Combinatorial properties of f-palindromes in the Thue-Morse sequence*

A. Blondin-Massé, S. Brlek†, A. Garon, S. Labbé

Laboratoire de Combinatoire et d'Informatique Mathématique,
Université du Québec à Montréal,
C. P. 8888 Succursale "Centre-Ville", Montréal (QC), CANADA H3C 3P8
alexandre.blondin.masse@gmail.com, Brlek.Srecko@uqam.ca,
garon.ariane@hotmail.com, slabqc@gmail.com

Abstract

The palindromic defect of a finite word w is defined as $D(w) = |w| + 1 - |\mathrm{Pal}(w)|$ where $\mathrm{Pal}(w)$ is the set of its palindromic factors. In this paper we study the problem of computing the palindromic defect of finite and infinite words. Moreover, we completely describe the palindromic complexity and the lacunas (the positions where the longest palindromic suffix is not uni-occurrent) in the Thue-Morse word, showing that there exist infinite words with infinitely many palindromes but infinite defect. Finally, we extend the results to f-palindromes, i.e. words w satisfying $w = f(\widetilde{w})$ for some involution f on the alphabet.

Keywords Palindromic complexity, generalized palindromes, defect, Thue-Morse, lacunary words.

1 Introduction

Among the many ways of measuring the information content of a finite word, counting the number of distinct factors of given length occurring in it has been widely used and known as its complexity. A refinement of this notion amounts to restrict the factors to palindromes. The motivations for the study of palindromic complexity comes from many areas ranging from the study of Schrödinger operators in physics [1, 5, 16] to number theory [3] and combinatorics on words where it appears as a powerful tool for understanding the local structure of words. It has been recently studied in various classes of infinite words, an account of which may be found in the survey provided by Allouche et al. [2].

In particular, the palindromic factors give an insight on the intrinsic structure of many classes of words, due to its connection with the usual complexity.

*with the support of NSERC (Canada)

†Corresponding author.

For instance, they completely characterize Sturmian words [20], and for the class of smooth words they provide a connection with the notion of recurrence [9, 10]. Droubay, Justin and Pirillo [13] noted that the palindrome complexity $|\mathrm{Pal}(w)|$ of a word w is bounded by $|w| + 1$, and that finite Sturmian (and even episturmian) words realize the upper bound. Moreover they show that the palindrome complexity is computed by an algorithm listing the longest palindromic suffixes that are uni-occurrent. Words that realize that bound are called full in Brlek et al. [8]. In that paper it is also shown that there exist periodic full words, and an optimal algorithm is provided to check if an infinite periodic word is full or not. Moreover, a characterization by means of a rational language is given for the language L_P of words whose palindromic factors belong to a fixed and finite set P of palindromes. A finite automaton recognizing L_P is therefore easy to obtain, and consequently, if there exists a recurrent infinite word having P for palindromic factors, then there exists a periodic one sharing exactly the same palindromic factors. Furthermore, by using a representation of periodic words by circular words, a geometric characterization of those having an infinite set of palindromic factors, as well as those having a finite one, is furnished. More precisely, an infinite periodic word $\mathbf{w} = w^\omega$, where w is primitive, has an infinite set of palindromes if and only if w is the product of two palindromes. Consequently the periodic words having a finite set of palindromes are the words whose smallest periodic pattern is *asymmetric*. An enumeration formula for asymmetric words was also provided. Still in [8], the authors considered the defect $D(w) = |w| + 1 - |\mathrm{Pal}(w)|$ of finite words, and showed in particular that for any $k \in \mathbb{N}$ there exist periodic words having finite defect k.

In this paper, we study in more details the palindromic defect of infinite words, and in particular the case of the Thue-Morse word $\mathbf{t}$ defined as the fixed point starting with a of the morphism μ defined by

$$\mu(a) = ab \quad ; \quad \mu(b) = ba$$

Note that $\mathbf{t}$ is also a fixed point of the morphism $\theta = \mu^2$, that is

$$\mathbf{t} = \mu^\omega(a) = abbabaabbaababba \cdots .$$

While the complexity function $F_{\mathbf{t}}(n) = |\mathrm{Fact}_n(\mathbf{t})|$ has been established independently by Brlek [7] and de Luca and Varricchio [21], the recent interest in palindromic complexity begs for a description of the palindromic structure of $\mathbf{t}$, suggesting further study on fixed points of morphisms.

For that purpose, in Section 2 we recall from Lothaire's book [18] the basic terminology on words. In Section 3 we borrow from Brlek et al. [8] the necessary definitions and results about the palindromic defect, full words and constructions on periodic infinite words. In section 4, we study the f-palindromes, and derive a general bound for its complexity similar to the bound for palindromic complexity. Finally, in Section 5, we study the f-palindromic lacunas of words that are not full, and show that the Thue-Morse word $\mathbf{t}$ is an infinite word having an infinite f-palindromic complexity but infinite f-defect. Moreover we give explicit formulas for its f-lacunas.

2 Definitions and notation

Terminology for words is taken from M. Lothaire [18]. In what follows, Σ is a finite *alphabet* whose elements are called *letters*. By *word* we mean a finite sequence of letters $w : [0..n-1] \longrightarrow \Sigma$, where $n \in \mathbb{N}$. The length of w is $|w| = n$ and $w[i]$ or w_i denote its i-th letter. By convention, the *empty* word is denoted ε and its length is 0. The free monoid generated by Σ is defined by $\Sigma^* = \bigcup_{n\geq 0} \Sigma^n$. The set of right infinite words is denoted by Σ^ω and we set $\Sigma^\infty = \Sigma^* \cup \Sigma^\omega$. Given a word $w \in \Sigma^\infty$, a *factor* u of w is a word $u \in \Sigma^*$ satisfying

$$\exists x \in \Sigma^*, \exists y \in \Sigma^\infty, w = xuy.$$

If $x = \varepsilon$ (resp. $y = \varepsilon$) then u is called *prefix* (resp. *suffix*). The set of all factors of w is denoted by $\text{Fact}(w)$, those of length n is $\text{Fact}_n(w) = \text{Fact}(w) \cap \Sigma^n$, and $\text{Pref}(w)$ is the set of all prefixes of w. The number of occurrences of a factor $u \in \Sigma^*$ is $|w|_u$. A *period* of a word w is an integer $p < |w|$ such that $w[i] = w[i+p]$, for all $i < |w| - p$. An infinite word w is *recurrent* if it satisfies the condition $u \in \text{Fact}(w) \implies |w|_u = \infty$. If $w = pu$, with $|w| = n$ and $|p| = k$, then $p^{-1}w = w[k..n-1] = u$ is the word obtained by erasing from w its prefix p. A word is said to be *primitive* if it is not a power of another word. Moreover, the *reversal* of $u = u_0u_1 \cdots u_{n-1} \in \Sigma^n$ is the word $\widetilde{u} = u_{n-1}u_{n-2} \cdots u_0$. A *palindrome* is a word p such that $p = \widetilde{p}$, and for a language $L \subseteq \Sigma^\infty$, the set of its palindromic factors is denoted by $\text{Pal}(L)$. Every word contains palindromes, the letters and ε being necessarily part of them. This justifies the introduction of the function $\text{LPS} : \Sigma^* \longrightarrow \Sigma^*$ which associates to any word w its longest palindromic suffix $\text{LPS}(w)$.

