Codings of rotations are full

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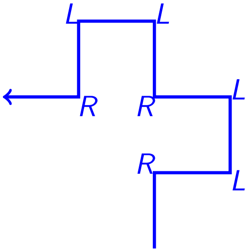
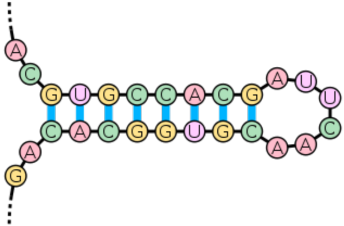
EUROCOMB 2009
September 11th, 2009

Palindromes





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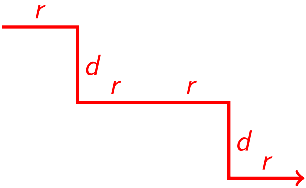

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The Fibonacci word

We define $f_{-1} = b$, $f_0 = a$ and, for $n \geq 1$,

$$f_n = f_{n-1} f_{n-2}.$$

Therefore, we have

$$\begin{array}{rcl} f_0 & = & a \\ f_1 & = & ab \\ f_2 & = & aba \\ f_3 & = & abaab \\ f_4 & = & abaababa \\ f_5 & = & abaababaabaab \\ \vdots & & \vdots \end{array}$$

The infinite word f_∞ is called the Fibonacci word.

The Thue-Morse word

We define $t_0 = a$ and, for $n \geq 1$,

$$t_n = t_{n-1}\overline{t_{n-1}}.$$

so that

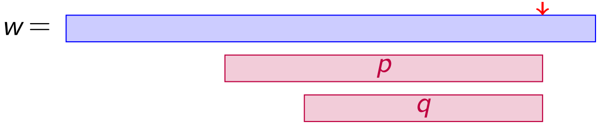
$$\begin{array}{rcl} t_0 & = & a \\ t_1 & = & ab \\ t_2 & = & abba \\ t_3 & = & abbabaab \\ t_4 & = & abbabaabbaababba \\ t_5 & = & abbabaabbaababbabaababbaabbabaab \\ \vdots & & \vdots \end{array}$$

The infinite word t_∞ is called the Thue-Morse word.

Number of distinct palindromic factors

Theorem (Droubay, Justin and Pirillo, 2001)

Let w be a finite word. Then $|\mathrm{Pal}(w)| \leq |w| + 1$.



- Assume that the first occurrence of some palindromes p and q ends at the same position.

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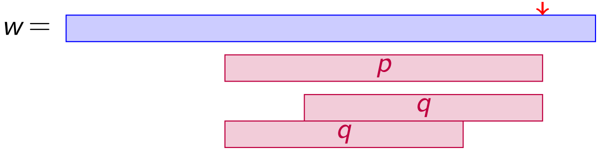

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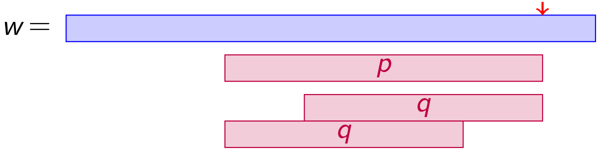

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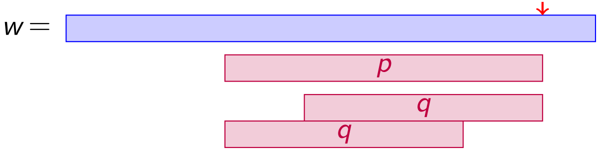

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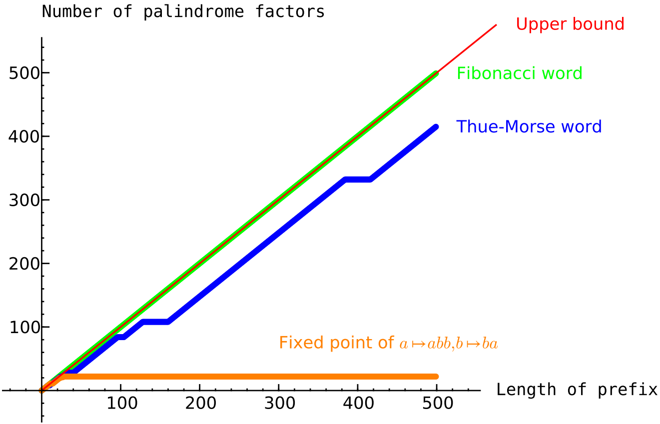
Theorem (Droubay, Justin and Pirillo, 2001)

Sturmian words are full, i.e. they realize the upper bound.

