

(iii) *$B_1m = gB_2$ and $n\widetilde{A_1} = \widetilde{A_2}h$,*

(iv) *$\widetilde{A_1}g = m\widetilde{A_2}$ and $hB_1 = B_2n$.*

Proof. (i) Since $A_1rsB_2 = A_1B_2A_1B_2 = uvuv = uA_2B_1v$, we have that $A_1rsB_2 = uA_2B_1v$. In particular, the prefixes of length $|r| + d_1$ on both sides of the equality are equal. Since $|u| = |r|$ and $|A_1| = |A_2| = d_1$, then $A_1r = uA_2$ and $sB_2 = B_1v$ which proves the claim. The argument is similar for proving (ii), (iii) and (iv), by starting respectively from B_2uvA_1, $B_1mn\widetilde{A_1}$ and $\widetilde{A_1}ghB_1$. □

Corollary 20. *Let k and l be such that $|u| = |r| = k$ and $|g| = |m| = l$. Then A_1, A_2, B_1 and B_2 have periods $k + l$ and $2d - (k + l)$.*

Proof. From Lemma 19 (i) and (iv), we have respectively that $uA_2 = A_1r$ and $\widetilde{g}A_1 = A_2\widetilde{m}$. Since these two equations satisfy the hypothesis of Lemma 8, then A_1 and A_2 both have period $|u\widetilde{g}| = |u| + |g| = k + l$. Now, from Corollary 19 (i) and (iv), we have $vA_1 = A_2S$ and $\widetilde{h}A_2 = A_1\widetilde{n}$. Then, Lemma 8 also applies with this pair of equations, so that both A_1 and A_2 have period $|v\widetilde{h}| = |v| + |h| = (d - k) + (d - l) = 2d - (k + l)$. The argument is the same for B_1 and B_2, using the pairs of equations ($hB_1 = B_2n$; $sB_2 = B_1v$) and ($rB_1 = B_2u$; $gB_2 = B_1m$). □

Corollary 21. *Let $A' = uvA = Asr$ and $B' = ghB = Bnm$. Then (A', B, d_1, d_2), (A, B', d_1, d_2) and (A', B', d_1, d_2) are admissible configurations.*

Proof. First, let $X' = SsrB_1$ and $Y' = Vmn\widetilde{A_2}$. Using the fact that $S = (rs)^i r$ and $V = (mn)^j m$, we verify that they also satisfy $X' = B_2uvT$ and $Y' = \widetilde{A_1}ghW$. Now, to prove the first assertion, we show that $A'B\widetilde{A'}\widetilde{B} \equiv X'Y\widetilde{X'}\widetilde{Y}$ with delays d_1 and d_2. Indeed, we have

$$A'B\widetilde{A'}\widetilde{B} \cdot A_1 = A_1Ssr \cdot B_1V \cdot \widetilde{A_2}\widetilde{T}\,\widetilde{v}\,\widetilde{u} \cdot \widetilde{B_2}\widetilde{W} \cdot A_1$$

which may be rewritten as

$$A_1 \cdot SsrB_1 \cdot V\widetilde{A_2} \cdot \widetilde{T}\,\widetilde{v}\,\widetilde{u}\widetilde{B_2} \cdot \widetilde{W}A_1 = A_1 \cdot X'Y\widetilde{X'}\widetilde{Y},$$

which proves the assertion. Similarly, we have $AB'\widetilde{A}\widetilde{B'} \equiv XY'\widetilde{X}\widetilde{Y'}$ and $A'B'\widetilde{A'}\widetilde{B'} \equiv X'Y'\widetilde{X'}\widetilde{Y'}$ with delays d_1 and d_2, which establishes the second and third assertions. □

Corollary 22. *Suppose that $r, m \neq \varepsilon$ (resp. $s, n \neq \varepsilon$), then (A_1r, B_1m, d_1, d_2) (resp. (A_2s, B_2n, d_1, d_2)) is an admissible configuration.*

Proof. From the first equalities in Corollary 19 (i) and (iii), we have

$$A_1r \cdot B_1m \cdot \widetilde{A_1r} \cdot \widetilde{B_1m} \cdot A_1 = A_1r \cdot B_1m \cdot \widetilde{A_2}\,\widetilde{u} \cdot \widetilde{B_2}\,\widetilde{g} \cdot A_1,$$

which in turn, by using the first equalities in Corollary 19 (ii) and (vi), can be factorized as

$$A_1 \cdot rB_1 \cdot m\,\widetilde{A_2} \cdot \widetilde{u}\,\widetilde{B_2} \cdot \widetilde{g}A_1 = A_1 \cdot rB_1 \cdot m\,\widetilde{A_2} \cdot \widetilde{rB_1} \cdot \widetilde{m\,\widetilde{A_2}}.$$