A *morphism* is a function $\varphi : \Sigma^* \longrightarrow \Sigma^*$ compatible with concatenation, that is, such that $\varphi(uv) = \varphi(u)\varphi(v)$ for all $u, v \in \Sigma^*$. The identity morphism on Σ is denoted Id_Σ or Id when the context is clear. A morphism φ is primitive if $\forall \alpha \in \Sigma, \varphi^k(\alpha)$ contains each letter of Σ for some k. For $\alpha \in \Sigma$ we call φ*-block* (block for short if no confusion arises) a word of the form $\varphi(\alpha)$. A morphism is called *uniform* when the blocks have equal lengths. Clearly, every infinite fixed point of a primitive morphism is recurrent, and there exist recurrent but nonperiodic words, the Thue-Morse word $\mathbf{t}$ [23], and the Sturmian words being some of these. The *mirror-image* of a morphism φ, denoted by $\widetilde{\varphi}$, is the morphism such that $\widetilde{\varphi}(\alpha) = \widetilde{\varphi(\alpha)}$ for all $\alpha \in \Sigma$. It is easy to check that

$$\widetilde{\varphi(w)} = \widetilde{\varphi}(\widetilde{w}) \quad \text{for all} \quad w \in \Sigma^* \quad \text{and} \quad \widetilde{\varphi \circ \mu} = \widetilde{\varphi} \circ \widetilde{\mu}. \tag{1}$$

3 Defect

Recall that Droubay, Justin and Pirillo showed in [13] that for any finite word w of length n, its palindromic complexity $|\text{Pal}(w)|$ is bounded by $n+1$, and that Sturmian and also episturmian words realize the bound. Brlek *et al.* defined in [8], the *palindromic defect* of w to be

$$D(w) = |w| + 1 - |\text{Pal}(w)|. \tag{2}$$

If $D(w) = 0$, that is when w contains a maximal number of palindromic factors, then w is said *full*. The next statements follow easily from the definition.

Lemma 1. *Let $u, v \in \Sigma^*$ be such that $u \in \mathrm{Fact}(v)$, and let $\alpha \in \Sigma$. Then we have*

(i) $D(u) = D(\widetilde{u})$;

(ii) $D(u) \leq D(u\alpha)$, *and* $D(u) \leq D(\alpha u)$;

(iii) $D(u) \leq D(v)$;

(iv) *if v is full then u is full.*

Proof. (i) By definition $\mathrm{Pal}(u) = \mathrm{Pal}(\widetilde{u})$. (ii) We have $\mathrm{Pal}(u) \subseteq \mathrm{Pal}(u\alpha)$, and therefore $|\mathrm{Pal}(u\alpha)| - |\mathrm{Pal}(u)| \leq 1 = |u\alpha| - |u|$. It follows that $|u| - |\mathrm{Pal}(u)| \leq |u\alpha| - |\mathrm{Pal}(u\alpha)|$, so that $D(u) \leq D(u\alpha)$. For the second part, we have $D(u) = D(\widetilde{u}) \leq D(\widetilde{u}\alpha) = D(\widetilde{\alpha u}) = D(\alpha u)$. (iii) By induction and (ii). (iv) Follows directly from (iii). ∎

Observe that the defect is easily computed with the help of the LPS function and the characterization of Droubay et al. [13]. It goes like this:

Algorithm 1.
Input : $w \in \Sigma^*$; **Initialization :** $D := 0$;
if $w \neq \varepsilon$ **then**
for $i := 0$ **to** $|w| - 1$ **do**
  $s := \mathrm{LPS}(w[0..i])$;
  if s is uni-occurrent in $w[0..i]$ **then** $H[i] := |s|$
  else $H[i] := 0$; $D := D + 1$;
  end if
end for
end if
Return D, H

This algorithm also computes the function $H : \Sigma^+ \longrightarrow \mathbb{N}$ defined by

$$H(w)[i] = \begin{cases} |\mathrm{LPS}(w[0..i])| & \text{if } \mathrm{LPS}(w[0..i]) \text{ is uni-occurent,} \\ 0 & \text{otherwise,} \end{cases} \tag{3}$$

for all i such that $0 \leq i \leq |w| - 1$.

Example. For instance, for $w = bbaabbabaaba$, we have the following table

i	0	1	2	3	4	5	6	7	8	9	10	11
w	b	b	a	a	b	b	a	b	a	a	b	a
H	1	2	1	2	4	6	4	3	3	0	0	6
D	0	0	0	0	0	0	0	0	0	1	2	2

where the defect is given by the number of 0's in the table listing the values of H. In this case we have $D(w) = 2$. ♢

The sequence of values of H may or not contain null values. When H vanishes in some position k, the word w is said *lacunary* and the position k is called a *lacuna*. Moreover, we say that w is *end-lacunary* if $k+1 = |w|$. The word in the example above is lacunary and contains two lacunas, namely at positions 9 and 10. Observe that the lacunas of w can also be computed from right to left, the choice here is justified by the computation of lacunas for right infinite words. Moreover, D and H may be viewed as functions $D, H : \mathbb{N} \times \Sigma^* \longrightarrow \mathbb{N}$, and D is increasing with respect to $\mathbb{N}$ by virtue of Lemma 1 (ii). It is clear that if k is a lacuna of some word u, that is $H(k, u) = 0$, then for all $v \in \Sigma^*$, we have $H(k, uv) = 0$. This means that lacunas are preserved by suffix concatenation. This is no longer true for prefix concatenation as the next example shows.

Example. Let $w = u \cdot v = aabca \cdot acbcaacbaa$. Then we have the following table

i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
w	a	a	b	c	a	a	c	b	c	a	a	c	b	a	a
H	1	2	1	1	0	0	4	6	3	5	7	9	11	13	15

where the lacunas of v are $9, 10$, while the lacunas of uv are $4, 5$. $\diamondsuit$

One might ask whether a relation exists or not between $D(uv)$ and $D(u)+D(v)$. This is not the case since

$$2 = D(aabca \cdot acbcaacbaa) \leq D(aabca) + D(acbcaacbaa) = 1 + 2.$$

However, $D(aabca \cdot acbcaacb) = 2$ while $D(aabca) + D(acbcaacb) = 1 + 0 = 1$.

The notion of palindromic defect is extended to infinite words $\mathbf{w} \in \Sigma^\omega$ by setting $D(\mathbf{w})$ to be the supremum of the defect of its factors: it may be finite or infinite. Hence $D(\mathbf{w})$ is also equal to the supremum of the defect of the finite prefixes of $\mathbf{w}$.

3.1 Periodic words

We know from [8] that there exist periodic infinite words with a finite number of palindromes, and consequently there exist infinite words with infinite defect (equivalently finite words with arbitrarily large defects).

A periodic word may not have both an infinite defect and an infinite palindromic complexity. This is a direct consequence of the following results [8].

Proposition 1 (Theorem 4 [8]). *Let w be a primitive word. Then we have the following equivalent conditions:*

(i) *w is the product of two palindromes;*

(ii) $|\mathrm{Pal}(w^\omega)|$ *is infinite.* ■

A word satisfying the equivalent conditions of the theorem is called *symmetric*, and *asymmetric* otherwise. An immediate consequence is that if w is symmetric, then the defect of a periodic word w^ω is necessarily finite. The following theorem provides an optimal bound.

