Codings of rotations are full

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Palindromes

- tenet
- ressasser
- reconocer
- kisik
The Fibonacci word

We define \( f_{-1} = b, \ f_0 = a \) and, for \( n \geq 1 \),

\[
f_n = f_{n-1}f_{n-2}.
\]

Therefore, we have

\[
\begin{align*}
f_0 &= a \\
f_1 &= ab \\
f_2 &= aba \\
f_3 &= abaab \\
f_4 &= abaababa \\
f_5 &= abaababaabaab \\
& \vdots \\
\end{align*}
\]

The infinite word \( f_\infty \) is called the Fibonacci word.
The Thue-Morse word

We define $t_0 = a$ and, for $n \geq 1$,

$$t_n = t_{n-1}t_{n-1}.$$

so that

$$
\begin{align*}
t_0 &= a \\
t_1 &= ab \\
t_2 &= abba \\
t_3 &= abbabaab \\
t_4 &= abbabaabbaabababba \\
t_5 &= abbabaabbaababababaababbaababbaababbaababbaababbaab
\end{align*}
$$

The infinite word $t_\infty$ is called the **Thue-Morse word**.
Theorem (Droubay, Justin and Pirillo, 2001)

Let $w$ be a finite word. Then $|\text{Pal}(w)| \leq |w| + 1$.

- Assume that the first occurrence of some palindromes $p$ and $q$ ends at the same position.
Theorem (Droubay, Justin and Pirillo, 2001)

Let \( w \) be a finite word. Then \( |\text{Pal}(w)| \leq |w| + 1 \).

- Assume that the first occurrence of some palindromes \( p \) and \( q \) ends at the same position.
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Let $w$ be a finite word. Then $|\text{Pal}(w)| \leq |w| + 1$.

- Assume that the first occurrence of some palindromes $p$ and $q$ ends at the same position.
- Then $p = q$. 
**Theorem (Droubay, Justin and Pirillo, 2001)**

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Assume that the first occurrence of some palindromes \( p \) and \( q \) ends at the same position.

Then \( p = q \).

---

**Theorem (Droubay, Justin and Pirillo, 2001)**

*Sturmian words are full, i.e. they realize the upper bound.*
Palindromic complexity

Number of palindrome factors

Length of prefix

Fibonacci word

Thue-Morse word

Fixed point of $a \mapsto abb, b \mapsto ba$

Upper bound
The Fibonacci word is full

\[ w = a \]

---

Palindromes \( a \)
The Fibonacci word is full

\[ w = a \ b \]

Palindromes

\[ a \]

\[ b \]
The Fibonacci word is full

\[ w = a \ b \ a \]

---

Palindromes

\[
\begin{align*}
\text{a} \\
\text{b} \\
\text{a \ b \ a}
\end{align*}
\]
The Fibonacci word is full

\[ w = a \ b \ a \ a \]

\[
\begin{align*}
\text{Palindromes} & \\
& a \\
& \quad b \\
& a \ b \ a \\
& & a \ a
\end{align*}
\]
The Fibonacci word is full

\[ w = a \ b \ a \ a \ b \]

<table>
<thead>
<tr>
<th>Palindromes</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td></td>
</tr>
<tr>
<td>b</td>
<td>b</td>
</tr>
<tr>
<td>a b a</td>
<td>a</td>
</tr>
<tr>
<td>a a</td>
<td></td>
</tr>
<tr>
<td>b a a b</td>
<td></td>
</tr>
</tbody>
</table>
The Fibonacci word is full

\[ w = a \ b \ a \ a \ b \ a \]

Palindromes

\[
\begin{array}{c}
a \\
b \\
a \ b \ a \\
a \ a \\
b \ a \ a \ b \\
a \ b \ a \ a \ b \ a
\end{array}
\]
The Fibonacci word is full

\[ w = a \ b \ a \ a \ b \ a \ b \]

---

Palindromes

\[
\begin{align*}
& a \\
& \quad \ b \\
& a \ b \ a \\
& a \ a \\
& b \ a \ a \ b \\
& a \ b \ a \ a \ b \ a \\
& \quad \ b \ a \ b
\end{align*}
\]
The Fibonacci word is full

\[ w = a \ b \ a \ a \ b \ a \ b \ a \]

Palindromes

\[
\begin{array}{cccccccc}
\text{a} \\
\text{b} \\
\text{a} \ b \ a \\
\text{a} \ a \\
\text{b} \ a \ a \ b \\
\text{a} \ b \ a \ a \ b \\
\text{b} \ a \ b \\
\text{a} \ b \ a \ b \ a \\
\end{array}
\]
The Fibonacci word is full

\[ w = a \ b \ a \ a \ b \ a \ b \ a \ a \ a \ \ldots \]
The Thue-Morse word is lacunary

\[ w = a \]

Palindromes

\[ a \]
The Thue-Morse word is lacunary

\[ w = a \ b \]

Palindromes

\[
\begin{align*}
\text{a} \\
\text{b}
\end{align*}
\]
The Thue-Morse word is lacunary

\[ w = a \ b \ b \]

Palindromes

\[
\begin{array}{c}
\text{a} \\
\text{b} \\
\text{b} \\
\text{b} \\
\end{array}
\]

There is no new palindrome at this position!
The Thue-Morse word is lacunary

\[ w = a \ b \ b \ a \]

Palindromes

\[
\begin{array}{c}
a \\
b \\
b \\
b \\
a \ b \ b \ a
\end{array}
\]
The Thue-Morse word is lacunary

\[ w = a \ b \ b \ a \ b \]