Then $A_1r \cdot B_1m \cdot \widetilde{A_1r} \cdot \widetilde{B_1m} \equiv rB_1 \cdot m\widetilde{A_2} \cdot \widetilde{rB_1} \cdot \widetilde{m\,\widetilde{A_2}}$ with delays d_1 and d_2. Since $|A_1r| > d_1$ and $|B_1m| > d_2$, this proves the first assertion. The proof is similar in the second case, using the four right hand equalities in Corollary 19. □

5. Application to tilings: characterization of double squares

A polyomino P is represented by a word $b(P)$ on the 4-letter alphabet $\mathcal{E} = \{a, b, \bar{a}, \bar{b}\}$ encoding the elementary steps (right, up, left and down respectively) on the square grid $\mathbb{Z} \times \mathbb{Z}$. There is a natural involution $\bar{\cdot} : \mathcal{E} \to \mathcal{E}$ defined by

$$\bar{\cdot} : a \mapsto \bar{a}, \bar{a} \mapsto a, b \mapsto \bar{b}, \bar{b} \mapsto b,$$

which amounts to swap the elementary directions. The composition $\widehat{\cdot} = \bar{\cdot} \circ \widetilde{\cdot}$ is an antimorphism interpreted as follows: if $w \in \mathcal{E}^*$ is a path, then $\widehat{w}$ is the same path traversed in the opposite direction. A polyomino *P tiles the plane (by translation)* if it is possible to cover the plane with nonoverlapping translated copies of P. Deciding if P tiles the plane by translation amounts to check if the circular word $b(P)$ can be factorized as

$$b(P) \equiv ABC\widehat{A}\widehat{B}\widehat{C}, \tag{2}$$

where at most one variable is empty (Beauquier and Nivat [16]). P is called a *square tile* if one variable is empty, otherwise it is called a *hexagon tile*.

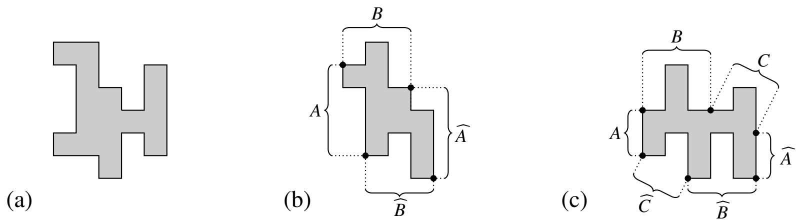


Figure 4: (a) A polyomino P, (b) a square tile S and (c) an hexagon tile H.

For instance, in Figure 4, the polyomino P is completely defined by the boundary word $b(P) = abba\bar{b}\bar{b}\bar{b}\bar{b}\bar{a}b\bar{a}\bar{b}\bar{b}\bar{a}b\bar{a}babbb\bar{a}baa\bar{b}\bar{b}a\bar{b}$. The boundary word of the square tile S can be factorized as $b(S) = bbb\bar{a}b \cdot aba\bar{b}\bar{b}a \cdot \bar{b}a\bar{b}\bar{b}\bar{b} \cdot \bar{a}bb\bar{a}\bar{b}\bar{a}$, while the factorization of the boundary word of the hexagon tile H is $b(H) = bb \cdot abba\bar{b}\bar{b}a \cdot aba\bar{b}\bar{b} \cdot \bar{b}\bar{b} \cdot \bar{a}bb\bar{a}\bar{b}\bar{b}\bar{a} \cdot bb\bar{a}\bar{b}\bar{a}$.

Moreover, although some polyominoes admit $O(|b(P)|)$ distinct hexagon factorizations, square tiles admit at most two distinct factorizations. This fact was conjectured in [11] and proved in [17]. Polyominoes having exactly two distinct square factorizations yield distinct tilings as illustrated in Figure 1, and are called *double squares*.

All results of Section 4 obtained from equations of type $A\,B\,\widetilde{A}\,\widetilde{B} \equiv X\,Y\,\widetilde{X}\,\widetilde{Y}$, are also valid for equations of the form $A\,B\,\widehat{A}\,\widehat{B} \equiv X\,Y\,\widehat{X}\,\widehat{Y}$ by replacing $\widetilde{\cdot}$ with $\widehat{\cdot}$ everywhere. Since the problem of characterizing double squares requires solving equations of this last form, the preceding section provides information about their periodic structure and a way to generate some of them. In particular, this application to double squares motivates the condition $d_1, d_2 > 0$ imposed at the beginning of our study. Indeed, it has been proved in [11] that if an admissible configuration represents a double square, then none of the factors A, B, X and Y is included in some other one.