Theorem 1 (Theorem 6 [8])**.** *Let $w = uv$, with $|u| \geq |v|$ and $u, v \in \mathrm{Pal}(\Sigma^*)$, be a primitive symmetric word. Then the defect of $\mathbf{w} = w^\omega$ is bounded by the defect of its prefix of length $|uv| + \lfloor \frac{|u|-|v|}{3} \rfloor$.* ∎

We exhibit now a few examples showing that the bound is not far from being sharp. For instance, let $w = u \cdot v = aabaa \cdot bab$. Then, $|uv| + \lfloor \frac{|u|-|v|}{3} \rfloor = |uv|$. It is easy now to check that $D(aabaa \cdot bab) = 0$. More generally, for any integer $k > 1$,

$$D((aab^k aa \cdot bab)^\omega) = D((aab^k aa \cdot bab) \cdot aab^k) = 0,$$

showing that there exist an infinite family of full infinite periodic words, a result in the spirit of that of Droubay, Justin and Pirillo on Sturmian words [13]. If $w = u \cdot v = a^2 bacaba^2 \cdot c$, we have $\lfloor \frac{|u|-|v|}{3} \rfloor = 2$. But

$$D(w^\omega) = D((aabacabaa \cdot c) \cdot a).$$

Consider now the word $w = u \cdot v = a^{k+1} ba^k ca^k ba^{k+1} \cdot c$. Then $\lfloor \frac{|u|-|v|}{3} \rfloor = \lfloor \frac{4k+4}{3} \rfloor$ and we have $D(w) = 1$ and $D(w^\omega) = D(w \cdot a^{k+1} ba^k) = D(w \cdot a^k) = k + 1$.

On the other hand, for $w = u \cdot \epsilon = a^2 babba^2 bbaba^2$, we have $D(w) = 1$ and $D(w^\omega) = D(w \cdot a^2 ba) = 1$. More generally, let $u = a^{k+1} babba^{k+1} bbaba^{k+1}$ where $k \geq 0$. Then w is palindromic) and $D(w) = D(w^\omega) = k$. This example solves the problem of constructing both a finite word and an infinite periodic word having a fixed finite defect value $k \geq 0$.

Moreover, by taking a convenient periodic sequence having defect value 0, preceded by w one obtains a nonperiodic sequence (ultimately periodic indeed) having defect k, so that we have $D(ww \cdot a^\omega) = k$. By choosing a convenient Sturmian infinite word $\mathbf{v}$ (which is necessarily full), one easily shows that $D(ww.\mathbf{v}) = k$ as well. Summarizing the preceding discussion we have

Proposition 2. *Let Σ be an alphabet such that $|\Sigma| \geq 2$ and let $k \geq 0$. Then*

(i) *the language $\{w \in \Sigma^* : D(w) = k\}$ is infinite;*

(ii) *the language $\{\mathbf{w} \in \Sigma^\omega : D(\mathbf{w}) = k\}$ is infinite.*

4 f-palindromes

In this section, we study a natural generalization of palindromes. Let $f : \Sigma \to \Sigma$ be an involution which extends to a morphism on Σ^*. A word $w \in \Sigma^*$ is an *f-pseudo-palindrome* [4, 11, 15], or simply an *f-palindrome* [17], if $w = f(\widetilde{w})$. Clearly the empty word is an f-palindrome, and the set of all f-palindromes of a language $L \subseteq \Sigma^\infty$ is noted f-Pal(L). Moreover, for any $w \in \Sigma^*$, we define f-Pal$(w) = f$-Pal$(\Sigma^*) \cap \mathrm{Fact}(w)$. The longest f-palindromic suffix of a word w is denoted $\mathrm{LPPS}(f, x)$.

Examples. The notion of f-palindrome is very rich as shown below.

1. A palindrome on an arbitrary alphabet Σ is an Id_Σ-palindrome.

2. Let f be an involution without fixed points. Then each f-palindrome is of even length: indeed, if an odd-length word is an f-palindrome, then its central letter is mapped on itself, contradicting the assumption.

3. On a 2-letter alphabet, say $\Sigma = \{a, b\}$, the only nontrivial involution is the exchange of letters defined by $E : a \mapsto b, b \mapsto a$. In this particular case, that is when $|\Sigma| = 2$ and $f \neq \mathrm{Id}_\Sigma$, an f-palindrome of Σ^* is called an *antipalindrome* [17]. The words

$$\varepsilon, ab, ba, abab, aabb, baba, bbaa, abbaab, bababa$$

are E-palindromes. Note that the length of an antipalindrome is always even.

4. Let $w = a^k$ for some $k \geq 0$. Then $E\text{-Pal}(w) = \{\varepsilon\}$, that is $|E\text{-Pal}(w)| = 1$, while $\mathrm{Pal}(w) = k + 1$.

5. Let $\Sigma = \{a, b, c\}$ and f be defined by $(a \mapsto a, b \mapsto c, c \mapsto b)$. Let $w = a^k.u.a^k$ where u is an f-palindrome such that $\mathrm{Alph}(u) = \{b, c\}$. Then again $\mathrm{LPPS}(a^k.u_0) = \varepsilon$. $\diamondsuit$

It is easy to adapt Algorithm 1 for taking into account f-palindromicity: indeed, it suffices to replace line 3 by

$\rightarrow 3 :$ $\quad s := \mathrm{LPPS}(f, w[0..i])\,;$

Therefore, f-palindromic and palindromic complexities share the same bounds.

Proposition 3. *Let $f : \Sigma \longrightarrow \Sigma$ be an involution. Then, for all $w \in \Sigma^*$, we have $|f\text{-Pal}(w)| \leq |w| + 1$.*

Proof. Let p be a nonempty prefix of w. It suffices to show that there exists at most one f-palindromic suffix of p unioccurrent in p. By contradiction, assume that there exist two f-palindromic suffixes of p, say u and v, that are unioccurrent in p, with $|u| < |v|$. Then $v = xu$ for some nonempty word x. Therefore, $v = f(\widetilde{v}) = f(\widetilde{xu}) = f(\widetilde{u}\widetilde{x}) = f(\widetilde{u})f(\widetilde{x}) = uf(\widetilde{x})$, so that u is not unioccurrent in p, contradiction. Thus $|f\text{-Pal}(w)| \leq |w| + 1$. ∎

In the proof above, the fact that f is an involution is not used. Therefore the result remains true for an arbitrary bijection (permutation) on Σ. As a special case, we have a sharper bound if f has no fixed point.

Corollary 1. *Let w be a nonempty word on Σ and f an involution without fixed point. Then $|f\text{-Pal}(w)| \leq |w|$.*

Proof. Since f has no fixed point, the first letter w_0 of w is not an f-palindrome, so that the longest f-palindromic suffix of w_0 is not uni-occurrent in w. Hence $|f\text{-Pal}(w)| \leq |w|$. ∎

In view of Example 5 above, we also have the following property.

