Université de Bordeaux
Laboratoire Bordelais de Recherche en Informatique

Aperiodic order: from combinatorics to geometry via symbolic dynamics, number theory and algorithms

Mémoire d'habilitation à diriger des recherches de

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soutenu publiquement le 4 juin 2025 devant le jury constitué de

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(version: April 9, 2025, two-sided)

À Renée, Louis et Oscar,

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Chapter 1

Introduction

The first conclusion (A1) of the 2023 report of the Intergovernmental Panel on Climate Change (IPCC, ou GIEC en français) says that [100]:

> "*Human activities, principally through emissions of greenhouse gases, have unequivocally caused global warming, with global surface temperature reaching 1.1° C above 1850-1900 in 2011-2020.*"

While also being a source of terrible events[1], the usage of oil for combustion engines increases the quantity of CO_2 emission in the atmosphere (see Figure 1.1) with many long-term undesirable effects on climate change.

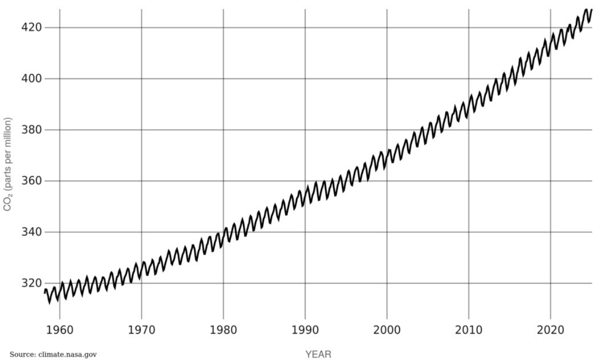


Figure 1.1: Atmospheric CO_2 levels measured by NOAA at Mauna Loa Observatory, Hawaii, since 1958 indicate that the concentratration went from below 320 parts per million to 427 as of February 2025. The annual rise and fall of CO_2 levels is caused by seasonal cycles in photosynthesis on a massive scale. Image credit: `https://climate.nasa.gov/vital-signs/carbon-dioxide/`

Among all the way to make carbon return to the soil is the *mycorrhizal fungi* and its symbiotic association with a green plant. The plant makes organic molecules from carbon dioxide by photosynthesis and supplies them to the fungus in the form of sugars or lipids, while the fungus supplies the plant with water and mineral nutrients, such as phosphorus, taken from the soil[2]. Thanks to this symbiotic association, 13 billion tons of atmospheric carbon dioxide, one-third of fossil-fuel emissions worldwide, enter the soil each year quietly helping regulate Earth's climate [3].

To exchange nutrients for carbon molecules, *mycorrhizal fungi* builds an underground fungal network following rules shaped by natural selection for over 450 million years. The growth of these networks, composed of one continuous cytoplasm, was studied recently [225]. The group of

[1] `https://en.wikipedia.org/wiki/Lac-Mégantic_rail_disaster`
[2] `https://en.wikipedia.org/wiki/Mycorrhiza`
[3] `https://www.nytimes.com/2025/03/01/science/climate-mycorrhizal-fungus-networks.html`

researchers found that mycorrhizal fungi builds a self-regulating network depending on the density of nutrient in the soil and which root of the plant offers the most carbon. The network grows as a living algorithm playing an economic trade game while obeying some basic rules. For instance, as the growing tips progress, the authors observed that new branches form behind them at a steady rate. Also, when one tip hits another, they fuse and form a loop thus avoiding dead ends. Inside the network, they observed that the flow of molecules in a tube of the network is often bidirectional simultaneously. But, the authors write that it is still unclear how flows are modulated across networks built by mycorrhizal fungi. Does the their global behavior emerges from local rules?

Figure 1.2: Two types of snow crystals among the many others listed in [197]. After living 20 years in Canada, the author could not believe the existence of *capped columns* shown on the right until he saw them with his own eyes for the first time in February 2019 at the Canada familly's sugar shack. Image credits:`https://www.snowcrystals.com/`.

Another domain where local interactions define global structures is the growth of snowflakes. Why every snowflakes have six branches? Why are two different branches of the same snowflake the same while any two snowflakes different? We now know the answers to these questions. In particular, the science of snow crystals evolved greatly in the recent decades due to the work of Kenneth Libbrecht [197] following the earlier work of Ukichiro Nakaya [218]. What we may observe with our eyes on a snowflake is a consequence of how water molecules combine at the molecular level; see Figure 1.2. Depending on the temperature and on the saturation level (humidity), water molecules tend to attach to the falling snowflake in a certain way (on top to form columns or on the side to form flat plates). These parameters evolve over time during the fall of a snowflake impacting its final shape. Since the branches of a single snowflake follow the same path in the space, they experience the same conditions at the same moment. This explains why two branches of the same snowflake grow the same independently.