Palindromic complexity



The Fibonacci word is full

$$w = a$$

Palindromes a

The Fibonacci word is full

w	=	a	b
Palindromes		a	
			b

The Fibonacci word is full

$$w = a\ b\ a$$

Palindromes

a

b

aba

The Fibonacci word is full

$$w = a \quad b \quad a \quad a$$

Palindromes

a

b

$a \quad b \quad a$

$a \quad a$

The Fibonacci word is full

$$w = a\ b\ a\ a\ b$$

Palindromes

a

b

$a\ b\ a$

$a\ a$

$b\ a\ a\ b$

The Fibonacci word is full

$$w = a\ b\ a\ a\ b\ a$$

Palindromes

a

b

aba

aa

$baab$

$abaaba$

The Fibonacci word is full

$$w = a \quad b \quad a \quad a \quad \color{red}{b} \quad \color{red}{a} \quad \color{red}{b}$$

Palindromes

$$\begin{array}{ccccccc} a & & & & & & \\ & b & & & & & \\ a & b & a & & & & \\ & & a & a & & & \\ & b & a & a & b & & \\ a & b & a & a & b & a & \\ & & & & \color{red}{b} & \color{red}{a} & \color{red}{b} \end{array}$$

The Fibonacci word is full

$$w = a\ b\ a\ a\ b\ a\ b\ a$$

Palindromes								
	a							
		b						
	a	b	a					
			a	a				
		b	a	a	b			
	a	b	a	a	b	a		
					b	a	b	
				a	b	a	b	a

The Fibonacci word is full

$$w = a \; b \; a \; a \; b \; a \; b \; a \; a \; \cdots$$

Palindromes	a								
		b							
	a	b	a						
			a	a					
		b	a	a	b				
	a	b	a	a	b	a			
					b	a	b		
				a	b	a	b	a	
			a	a	b	a	b	a	a
									$\ddots$

The Thue-Morse word is lacuna ry

$$\mathbf{w} = a$$

Palindromes a

The Thue-Morse word is lacunary

$$\mathrm{w} = a \quad b$$

Palindromes	a
	b

The Thue-Morse word is lacunary

$$w = a \; b \; b$$

Palindromes

a

b

$b \; b$

The Thue-Morse word is lacunary

$$w = a \quad b \quad b \quad a$$

Palindromes				
	a			
		b		
		b	b	
	a	b	b	a

The Thue-Morse word is lacunary

$$w = a\ b\ b\ a\ b$$

Palindromes	a				
		b			
		b	b		
	a	b	b	a	
			b	a	b

The Thue-Morse word is lacunary

$$w = a\ b\ b\ a\ b\ a$$

Palindromes

Palindromes	a					
		b				
		b	b			
	a	b	b	a		
			b	a	b	
				a	b	a

The Thue-Morse word is lacunary

w	=	a	b	b	a	b	a	a
Palindromes		a						
			b					
			b	b				
		a	b	b	a			
				b	a	b		
					a	b	a	
							a	a

The Thue-Morse word is lacunary

$$w = a \quad b \quad b \quad a \quad \color{red}{b \quad a \quad a \quad b}$$

Palindromes	a							
		b						
		b	b					
	a	b	b	a				
			b	a	b			
				a	b	a		
						a	a	
					$\color{red}{b}$	$\color{red}{a}$	$\color{red}{a}$	$\color{red}{b}$

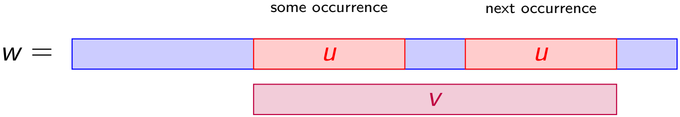
The Thue-Morse word is lacunary

$$w = a \quad b \quad b \quad a \quad b \quad a \quad a \quad b \quad b \quad \cdots$$

Palindromes									
	a								
		b							
		b	b						
	a	b	b	a					
			b	a	b				
				a	b	a			
						a	a		
					b	a	a	b	
									—
									$\ddots$

There is no new palindrome at this position!