Corollary 2. *Let w be a nonempty word on Σ and $f \neq \mathrm{Id}$. If $\mathrm{Alph}(w) = \Sigma$ then $|f\text{-Pal}(w)| \leq |w|$.*

Proof. Since $f \neq \mathrm{Id}$, there exists at least one letter, say α, which is not a fixed point of f. Since $\alpha \in \mathrm{Alph}(w)$, let k be the first occurrence of α. Then $\mathrm{LPPS}(w[0..k]) = \varepsilon$. ∎

Let Σ_{fix} be the subset of letters that are fixed by f, i.e. the restriction of f on Σ_{fix} is the identity:

$$f|_{\Sigma_{\mathrm{fix}}} = \mathrm{Id}_{\Sigma_{\mathrm{fix}}}.$$

Then, as a consequence, a necessary condition for a word w to be f-full is that $\mathrm{Alph}(w) \subseteq \Sigma_{\mathrm{fix}}$, which amounts to check its palindromic fullness.

Given a fixed involution f, we say that w has maximal f-palindromic complexity if there is no word $w' \in \Sigma^*$ of the same length such that $|f\text{-Pal}(w')| > |f\text{-Pal}(w)|$. Indeed, this implies that if f is without fixed point then the bound is precisely $|w|$, and $|w| + 1$ otherwise.

The problem of characterizing Id-full words seems hard [6, 13, 17]. On the other hand, it is possible to describe exactly the words having maximal E-palindromicity.

Proposition 4. *Let w be a nonempty word on $\{a, b\}$. Then $|E\text{-Pal}(w)| = |w|$ if and only if $w = (\alpha\beta)^n$ or $(\alpha\beta)^n\alpha$ for some distinct letters $\alpha, \beta \in \{a, b\}$ and $n \geq 1$.*

Proof. Since E admits no fixed point, Theorem 1 applies, thus $|E\text{-Pal}(w)| \leq |w|$.

($\Rightarrow$) It is sufficient to show that for all $\alpha \in \{a, b\}, \alpha\alpha$ does not occur in w. We proceed by contradiction and assume that $\alpha\alpha \in \mathrm{Fact}(w)$. We already know from the proof of Proposition 3 that there is at most one new antipalindrome at each position and that there is no new antipalindrome at position 1. Let p be a prefix of w such that either $\alpha\alpha$ or $\beta\beta$ is a suffix of p and $|p|_{\alpha\alpha} + |p|_{\beta\beta} = 1$, i.e. p ends with the first occurrence of $\alpha\alpha$ or $\beta\beta$. Then $\mathrm{LPPS}(E, p) = \varepsilon$ and consequently $|E\text{-Pal}(p) = |p| - 1$ so that $|E\text{-Pal}(w) < |w|$.

($\Leftarrow$) On the one hand, suppose that $w = (\alpha\beta)^n$. Then

$$E\text{-Pal}(w) = \{\varepsilon\} \cup \{(\alpha\beta)^m \mid 1 \leq m \leq n\} \cup \{(\beta\alpha)^m \mid 1 \leq m \leq n-1\}.$$

On the other hand, suppose that $w = (\alpha\beta)^n\alpha$. Then

$$E\text{-Pal}(w) = \{\varepsilon\} \cup \{(\alpha\beta)^m \mid 1 \leq m \leq n\} \cup \{(\beta\alpha)^m \mid 1 \leq m \leq n\}.$$

In both cases, $|E\text{-Pal}(w)| = |w|$. ∎

Clearly the set $M = \{\mathrm{Id}, E\}$ of idempotent morphisms on $\{a, b\}$ forms a commutative group for the composition of morphisms, and one defines the *transposition* on M by $\overline{E} = \mathrm{Id}$ and $\overline{\mathrm{Id}} = E$.

Proposition 5. *For all $f \in M$ we have $f \circ E = E \circ f = \overline{f}$.* ∎

5 The Thue-Morse word

Among all binary words, the Thue-Morse word is one of those having very remarkable palindromic properties. Moreover, there exists an interesting link between its palindromes and its antipalindromes.

Indeed, one shows easily that **t** contains infinitely many palindromes and antipalindromes, since for all odd n, $\mu^n(a)$ is an antipalindromic prefix of **t** while for all even n, $\mu^n(a)$ is a palindromic prefix of **t**. A very convenient way of studying the structure of those special factors is to represent them as trees (see Figures 1 and 2). In particular, we notice that they both present a symmetry axis since the languages Pal(**t**) and E-Pal(**t**) are closed under the exchange of letters involution E.

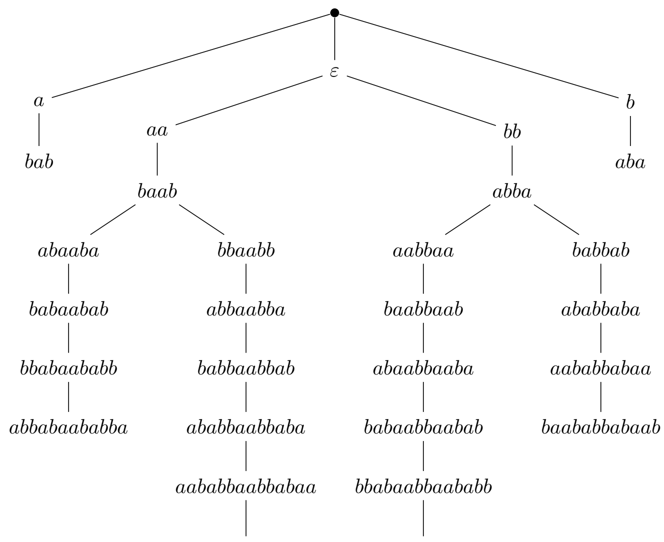

Figure 1: Palindromic factors of **t**

However **t** is not full as can be seen in the table below, where there are lacunas at positions 8 and 9.

i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
w	a	b	b	a	b	a	a	b	b	a	a	b	a	b	b	a
H	1	1	2	4	3	3	2	4	0	0	6	8	10	12	14	16

The main goal of this section is to give a complete characterization of the f-lacunas in **t**, and for that purpose the following definition, equivalent to that of Mignosi [22](see also Frid [14]), is useful.

Definition 1. *We say that* v *is an* ancestor of w with respect to μ *if there exist a proper block-prefix* x *and a proper block-suffix* y *such that* $xwy = \mu(v)$*. We denote by* Anc(w) *the set of ancestors of* w*. Moreover, we say that* w *is* centered (with respect to v) *if* $|x| = |y|$.