Passing from equations involving the operator $\widetilde{\cdot}$ to the study of polyominoes, it is crucial to keep in mind that all admissible configurations do not represent a double square. Indeed, the boundary word of a polyomino is a non-crossing closed path. Thus, admissible configurations require additional verifications and hypothesis. In this section, we show how the results of the preceding section concretely apply on double squares by simply looking at examples.

Consider first Corollary 16. It ensures that each factor of the factorization describing a double square is periodic with period d. An instance of this situation is shown in Figure 5, where the black and white dots distinguish the $A\,B\,\widehat{A}\,\widehat{B}$ and $X\,Y\,\widehat{X}\,\widehat{Y}$ factorizations of the tile. In this specific case $d_1 = 2$, $d_2 = 3$, the polyomino is a realization of the admissible configuration $(bbababbababb, aba\bar{b}\bar{b}aba\bar{b}\bar{b}aba, 2, 3)$ and each factor has period $d = d_1 + d_2 = 5$.

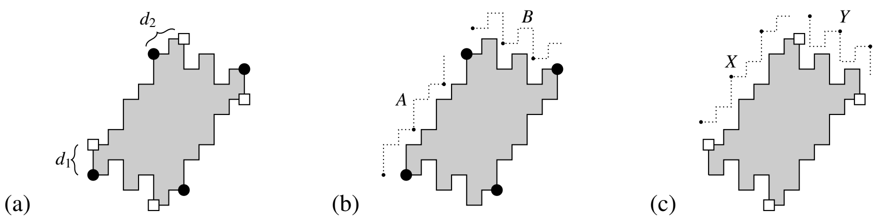


Figure 5: (a) A double square with the $A\,B\,\widehat{A}\,\widehat{B}$ (black dots) and $X\,Y\,\widehat{X}\,\widehat{Y}$ (white dots) factorizations. The periodic factors of length $d_1 + d_2 = 5$ are emphasized: (b) in factors A and B; (c) in factors X and Y.

Corollary 21 provides an operator acting on a double square by stretching it according to the periods found in its factors in a consistent way. Figure 6 shows three possible extensions of a double square: in the first case, the factor A has been stretched while in the second case, only the factor B has been stretched. In the last case, both factors A and B have been stretched.

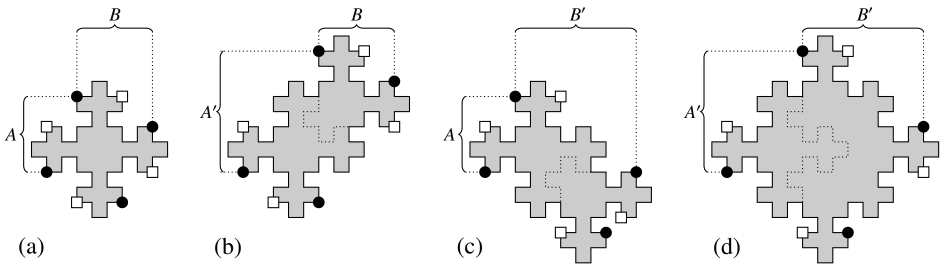


Figure 6: (a) A double square; extended by stretching (b) the factor A, (c) the factor B and (d) both A and B.

Finally, Corollary 22 provides two operators that yield in some cases smaller double squares, as illustrated in Figure 7.

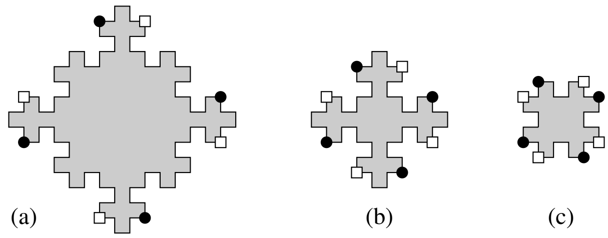


Figure 7: (a) A double square and (b,c) its two respective reductions according to Corollary 22.