Complete return words



We say that v is a complete return word of u in w, if v starts at an occurrence of u and ends at the end of the next occurrence of u.

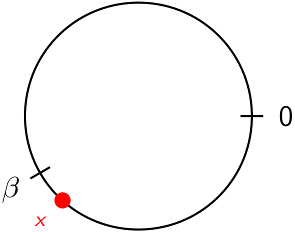
Fact

A word w is full if and only if every complete return word of a palindrome factor of w is a palindrome.

Codings of rotations (1/2)

The coding of rotations of parameters (x, α, β) is the word $\mathbf{C} = c_0 c_1 c_2 \cdots$ such that

$$c_i = \begin{cases} 0 & \text{if } x + i\alpha \in [0, \beta) \\ 1 & \text{if } x + i\alpha \in [\beta, 1) \end{cases}$$



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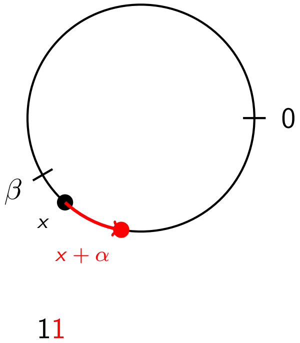

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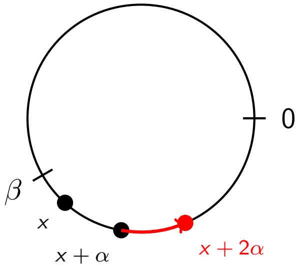

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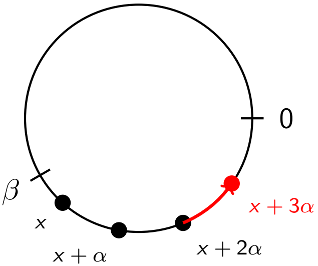

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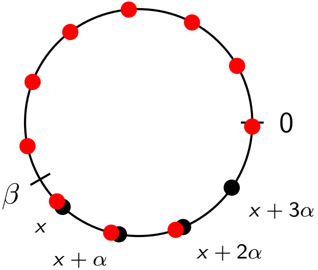

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Codings of rotations (2/2)

Many interesting problems related to codings of rotations:

- Density of the letters 0 and 1,
- Complexity, i.e. the number of factors of length n, or palindromic and f-palindromic complexity,
- Applications to number theory [Adamczewski, 2002],
- etc.

In particular, Rote (1994) expressed sequences of complexity $2n$ with respect to codings of rotations.

The different cases

Let **C** be a coding of rotations of parameters (x, α, β).

- If α is rational, then **C** is periodic.
- If $\alpha = \beta$ is irrational, then **C** is Sturmian

$$f(n) = n + 1.$$

- If α and β are rationally dependent, then **C** is quasi-Sturmian.

$$f(n) = n + k, \quad \text{for some constant } k.$$

- Otherwise, **C** is a Rote sequence

$$f(n) = 2n, \quad \text{for large enough } n.$$

Main result

Theorem

Every coding of rotations is full.