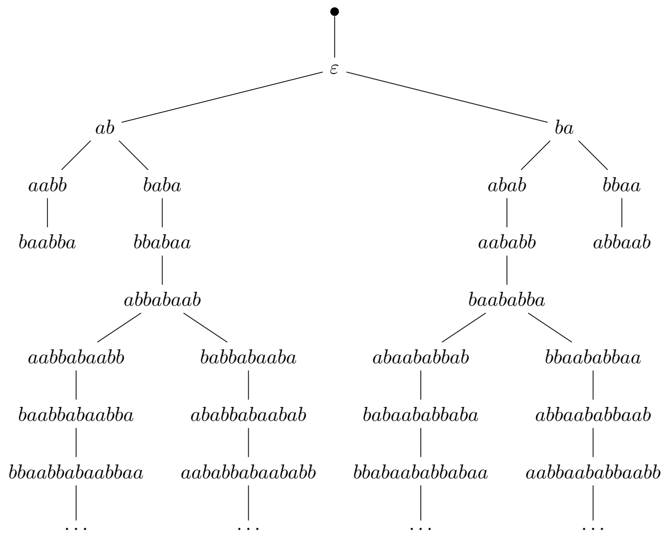

Figure 2: Antipalindromic factors of $\mathbf{t}$

Let v be an ancestor of w such that $xwy = \mu(v)$ as above. Let $p, s \in \Sigma^*$ be such that $w = ps$. Then we say that there is a *bar line* between p and s, written $p|_\mu s$ (or simply $p|s$ when the context is clear), if there exist $v_1, v_2 \in \Sigma^*$ such that $v = v_1 v_2$, $xp = \mu(v_1)$ and $sy = \mu(v_2)$. We start by listing some useful properties of $\mathbf{t}$.

Proposition 6. *Let $f \in \{\mathrm{Id}, E\}$ and u and v be any words such that $\mu(v) = u$. Then u is an f-palindrome if and only if v is an $\overline{f}$-palindrome.*

Proof. We have that u is an f-palindrome if and only if $u = f(\widetilde{u})$ if and only if $\mu(v) = f(\widetilde{\mu(v)}) = f(E(\mu(\widetilde{v}))) = \mu(\overline{f}(\widetilde{v}))$ if and only if $v = \overline{f}(\widetilde{v})$ if and only if v is a $\overline{f}$-palindrome. ■

Note that the words u and v are not necessarily factors of $\mathbf{t}$.

Lemma 2. *Let $u \in f\text{-}\mathrm{Pal}(\mathbf{t})$. If $|u| \geq 4$, then $|u|$ is even.*

Proof. It is clear for $f = E$ so that we only have to consider the case $f = \mathrm{Id}$. Assume by contradiction that such a palindromic factor exists. Then $u = zv\tilde{z}$ for some word z and some palindrome v of length 5. But the only possibilities for v over $\Sigma = \{a, b\}$ are *aaaaa*, *aabaa*, *ababa*, *abbba*, *baaab*, *babab*, *bbabb*, *bbbbb*, which are not factors of $\mathbf{t}$. ■

Lemma 3. *Let $u \in f\text{-}\mathrm{Pal}(\mathbf{t})$. Then the following properties hold.*

(i) *If $|u|$ is even, then all ancestors v of u with respect to μ are $\overline{f}$-palindromes such that u is centered.*

(ii) *If $|u| \geq 4$, then u has a unique ancestor.*

Proof. (i) Let v be an ancestor of u. Then there exist words x and y such that $|x|, |y| \leq 1$ and $\mu(v) = xuy$. First, since $|\mu(v)|$ and $|u|$ are both even, it follows that $|x| + |y|$ is even, so that $|x| = |y|$. Hence, u is centered.

We now show that $\mu(v)$ is an f-palindrome. If $|x| = |y| = 0$, then $\mu(v) = u$ and the claim is true. Otherwise, assume that $|x| = |y| = 1$ and let α be the first letter of u. Then $x\alpha$ and $f(\alpha)y$ are both μ-blocks. But μ-blocks are either ab or ba so that $x = E(\alpha)$ and $y = E(f(\alpha)) = f(E(\alpha)) = f(x)$. Hence, $\mu(v) = xuf(x)$, i.e. $\mu(v)$ is an f-palindrome. In both cases, we conclude from Proposition 6 that v is a $\overline{f}$-palindrome.

(ii) We know from Lemma 2 that $|u|$ is even. Now, if u contains a square, i.e. $u = u'\alpha\alpha u''$ for some words u', u'' and some letter α, then we must have $u'\alpha|\alpha u''$ and all the other bar-lines are determined, so that v is unique. On the other hand, if there is no square in u and since $\mathbf{t}$ is overlap-free, we have $u = \alpha\beta\beta$ whose unique ancestor is $\alpha\alpha$, for some distinct letters α and β. ■

Remark 1. *It follows from Lemma 3 that the only non empty f-palindromes in $\mathbf{t}$ whose ancestors need not be unique have length 1, 2 or 3. The only possible cases are a, b, aa, bb, ab, ba, aba and bab and their ancestors are* $\mathrm{Anc}(a) = \mathrm{Anc}(b) = \{a, b\}$, $\mathrm{Anc}(aa) = \{ba\}$, $\mathrm{Anc}(bb) = \{ab\}$, $\mathrm{Anc}(ab) = \{a, bb\}$, $\mathrm{Anc}(ba) = \{b, aa\}$ *and* $\mathrm{Anc}(aba) = \mathrm{Anc}(bab) = \{aa, bb\}$.

It is now easy to compute the f-palindromic complexity $f\text{-}P_{\mathbf{t}}(n) = |f\text{-Pal}(\mathbf{t}) \cap \mathrm{Fact}_n(\mathbf{t})|$ of the Thue-Morse sequence for $f \in \{\mathrm{Id}, E\}$.

Proposition 7. *The palindromic and E-palindromic complexities of $\mathbf{t}$ satisfy the following recurrences :*

(i) $P_{\mathbf{t}}(0) = 1$, $P_{\mathbf{t}}(1) = P_{\mathbf{t}}(2) = P_{\mathbf{t}}(3) = P_{\mathbf{t}}(4) = 2$,

(ii) $E\text{-}P_{\mathbf{t}}(0) = 1$, $E\text{-}P_{\mathbf{t}}(1) = E\text{-}P_{\mathbf{t}}(3) = 0$, $E\text{-}P_{\mathbf{t}}(2) = 2$, $E\text{-}P_{\mathbf{t}}(4) = E\text{-}P_{\mathbf{t}}(6) = 4$, $E\text{-}P_{\mathbf{t}}(8) = 2$

(iii) *for all* $n \geq 2$, $f\text{-}P_{\mathbf{t}}(2n+1) = 0$,

(iv) *for all* $n \geq 3$, $\overline{f}\text{-}P_{\mathbf{t}}(2n) = f\text{-}P_{\mathbf{t}}(n) + f\text{-}P_{\mathbf{t}}(n+1)$, *and*

(v) *for all* $n \geq 3$, $f\text{-}P_{\mathbf{t}}(4n) = f\text{-}P_{\mathbf{t}}(4n-2) = \overline{f}\text{-}P_{\mathbf{t}}(2n)$.

Proof. First, by inspection, (i) and (ii) are satisfied. Part (iii) follows from Lemma 2.