Even more structured are crystals where molecules are organized periodically into a lattice. For example, sodium chloride and pyrite are known to organize into cubic crystal systems at the molecular level and this can be seen macroscopically; see Figure 1.3 (left). Positions of molecules in the space are represented mathematically by discrete point sets $\Lambda \subset \mathbb{R}^d$. To describe a point set, it is natural to define the *set of periods* of Λ as

$$\mathrm{per}(\Lambda) := \{t \in \mathbb{R}^d \mid \Lambda + t = \Lambda\}.$$

The set of periods always contain 0 and is a subgroup of $\mathbb{R}^d$. A point set is said *crystallographic* if its set of periods is a module of rank d in $\mathbb{R}^d$. Crystallographic point sets have a inherent restriction among their possible symmetries [51, Section 3.2]. If a crystallographic point set $\Lambda \subset \mathbb{R}^d$ is invariant under a n-fold rotational symmetry and $d = 2$ or $d = 3$, then it must be that $n \in \{1, 2, 3, 4, 6\}$. In particular $n \neq 5$. Thus, it was a surprise when 5-fold rotational symmetry were observed for an aluminium-manganese alloy, Al_6Mn, by Shechtman and his team in the early 1980's [255]. This observation implied that the positions of the molecules could not have any translational symmetries. These non-periodic but very ordered materials are now called *quasicrystals*. A quasicrystal is shown in Figure 1.3 (right). The pentagonal faces of a dodecahedron illustrate that the global structure is explained by a non-periodic arrangement of the molecules. The discovery of quasicrystals led Shechtman to be awarded the Nobel Prize in Chemistry in 2011. The research on quasicrystals and aperiodic order is quite rich and transdisciplinary involving different sciences including Chemistry, Physics, Theoretical Computer Science, Mathematics and its sub-disciplines (geometry, combinatorics, topology, dynamical systems, number theory) and, none the least, Arts [148, 252, 51, 52].

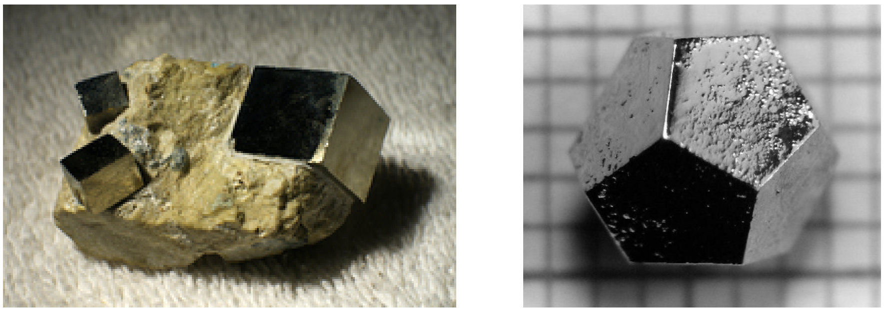


Figure 1.3: Left: A rock containing three crystals of pyrite (FeS_2). The crystal structure of pyrite is primitive cubic, and this is reflected in the cubic symmetry of its natural crystal facets. Right: A Ho-Mg-Zn dodecahedral quasicrystal formed as a pentagonal dodecahedron. Unlike the pyrite, the quasicrystal has faces that are regular pentagons. Image credits: `https://en.wikipedia.org/wiki/Cubic_crystal_system` and `https://en.wikipedia.org/wiki/Quasicrystal`.

On the geometrical side, quasicrystals and aperiodic order are often studied from the point of view of 2-dimensional tilings of the plane. One of the most well-known aperiodic tiling was discovered by Penrose. In its original version, four shapes derived from the regular pentagon can be used to tile the plane and none of the allowed tilings are periodic [227]. Another version uses thin and thick rhombi of angle $\frac{\pi}{5}$ and $\frac{2\pi}{5}$; see Figure 1.4. The aperiodic structure of Penrose tilings is explained by the properties of a specific irrational number: the positive root φ of the polynomial $x^2 - x - 1$, also known as the golden ratio or golden mean. For example, in the kite-and-dart version of the Penrose tilings, the ratio of kites to darts is equal to the golden ratio [228]. In a Physics laboratory, the structure of a crystal or quasicrystals is guessed from the image generated from a diffraction experiment. The same can be done with a tiling. Michael Baake made such an experiment with a Penrose tiling during a conference at CIRM in April 2024; see Figure 1.4. Such a five-fold symmetric patterns for the diffraction of a Penrose tiling was theoretically suggested by Mackey as early as 1982 [205]. This was confirming a relation between quasicrystals discovered by Shechtman and Penrose tilings.