Clearly, solving equations on circular words is a powerful technic for describing, for instance double squares. It is likely that it could also provide a mean for solving a conjecture about prime double squares ([11], p. 97). Recall that, given a polyomino Q and a square S, the composition $Q \circ S$ is the polyomino defined by replacing each unit cell of Q by S. A polyomino is said to be *prime* if it is not obtained by a trivial composition (see [11] for more details). The conjecture states that in the factorization $A\,B\,\widehat{A}\,\widehat{B} \equiv X\,Y\,\widehat{X}\,\widehat{Y}$ of a prime double square, the factors A, B, X and Y all are palindromes. Therefore, a first approach to solve it is to describe polyominoes that satisfy the following equation:

$$A\,B\,\widehat{A}\,\widehat{B} \equiv X\,Y\,\widehat{X}\,\widehat{Y}, \quad \text{for some } A, B, X, Y \in \mathrm{Pal}(\mathcal{E}^*). \tag{3}$$

In particular, if a double square satisfies this palindromicity hypothesis, we obtain extra conditions on the length of some factors. Indeed, with the decomposition given by Proposition 15 we have the following elementary fact, whose proof is left to the reader.

Proposition 23. *Assume that* $A, B, X, Y \in \mathrm{Pal}(\mathcal{E}^*)$. *Then* $|r| + |g| \neq d$.

To describe exhaustively the solutions to Equation (3), we used the combinatorics on words library from the open-source mathematical software `Sage` [18]. More precisely, we implemented an equation solver, which is now distributed with `Sage`, that returns the general solution to a given set of equations. For instance, for $|A| = |B| = |X| = |Y| = 43$ and all possible delays $D = d_1 = d_2$ such that $1 \leq D \leq 42$ in Equation (3), prime double squares are obtained exactly when

$$D = 1, 2, 4, 5, 6, 9, 14, 17, 26, 29, 34, 37, 38, 39, 41, 42.$$

Also, note that each double square obtained with delay D is isometric to the one obtained with delay $43 - D$. The eight polyominoes in question are shown in Figure 8. A description of two infinite families of double squares appears in [19].

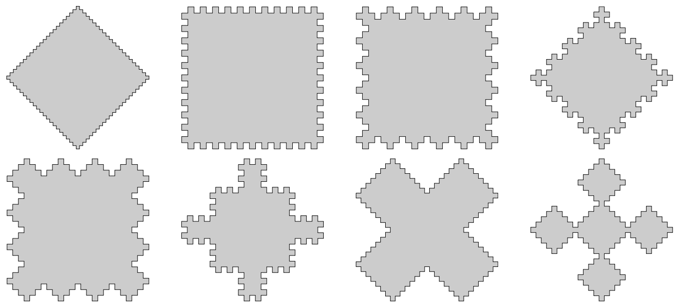


Figure 8: Prime double squares with $|A| = |B| = |X| = |Y| = 43$.

6. Concluding remarks

In this article, we considered different equations involving palindromes and circular words. In all cases, we deduced local periodicity conditions that cast a new light on the structure of objects encoded by such equations. These observations raise several challenging problems that we briefly describe now. First, since equations of the form $ABC = \sigma(A)\sigma(B)\sigma(C)$ appear naturally in many problems expressed in term of combinatorics on words, it would be interesting to generalize Proposition 5. For instance, one could allow the reversal of some of the variables in the permutation, like in the equation: $ABC = \widetilde{\sigma(A)}\sigma(B)\sigma(C)$. It would also be pertinent to extend the results by considering more than three variables. Moreover, the study of equations of the form $A\,B\,\widetilde{A}\,\widetilde{B} \equiv X\,Y\,\widetilde{X}\,\widetilde{Y}$ suggests a new starting point to the problem of characterizing double squares. As discussed in Section 5, one could establish the conditions required to generalize all results on admissible configurations to double squares, or at least to find for which classes it is possible. Figure 9 shows a simple example of a double square for which the result of a reduction is not a polyomino.

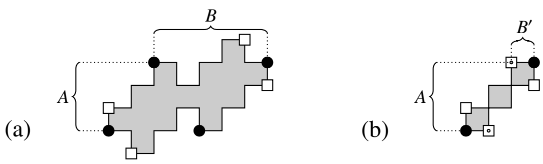


Figure 9: (a) A double square (b) and its reduction obtained by shrinking the factor B according to its period $d = 8$. Remark that some of the factorization points are merged together.

This naturally leads to the question of finding which properties are preserved by the stretching and shrinking operations. Figure 9 already shows that such is not the case for the "being a self-avoiding path" property. Finally, the problem of characterizing and generating double squares is still open, as well as the problem of efficiently generating random tiles.

Acknowledgements

Some results in this paper were discovered by computer exploration using the open-source mathematical software `Sage` [18] and its algebraic combinatorics features developed by the `Sage-Combinat` community [20]. Also, the pictures have been produced using `Sage` and the `pgf/tikz` package [21].

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