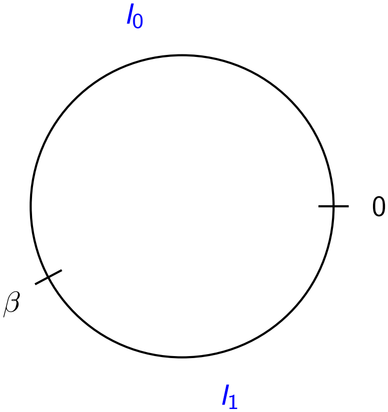
The proof is based on the following ideas:

1. Return words
2. Interval exchange transformations
3. Poincaré's first return function
4. Many results on those dynamical systems

Idea of the proof

Let $x = 0.102$, $\alpha = 0.135$ and $\beta = 0.578$. Then

$$\mathbf{C} = 00001110000011110000011100000111000000111000\cdots$$



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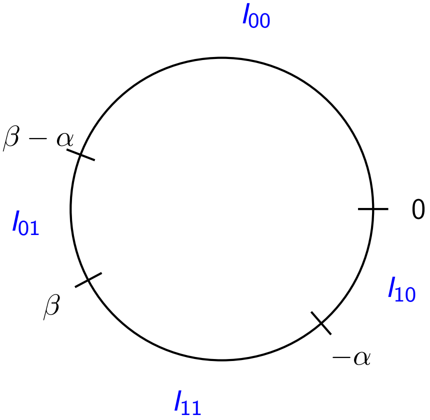

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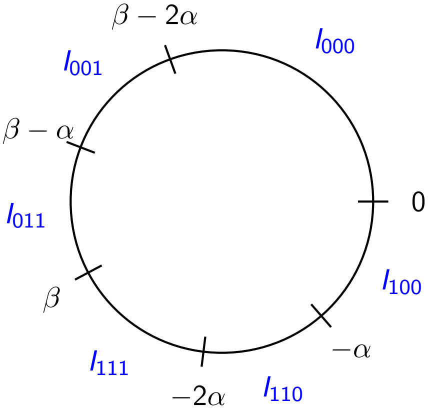

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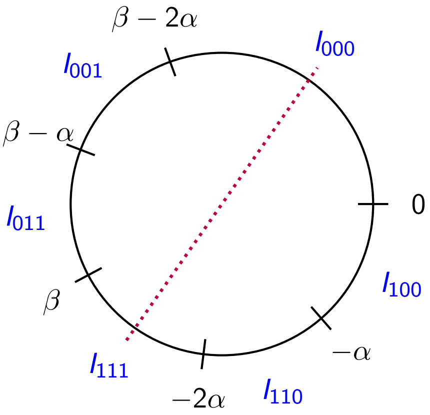

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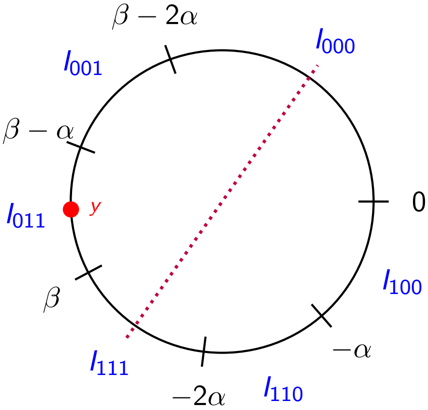

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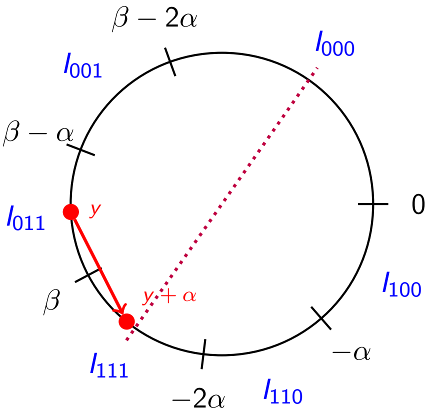

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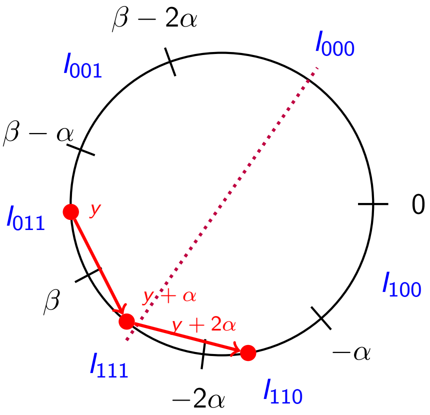