Palindromes

\[
\begin{aligned}
& a \\
& \quad b \\
& \quad \quad b \\
& \quad \quad b \\
& a \ b \ b \ a \\
& \quad \quad b \\
& \quad \quad a \ b \ b \ a \\
& \quad \quad \quad b \\
& \quad \quad \quad a \ b \ b \ a \\
& \quad \quad \quad \quad b \\
& \quad \quad \quad \quad a \ b \ b \ a
\end{aligned}
\]
The Thue-Morse word is lacunary

\[ w = a \ b \ b \ a \ b \ a \]

Palindromes

\[
\begin{align*}
\text{a} \\
\ b \\
\ b \ b \\
\ a \ b \ b \ a \\
\ b \ a \ b \\
\ a \ b \ a
\end{align*}
\]
The Thue-Morse word is lacunary

\[ w = a \ b \ b \ a \ b \ a \ a \]

Palindromes

\[
\begin{align*}
& a \\
& b \\
& b \\
& b \\
& a \\
& b \\
& a \\
& b \\
& a \\
& a \\
& a \\
\end{align*}
\]
The Thue-Morse word is lacunary

\[ w = a \ b \ b \ a \ b \ a \ a \ b \]

Palindromes

\[
\begin{array}{c}
\ a \\
\ b \\
\ b \ b \\
\ a \ b \ b \ a \\
\ b \ a \ b \\
\ b \ a \ b \\
\ a \ b \ a \\
\ a \ a \\
\ b \ a \ a \ b \\
\end{array}
\]
The Thue-Morse word is lacunary

\[ w = a \ b \ b \ a \ b \ a \ a \ b \ b \ \ldots \]

---

Palindromes

\[
\begin{align*}
& a \\
& \quad b \\
& \quad \quad b \ b \\
& \quad \quad a \ b \ b \ a \\
& \quad \quad \quad b \ a \ b \\
& \quad \quad \quad \quad a \ b \ a \\
& \quad \quad \quad \quad \quad a \ a \\
& \quad \quad \quad \quad \quad \quad b \ a \ a \ b
\end{align*}
\]

There is no new palindrome at this position!
Complete return words

We say that $v$ is a complete return word of $u$ in $w$, if $v$ starts at an occurrence of $u$ and ends at the end of the next occurrence of $u$.

Fact
A word $w$ is full if and only if every complete return word of a palindrome factor of $w$ is a palindrome.
The **coding of rotations** of parameters \((x, \alpha, \beta)\) is the word \(C = c_0 c_1 c_2 \cdots\) such that

\[
c_i = \begin{cases} 
0 & \text{if } x + i\alpha \in [0, \beta) \\
1 & \text{if } x + i\alpha \in [\beta, 1) 
\end{cases}
\]
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\]
The coding of rotations of parameters \((x, \alpha, \beta)\) is the word \(C = c_0c_1c_2 \cdots\) such that

\[
c_i = \begin{cases} 
0 & \text{if } x + i\alpha \in [0, \beta) \\
1 & \text{if } x + i\alpha \in [\beta, 1) 
\end{cases}
\]
Many interesting problems related to codings of rotations:

- **Density** of the letters 0 and 1,
- **Complexity**, i.e. the number of factors of length $n$, or **palindromic** and **$f$-palindromic** complexity,
- Applications to **number theory** [Adamczewski, 2002],
- etc.

In particular, Rote (1994) expressed sequences of complexity $2n$ with respect to codings of rotations.
The different cases

Let $C$ be a coding of rotations of parameters $(x, \alpha, \beta)$.

- If $\alpha$ is rational, then $C$ is periodic.
- If $\alpha = \beta$ is irrational, then $C$ is Sturmian
  \[ f(n) = n + 1. \]
- If $\alpha$ and $\beta$ are rationally dependent, then $C$ is quasi-Sturmian.
  \[ f(n) = n + k, \quad \text{for some constant } k. \]
- Otherwise, $C$ is a Rote sequence
  \[ f(n) = 2n, \quad \text{for large enough } n. \]
Main result

Theorem

Every coding of rotations is full.

The proof is based on the following ideas:

1. Return words
2. Interval exchange transformations
3. Poincaré’s first return function
4. Many results on those dynamical systems
Idea of the proof

Let $x = 0.102$, $\alpha = 0.135$ and $\beta = 0.578$. Then

$$C = 000011100001111000011100001100000111000 \cdots$$
Idea of the proof

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$C = 000011100001111000011100001110000111000001110000 \cdots$

\[ \beta - 2\alpha \]

\[ l_{001} \]

\[ l_{000} \]

\[ l_{011} \]

\[ l_{011} \]

\[ l_{111} \]

\[ l_{110} \]

\[ l_{100} \]

\[ l_{100} \]

\[ y \]

\[ y + \alpha \]

\[ \beta \]

\[ 0 \]

Blondin Massé et al. (UQÀM, U. Savoie)

Codings of rotations are full

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Idea of the proof

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$$C = 000011100011110000111000011100000111000\cdots$$
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$$C = 000011100001111000111000011100000111000\cdots$$
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This research was driven by computer exploration using the open-source mathematical software Sage [1] and its algebraic combinatorics features developed by the Sage-Combinat community [2], and in particular, F. Saliola, A. Bergeron and S. Labbé.

The pictures have been produced using Sage and pgf/tikz.