To prove (iv), let $p \in \overline{f}\text{-Pal}_{\mathbf{t}}(2n)$. Then by Lemma 3, there exist a unique x and a unique f-palindrome u such that $f(\widetilde{x})px = \mu(u)$, where p is centered, $|x| \in \{0, 1\}$ and $|u| \in \{n, n+1\}$. Let $\mathrm{Anc} : \overline{f}\text{-Pal}_{\mathbf{t}}(2n) \to f\text{-Pal}_{\mathbf{t}}(n) \cup f\text{-Pal}_{\mathbf{t}}(n+1)$ be the function defined by $p \mapsto u$. We first show that Anc is injective. Indeed, let $p, q \in \overline{f}\text{-Pal}_{\mathbf{t}}(2n)$ and assume that $\mathrm{Anc}(p) = \mathrm{Anc}(q)$. Then p and q are both centered in $\mu(\mathrm{Anc}(p)) = \mu(\mathrm{Anc}(q))$ so that $p = q$. To show that Anc is surjective, let $u \in f\text{-Pal}_{\mathbf{t}}(n) \cup f\text{-Pal}_{\mathbf{t}}(n+1)$. If $|u| = n$, then $u = \mathrm{Anc}(\mu(u))$. On the other hand, if $|u| = n+1$, then $u = \mathrm{Anc}(\overline{f}(\widetilde{x^{-1}})\mu(u)x^{-1})$. Hence Anc is a bijection and the result follows.

(v) Let $p \in f\text{-Pal}_{\mathbf{t}}(4n-2)$ be an f-palindrome. Then there exist a unique u and a unique x, $|x| \leq 1$, such that $\mu(u) = xpf(\widetilde{x})$ and $|\mu(u)| = 4n$. Let $g : \text{Pal}_{\mathbf{t}}(4n-2) \to \text{Pal}_{\mathbf{t}}(4n)$, be the function defined by $p \mapsto xpf(\widetilde{x})$. Clearly, g is a bijection.

Finally, by (iv), we have $f\text{-}P_{\mathbf{t}}(4n) = \overline{f}\text{-}P_{\mathbf{t}}(2n) + \overline{f}\text{-}P_{\mathbf{t}}(2n+1)$, and by (iii), we conclude that $f\text{-}P_{\mathbf{t}}(4n) = \overline{f}\text{-}P_{\mathbf{t}}(2n)$. ■

Closed formulas for $P_{\mathbf{t}}(n)$ and $E\text{-}P_{\mathbf{t}}(n)$ are easily obtained by induction :

$$P_{\mathbf{t}}(n) = \begin{cases} 1 & \text{if } n = 0, \\ 2 & \text{if } 1 \leq n \leq 4, \\ 0 & \text{if } n \text{ is odd and } n \geq 5, \\ 4 & \text{if } n \text{ is even and } 4^k + 2 \leq n \leq 3 \cdot 4^k, \text{ for } k \geq 1, \\ 2 & \text{if } n \text{ is even and } 3 \cdot 4^k + 2 \leq n \leq 4^{k+1}, \text{ for } k \geq 1. \end{cases} \tag{4}$$

$$E\text{-}P_{\mathbf{t}}(n) = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{if } n \text{ is odd}, \\ 2 & \text{if } n = 2, \\ 4 & \text{if } n \text{ is even and } 2 \cdot 4^k + 2 \leq n \leq 6 \cdot 4^k, \text{ for } k \geq 0, \\ 2 & \text{if } n \text{ is even and } 6 \cdot 4^k + 2 \leq n \leq 2 \cdot 4^{k+1}, \text{ for } k \geq 0. \end{cases} \tag{5}$$

Lemma 4. *Let $w \neq \varepsilon$ be a prefix of $\mathbf{t}$ and u be an f-palindromic suffix of w. Let $v \neq \varepsilon$ and x be words such that $\mu(u) = \overline{f}(\widetilde{x})vx$ and $|x| \leq 1$. Then $u = \text{LPPS}(f, w)$ if and only if $v = \text{LPPS}(\overline{f}, \mu(w)x^{-1})$.*

Proof. The overall situation is depicted in Figure 3.

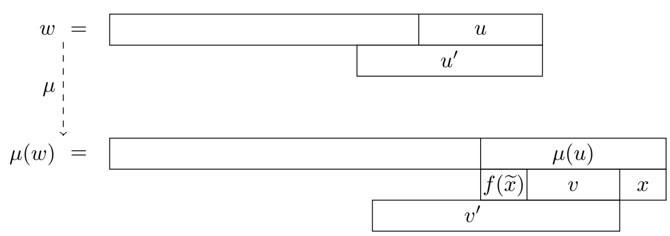

Figure 3: Schematic representation of the proof of Lemma 4

($\Rightarrow$) By contradiction, assume that v' is a $\overline{f}$-palindromic suffix of $\mu(w)x^{-1}$ and $|v'| > |v|$. Since $|\overline{f}(\widetilde{x})vx|$ is even, it follows that $|v|$ is even as well so that $|v| \geq 2$ and $|v'| \geq 3$. Moreover, the case $|v'| = 3$ is impossible: otherwise, we would have $v' = \alpha^3$ for some letter α. Therefore, $|v'| \geq 4$ and, by Lemma 3, v' is also centered and has an f-palindromic ancestor u' which is a suffix of w. This means that $\mu(u') = \overline{f}(\widetilde{x})v'x$ is a $\overline{f}$-palindromic suffix of $\mu(w)$. But $|v'| > |v|$ so that $|\mu(u')| = |\overline{f}(\widetilde{x})v'x| > |\overline{f}(\widetilde{x})vx| = |\mu(u)|$. Hence $|u'| > |u|$, contradicting the assumption that $u = \text{LPPS}(f, w)$.

($\Leftarrow$) Again by contradiction, assume that u' is an f-palindromic suffix of w, $|u'| > |u|$. Then $\mu(u')$ is a $\overline{f}$-palindrome suffix of $\mu(w)$. Moreover, there exists a $\overline{f}$-palindrome v' such that $\mu(u') = \overline{f}(\widetilde{x})v'x$. But $|u'| > |u|$, so that $|\overline{f}(\widetilde{x})v'x| = |\mu(u')| > |\mu(u)| = |\overline{f}(\widetilde{x})vx|$. Hence $|v'| > |v|$. This contradicts $v = \mathrm{LPPS}(\overline{f}, \mu(w)x^{-1})$. ■

Lemma 5. *Let w be a prefix of $\mathbf{t}$. Then we have*

(i) $|\mathrm{LPS}(w)| = 1$ *if and only if* $|w| = 1$ *or* $|w| = 2$,

(ii) $|\mathrm{LPS}(w)| = 3$ *if and only if* $|w| = 5$ *or* $|w| = 6$.

Proof. (i) ($\Leftarrow$) Clearly, $|\mathrm{LPS}(a)| = |\mathrm{LPS}(ab)| = 1$. ($\Rightarrow$) We show that if $|w| \geq 3$, then $|\mathrm{LPS}(w)| \geq 2$. For $|w| = 3$, it is true since $|\mathrm{LPS}(abb)| = 2$. Now, assume that $|w| \geq 4$ and let α be the last letter of w. Since $\mathbf{t}$ is overlap-free, one of the words in $\{\alpha\alpha, \alpha\beta\alpha, \alpha\beta\beta\alpha\}$ is a suffix of w, where $\beta \neq \alpha$ is a letter. Hence $|\mathrm{LPS}(w)| \geq 2$.