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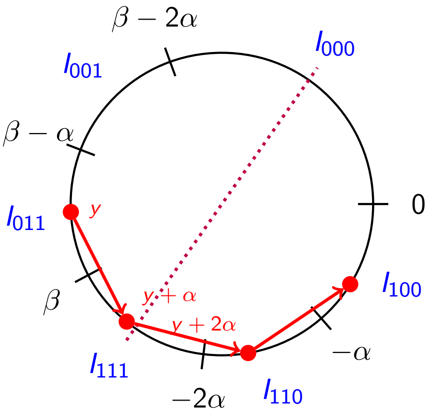

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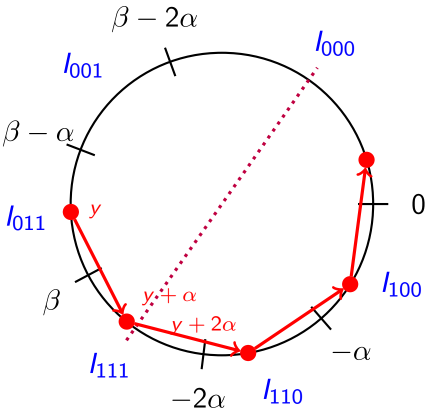

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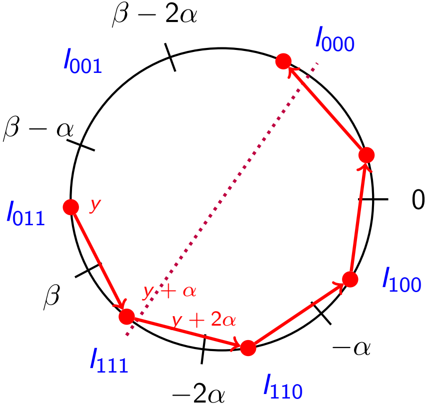

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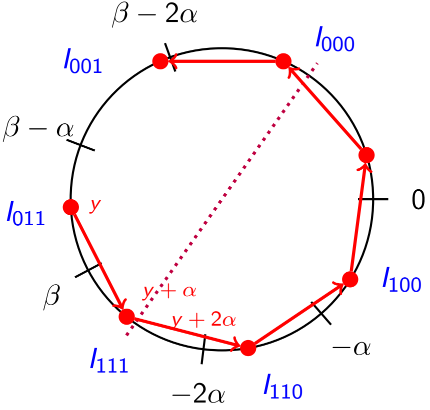

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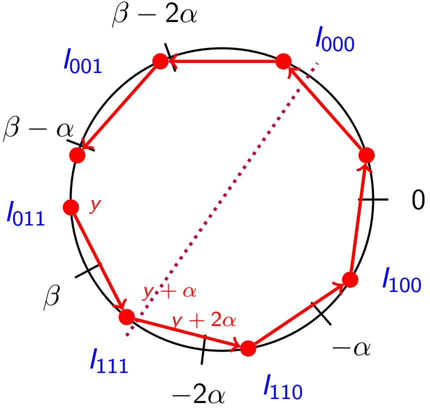

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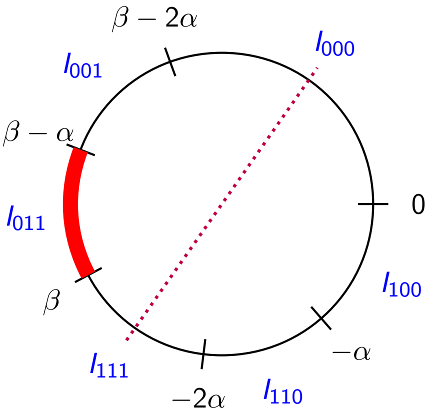

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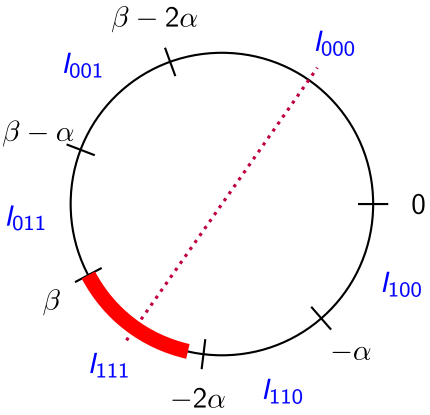