Penrose tilings were soon given an equivalent description in terms of multigrids [98]. A Penrose tiling can be lifted to a discrete surface in $\mathbb{R}^5$. Every edge of a Penrose tiling is parallel to one of the fifth root of unity. We associate some vertex of a Penrose tiling to the origin $0 \in \mathbb{R}^2$. The origin is lifted to the origin $(0,0,0,0,0)$ of $\mathbb{R}^5$. We proceed iteratively as follows. If a vertex $v \pm \xi^k$ is neighbor of a vertex v already lifted to a coordinate $p \in \mathbb{R}^5$, then $v \pm \xi^k$ is lifted to the coordinate $p \pm e_k \in \mathbb{R}^5$, see Figure 1.5. This process is well-defined as the lifted coordinates does not depend on the path taken from the origin. The set of lifted coordinates have the particularity of being pretty close to a 2-dimensional vector space. More precisely, they are the points of a lattice inside the Minkowski sum of a 5-dimensional hypercube with a 2-dimensional vector space in $\mathbb{R}^5$. This construction is nowadays called a cut and project schemes [148, §10] and [51, §6.2]. We say that

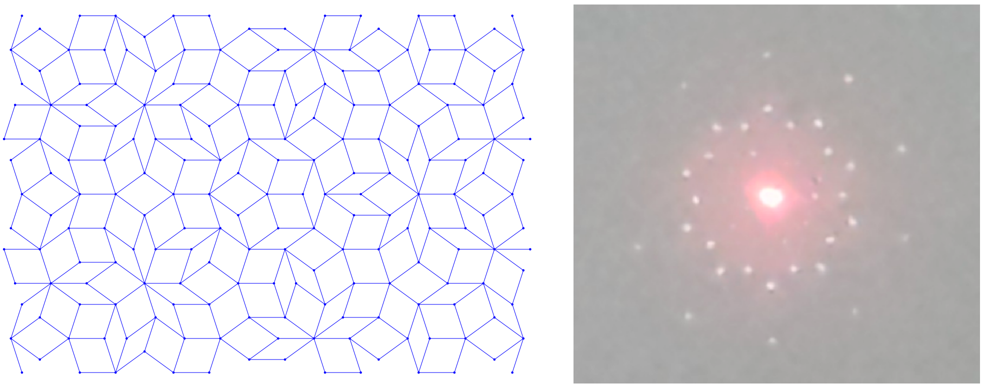


Figure 1.4: Left: a Penrose tiling. Right: the diffraction image of a laser beam going through a Penrose tiling. Photo taken by the author during the presentation of Michael Baake at the conference *Multidimensional symbolic dynamics and lattice models of quasicrystals* at CIRM, Marseille, April 4^{th}, 2024.

the cut and project scheme here is $5 \to 2$ (read 5-to-2) because the projection to the physical space goes from a 5-dimensional space to a 2-dimensional space. Notice that the fact that each edge in a Penrose tiling has its own angle is what makes this construction "easy". When tiles have all of their edges parallel (for example, with tilings by unit square Wang tiles), it becomes less evident how to lift a tiling to a higher-dimensional surface if at all possible.

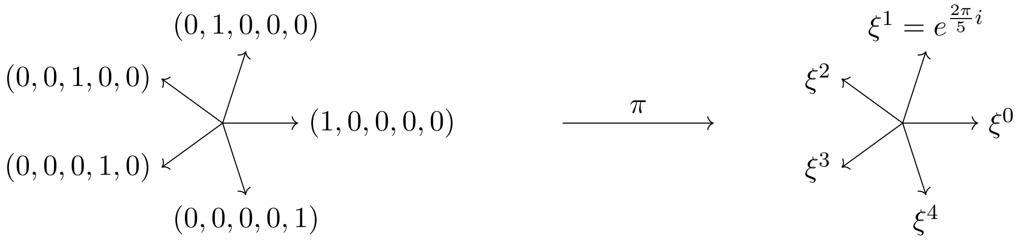


Figure 1.5: The vertices in a Penrose tiling can be lifted to a discrete surface in $\mathbb{R}^5$ using the preimage of the projection π.