(ii) ($\Leftarrow$) It is easy to see that $|\mathrm{LPS}(abbab)| = |\mathrm{LPS}(abbaba)| = 3$. ($\Rightarrow$) Let y be the word such that $wx = \mu^2(y)$, for some word x, $|x| \leq 3$. By inspection, if $|y| \leq 4$, the only possibilities satisfying $|\mathrm{LPS}(w)| = 3$ are $|w| = 5$ or $|w| = 6$. Now, assume that $|y| \geq 5$. Then one palindrome p among $\mu^2(\alpha\alpha)$, $\mu^2(\alpha\beta\alpha)$ and $\mu^2(\alpha\beta\beta\alpha)$ is a suffix of wx, for some distinct letters α and β. If $p \in \{\mu^2(\alpha\beta\alpha), \mu^2(\alpha\beta\beta\alpha)\}$, then $\widetilde{x^{-1}}px^{-1}$ is a palindromic suffix of w, but $|\widetilde{x^{-1}}px^{-1}| \geq 6$. Now, if $p = \mu^2(\alpha\alpha)$, then the suffix of length 3 of w is in $\{\beta\alpha\alpha, \alpha\alpha\beta, \alpha\beta\beta, \beta\beta\alpha\}$, i.e. $|\mathrm{LPS}(w)| \neq 3$. ■

Lemma 6. *Let w be a prefix of $\mathbf{t}$ such that $|w| \geq 8$ and let x be a suffix of $\mu(w)$ such that $|x| \leq 1$. Then w is f-end-lacunary if and only if $\mu(w)x^{-1}$ is $\overline{f}$-end-lacunary.*

Proof. ($\Rightarrow$) Let $u = \mathrm{LPPS}(f, w)$. Since w is end-lacunary, u is not uni-occurent in w so that $w = yuzu$ for some word y and some non empty word z. But $\mu(u)$ is a $\overline{f}$-palindrome, which means that there exists a $\overline{f}$-palindrome v such that $\mu(u) = \overline{f}(\widetilde{x})vx$. Then we have the situation depicted in Figure 4. Moreover, by Lemma 4, $v = \mathrm{LPS}(\overline{f}, \mu(w)x^{-1})$. Hence, $\mu(w)x^{-1}$ is $\overline{f}$-end-lacunary, since v is not unioccurent in $\mu(w)x^{-1} = \mu(y)\overline{f}(\widetilde{x})vx\mu(z)\overline{f}(\widetilde{x})v$.

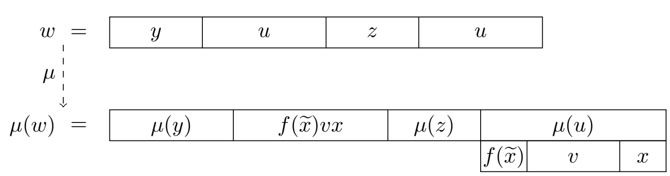

Figure 4: Schematic representation of the proof of Lemma 6

($\Leftarrow$) Let $v = \mathrm{LPPS}(\overline{f}, \mu(w)x^{-1})$. By Lemma 5, $|v|$ is even since $|w| \geq 8$. Moreover, Lemma 3 implies that v is centered with respect to some $\overline{f}$-palindromic ancestor u, i.e. $\overline{f}(\widetilde{x})vx = \mu(u)$. Therefore, since $\mu(w)x^{-1}$ is $\overline{f}$-end-lacunary, there exist some word y and some nonempty word z such that $\mu(w) = \mu(y)\overline{f}(\widetilde{x})vx\mu(z)\overline{f}(\widetilde{x})vx$ and $w = yuzu$. Hence u is not uni-occurrent in w. But, from Lemma 4, $u = \mathrm{LPPS}(f, w)$ showing that w is f-end-lacunary. ■

Remark 2. *Lemma 6 can be restated as follows. Let $i \geq 8$ be an integer. Then i is a f-lacuna of $\mathbf{t}$ if and only if $2i$ and $2i+1$ are $\overline{f}$-lacunas of $\mathbf{t}$. In particular, if $i, j \in \mathbb{N}$ and $8 \leq i \leq j$, then all integers in $[i..j]$ are f-lacunas if and only if all the integers in $[2i..2j+1]$ are $\overline{f}$-lacunas. Applying Lemma 6 twice, we get that all integers in $[i..j]$ are f-lacunas if and only if all the integers in $[4i..4j+3]$ are f-lacunas.*

An explicit description of the f-lacunas is given now. For $n \in \mathbb{N}^+$, let $L(n)$ (resp. E-$L(n)$) be the index where the n-th interval of lacunas (resp. E-lacunas) start and $\ell(n)$ (resp. E-$\ell(n)$) be its length.

Theorem 2. *The sequences L, E-L, ℓ and E-ℓ satisfy the following equations :*

(i) E-$L(1) = 0$, E-$L(2) = 2$, E-$L(3) = 4$ *and* E-$L(4) = 12$,

(ii) E-$\ell(n) = 1$ *for* $n = 1, 2, 3, 4$,

(iii) $L(n) = 2E$-$L(n+2)$, *for* $n \geq 1$,

(iv) $\ell(n) = 2E - \ell(n+2)$, *for* $n \geq 1$,

(v) E-$L(n) = 2L(n-4)$, *for* $n \geq 5$, *and*

(vi) E-$\ell(n) = 2\ell(n-4)$, *for* $n \geq 5$.

Proof. Let $f \in \{\mathrm{Id}, E\}$. An easy proof by induction shows that the sequence f-L is increasing and that the intervals described by f-L and f-ℓ are pairwise nonoverlapping.

On the other hand, if we consider the prefix of length 16 of $\mathbf{t}$, there are no Id-lacunas but only E-lacunas, which are 0, 2, 4 and 12. Therefore E-$L(1) = 0$, E-$L(2) = 2$, E-$L(3) = 4$, E-$L(4) = 12$ and E-$\ell(1) = E$-$\ell(2) = E$-$\ell(3) = E$-$\ell(4) = 1$. Now, for any prefix of $\mathbf{t}$ of length at least 16, every lacuna must come from a shorter prefix by Lemma 6. Therefore, by Remark 2, equations (iii), (iv), (v) and (vi) hold. ■

Closed formulas for L, E-L, ℓ and E-ℓ are easily obtained :

$$E\text{-}L(n) = \begin{cases} 2^{n+1}, & \text{if } n \text{ is odd,} \\ 2^{n+1} + 2^n, & \text{if } n \text{ is even.} \end{cases}$$

$$L(n) = \begin{cases} 2^{n+2}, & \text{if } n \text{ is odd,} \\ 2^{n+2} + 2^{n+1}, & \text{if } n \text{ is even.} \end{cases}$$

and

$$\ell(n) = \begin{cases} 2^n, & \text{if } n \text{ is odd,} \\ 2^{n-1}, & \text{if } n \text{ is even.} \end{cases}$$

$$E - \ell(n) = \begin{cases} 2^{n-1}, & \text{if } n \text{ is odd,} \\ 2^{n-2}, & \text{if } n \text{ is even.} \end{cases}$$

Moreover, the first intervals where E-lacunas occur are

$$[0], [2], [4], [12], [16..19], [48..51], [64..79], [192..207], \ldots$$

and those where Id-lacunas occur are

$$[8..9], [24..25], [32..39], [96..103], [128..159], [384..415], \ldots$$

The closed formulas above show that the lacunas do not intersect, leading to the following statement.