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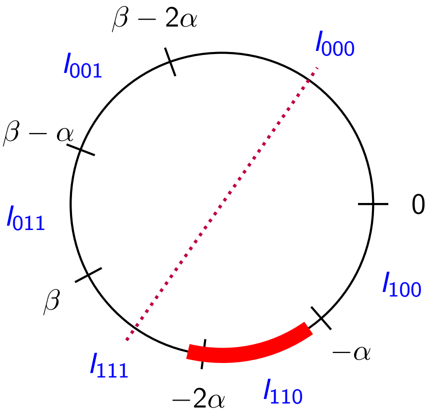

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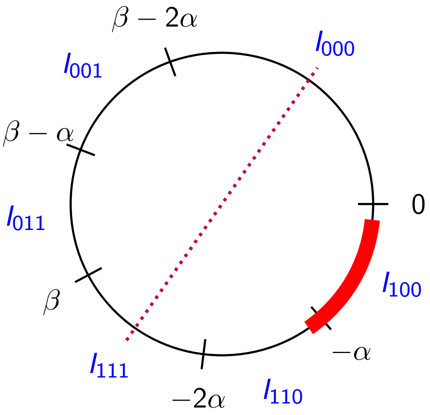

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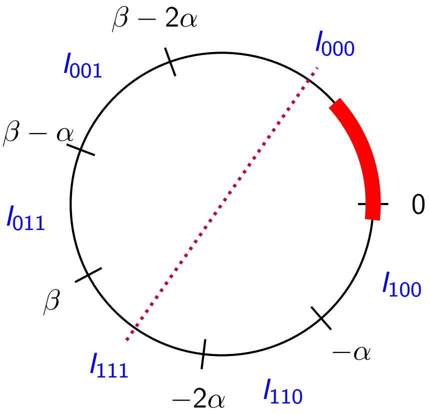

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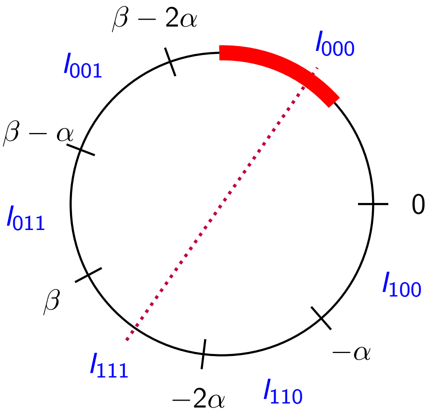

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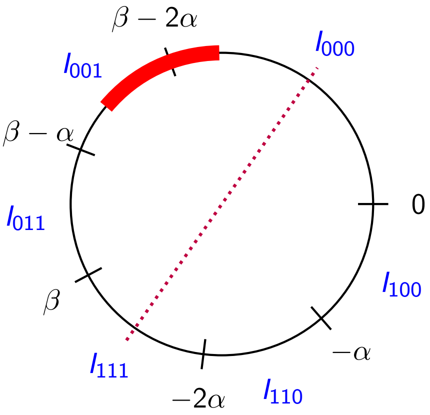

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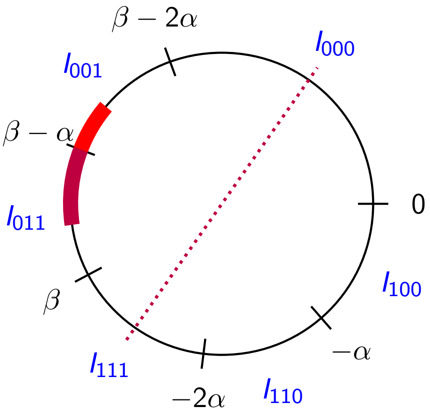

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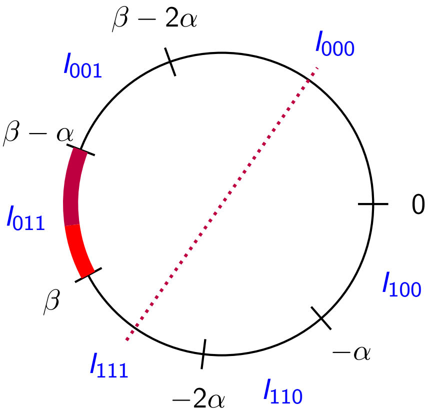

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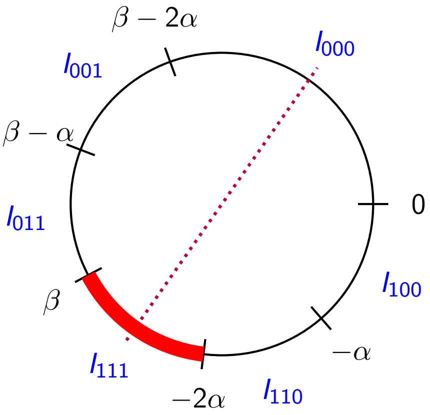

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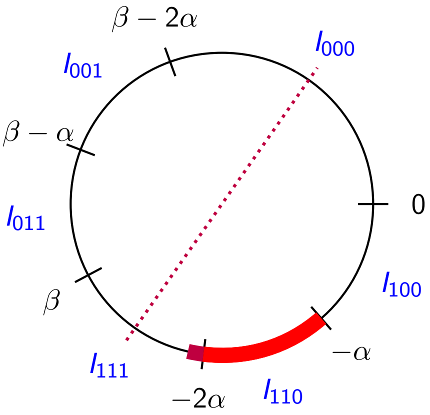
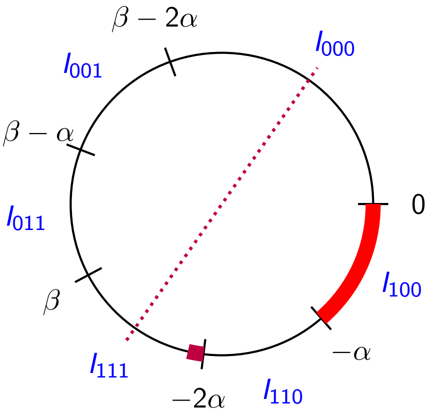


Idea of the proof

Let $x = 0.102$, $\alpha = 0.135$ and $\beta = 0.578$. Then

$$\mathbf{C} = 000011100001111000011100001110000011100 0\cdots$$



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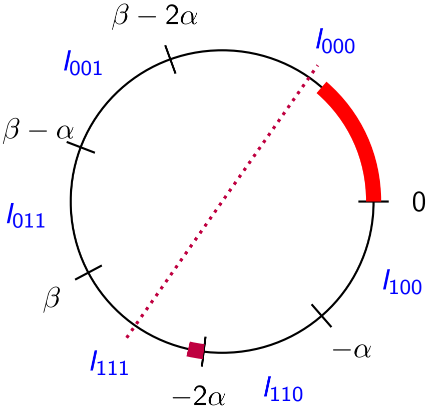

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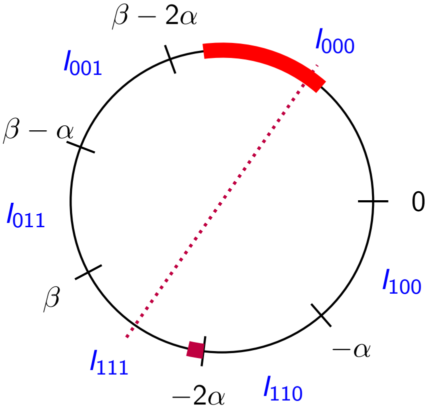