Proposition 8. *The* Id*-lacunas intervals and the E-lacunas intervals are pairwise disjoint.* ■

6 Concluding remarks

Now, recall from [14] that a morphism φ is *marked* if for every $\alpha, \beta \in \Sigma$ such that $\alpha \neq \beta$, the first letters and the last letters of $\varphi(\alpha)$ and $\varphi(\beta)$ are different. From this definition, we see that some results of Section 5 can easily be extended to uniform marked morphisms φ such that $\varphi(\alpha)$ is a palindrome, for every $\alpha \in \Sigma$. Obviously, Lemma 2 would have to be stated rather differently. In fact, the uniqueness of the ancestor, which is a simple matter for uniform marked morphisms (see Remark 3 of [14]), is not absolutely necessary. Moreover, it is possible to provide an algorithm to decide whether a uniform marked morphism generates full words or not : it suffices to generalize Lemma 6 accordingly. Finally, it would be easy to deduce from these Lemmas that a fixed point of a uniform marked morphism is either full or has an infinite defect. More precisely we state the following conjecture.

Conjecture 1. *Let* $\mathbf{u}$ *be the fixed point* $\mathbf{u} = \varphi(\mathbf{u})$ *of a primitive morphism* φ*. If the defect is such that* $0 < D(\mathbf{u}) < \infty$*, then* $\mathbf{u}$ *is periodic.*

Considering the remarks above, the next step would be to extend the results of Section 5 to any fixed point of a marked morphism, in order to obtain information on the f-palindromic complexity and the lacunas of a more general class of words.

We conclude by suggesting some open problems :

1. It would be interesting to extend the optimal algorithm deduced from Theorem 6 of [8] (restated in Theorem 1) for computing the defect of an infinite periodic word to fixed points of primitive morphisms. For this purpose, recall that in [16], Hof et al. introduced morphisms of class P, i.e. morphisms such that there exist palindromes p and q_α satisfying $\varphi(\alpha) = pq_\alpha$ for every $\alpha \in \Sigma$. They also conjectured that if a fixed point $\mathbf{u}$ of a primitive morphism has infinitely many palindromes, then there exists a morphism φ such that either φ or $\widetilde{\varphi}$ is of class P and $\mathbf{u} = \varphi(\mathbf{u})$. Recently, a constructive proof has been provided for binary alphabets by Tan [24]. Let $\mathbf{u} = \varphi(\mathbf{u})$ be a fixed point of a morphism φ. Does there exist an algorithm for deciding whether $\mathbf{u}$ is full or

not ? At first, it would be interesting to provide one for morphisms in class P. Indeed, the algorithm could be extended to any morphism having a conjugate of class P, and assuming that the Hof-Knill-Simon conjecture is true (which is the case for the binary alphabet), we would have an algorithm for any primitive morphism.

2. Is there a better algorithm than Algorithm 1 to compute the defect of a finite word ? In other words, is Algorithm 1 optimal ?

3. Let $f(n, k, d)$ be the number of words of length n over a k-letter alphabet having d lacunas. Is it possible to compute $f(n, k, d)$ in an efficient way ?

Acknowledgements The authors are grateful to the anonymous referees for their accurate and careful reading of the paper, for suggesting improvements that enhanced the overall presentation of the paper.

References

[1] J.P. Allouche, *Schrödinger operators with Rudin-Shapiro potentials are not palindromic*, J. Math. Phys. **38** (1997) 1843–1848.

[2] J.P. Allouche, M. Baake, J. Cassaigne and D. Damanik, *Palindrome complexity*, Theoret. Comput. Sci. **292** (2003) 9–31.

[3] J.P. Allouche and J. Shallit, *Sums of digits, overlaps, and palindromes*, Disc. Math. and Theoret. Comput. Sci. **4** (2000) 1–10.

[4] V. Anne, L. Q. Zamboni and I. Zorca, Palindromes and Pseudo-Palindromes in Episturmian and Pseudo-Palindromic Infinite Words, in : S. Brlek, C. Reutenauer (Eds.), Words 2005, Publications du LaCIM, Vol. 36 (2005) 91–100.

[5] M. Baake, *A note on palindromicity*, Lett. Math. Phys. **49** (1999) 217–227.

[6] A. Blondin Massé, Sur le défaut palindromique des mots infinis, Mémoire de maîtrise en Mathématiques, Montréal, UQAM, 2008, 74 p.

[7] S. Brlek, *Enumeration of factors in the Thue-Morse word*, Disc. Appl. Math. **24** (1989) 83–96.

[8] S. Brlek, S. Hamel, M. Nivat and C. Reutenauer, *On the Palindromic Complexity of Infinite Words*, in J. Berstel, J. Karhumäki, D. Perrin, Eds, Combinatorics on Words with Applications, Int. J. of Found. Comput. Sci. **15**: 2 (2004) 293–306

[9] S. Brlek and A. Ladouceur, *A note on differentiable palindromes*, Theoret. Comput. Sci. **302**:1-3 (2003) 167–178.

[10] S. Brlek, D. Jamet and G. Paquin, *Smooth Words on 2-letter alphabets having same parity*, Theoret. Comput. Sci. **393**:1-3 (2008) 166–181.

[11] A. de Luca and A. De Luca, Pseudopalindrome closure operators in free monoids, Theoret. Comput. Sci. 362 (2006) 282–300.

[12] X. Droubay and G. Pirillo, *Palindromes and Sturmian words*, Theoret. Comput. Sci. **223** (1999) 73–85.

[13] X. Droubay, J. Justin and G. Pirillo, *Episturmian words and some constructions of de Luca and Rauzy*, Theoret. Comput. Sci. **255** (2001) 539–553.

[14] A. Frid, *Applying a uniform marked morphism to a word*, Disc. Math. and Theoret. Comput. Sci. **3** (1999) 125–140.

[15] V. Halava, T. Harju, T. Kärki and L. Q. Zamboni, Relational Fine and Wilf words, in : P. Arnoux, N. Bédaride, J. Cassaigne (Eds.), Proceedings of WORDS 2007, 159–167.

[16] A. Hof, O. Knill and B. Simon, *Singular continuous spectrum for palindromic Schrödinger operators*, Commun. Math. Phys. **174** (1995) 149–159.

[17] S. Labbé, Propriétés combinatoires des f-palindromes, Mémoire de maîtrise en Mathématiques, Montréal, UQAM, 2008, 115 p.

[18] M. Lothaire, Combinatorics on words, Addison-Wesley, 1983.

[19] M. Lothaire, Algebraic Combinatorics on words, Cambridge University Press, 2002.

[20] A. de Luca, *Sturmian words: structure, combinatorics, and their arithmetics*, Theoret. Comput. Sci. **183** (1997) 45–82.

[21] A. de Luca and S. Varricchio, *Some combinatorial properties of the Thue-Morse sequence and a problem in semigroups*, Theoret. Comput. Sci. **63**:3 (1989) 333–348.

[22] F. Mignosi and P. Séébold, *If a D0L language is k-power free then it is circular*, Proc. ICALP'93, LNCS **700** (1993) 507–518.

[23] M. Morse and G. Hedlund, *Symbolic Dynamics*, Amer. J. Math. **60** (1938) 815–866.

[24] B. Tan, *Mirror substitutions and palindromic sequences*, Theoret. Comput. Sci. **389**:1-2 (2007), 118–124.