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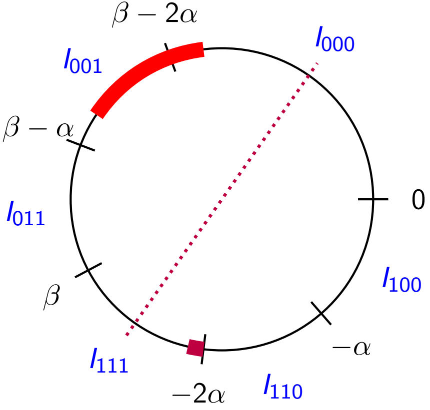

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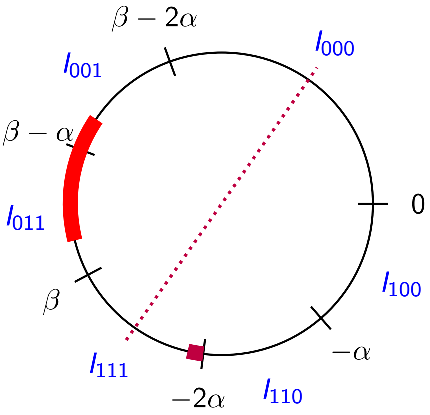

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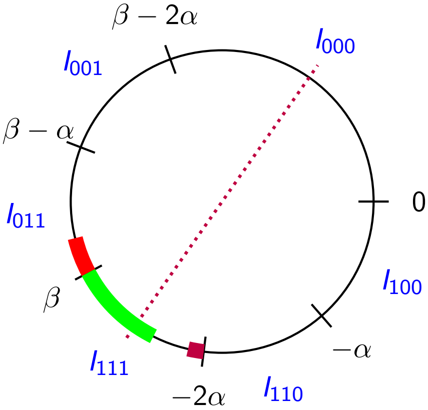

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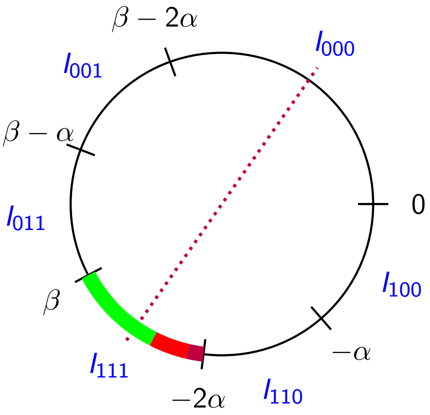

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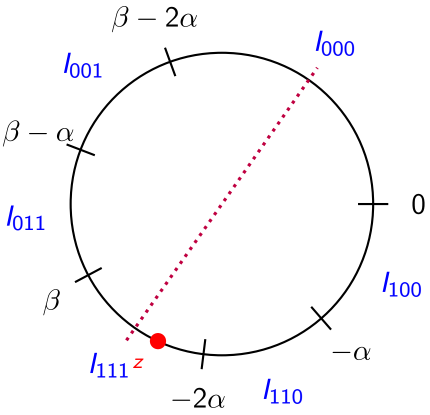

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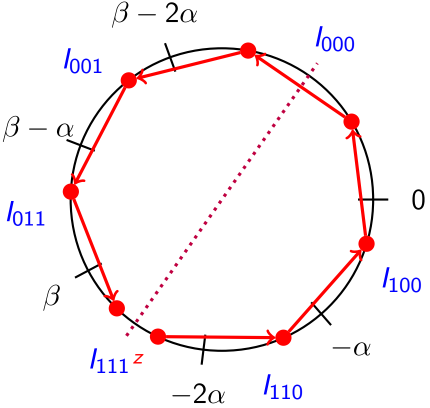

Useful software

This research was driven by computer exploration using the open-source mathematical software `Sage` [1] and its algebraic combinatorics features developed by the `Sage-Combinat` community [2], and in particular, F. Saliola, A. Bergeron and S. Labbé.
The pictures have been produced using Sage and `pgf/tikz`.

- W. A. Stein et al., *Sage Mathematics Software (Version 4.1.1)*, The Sage Development Team, 2009, `http://www.sagemath.org`.
- The Sage-Combinat community, Sage-Combinat: enhancing Sage as a toolbox for computer exploration in algebraic combinatorics, `http://combinat.sagemath.org`, 2009.