Symbolic dynamics, Combinatorics and Algorithms

One way to confirm we understand the snow crystal growth is to perform numerical simulations based on physical models working on different length- and time- scales [196] reproducing quantitative experimental observations. However, the remarkable morphological diversity observed in snow crystal growth remains generally inexplicable [198]. Among the most realistic computational models of snow crystal growth, are cellular automata. A complete chapter of Libbrecht's book is devoted to cellular automata models for snow crystals growth including 2D and 3D models [197]. Cellular automata are maps defined on shift spaces $A^{\mathbb{Z}^d}$ where the state of the image of a configuration at a certain position $p \in \mathbb{Z}^d$ depends only on the state of the surrounding positions at a bounded distance from p [168]. When $d = 1$, the terminology of sliding block code is also used [200]. The long-term behavior of configurations within a cellular automaton are thus defined solely from local rules.

Cellular automata theory is an important object of study within the subject of symbolic dynamics, which is the thematic of this habilitation thesis. Symbolic dynamics is the study of dynamics on a discrete space where each position in the space can take finitely many possible values from a finite set of states. One of the founding results in symbolic dynamics is the theorem of Curtis-Lyndon-Hedlund which asserts that the morphisms between any two shift spaces (that is, continuous mappings that commute with the shift) are exactly those mappings which can be defined uniformly by a sliding block code.

Another important result of Symbolic Dynamics is a theorem of Morse-Hedlund about Sturmian sequences, namely that a sequence is balanced if and only if it is a symbolic coding of a rotation, also known as a mechanical word [216]. Morse-Hedlund theorem and its extension made 30 years later by Coven and Hedlund to also include sequences of complexity $n+1$ are presented in details in Chapter 3 within Part I.

The subject of this Habilitation thesis

This habilitation thesis aims at improving our understanding of how global ordered structures emerge from local rules. The rules are often given in combinatorial terms by a restriction on the possible patterns that may appear locally: either a finite set of forbidden patterns (subshifts of finite type), or a limited number of patterns of a given size (low pattern complexity). From these local rules, an interesting phenomenon happens when some global order is forced without being periodic. This state is called aperiodic order [51]. Aperiodic order is a very interesting subject of study as it gathers many sciences and communities: theoretical computer science, physics, chemistry, geometry, dynamics, number theory, topology, algorithms and arts.

Sturmian sequences and their many characterizations are an important stepping stone for this habilitation thesis. Many of the results gathered in this habilitation thesis are different kinds of extensions to higher dimensions of the characterization of Sturmian sequences. Sturmian sequences involve the simplest case of cut and project schemes, those that are 2-to-1. Therefore, it was natural to split this thesis into the kind of cut and project schemes that are involved:

- Part II: Contributions within 2-to-1 cut and project schemes
- Part III: Contributions within 3-to-1 cut and project schemes
- Part IV: Contributions within 4-to-2 cut and project schemes
- Part V: Contributions within $(d+1)$-to-d cut and project schemes

Part II contains the new characterization of Sturmian sequences by indistinguishable asymptotic pairs done in collaboration with Sebastián Barbieri during his postdoctoral studies in Bordeaux and Štěpán Starosta (Czech Technical University in Prague). This results fits within 2-to-1 cut and project schemes. It also includes the application of these results to the Markoff injectivity conjecture, also known as the uniqueness conjecture, an open question open since more than 100 years. This was done in collaboration with Mélodie Lapointe (U. Moncton, Canada) and Wolfgang Steiner (Paris).

Part III presents a set of ternary sequences of complexity $2n+1$. These sequences are generated by 2 substitutions proposed by Julien Cassaigne and are associated to a multidimensional continued fraction algorithm. We proved that these sequences are almost always balanced. Thus, it also fits within 3-to-1 cut and project schemes. This works constitute a very nice generalization of Sturmian sequences to sequences over a ternary alphabets as it extends both its combinatorial and dynamical properties. This works was done in collaboration with Julien Leroy (Belgium) and Julien Cassaigne (Marseille).

Part IV presents our contributions to the study of Jeandel-Rao tilings and how it all started; see Figure 1.6. The work was split into 4 articles, the last three of them published in 2021. Basically, the articles prove that the aperiodic tilings by the set of 11 Wang tiles discovered by Jeandel and Rao are generated by 4-to-2 cut and project schemes. Since the Wang tiles are unit square, the

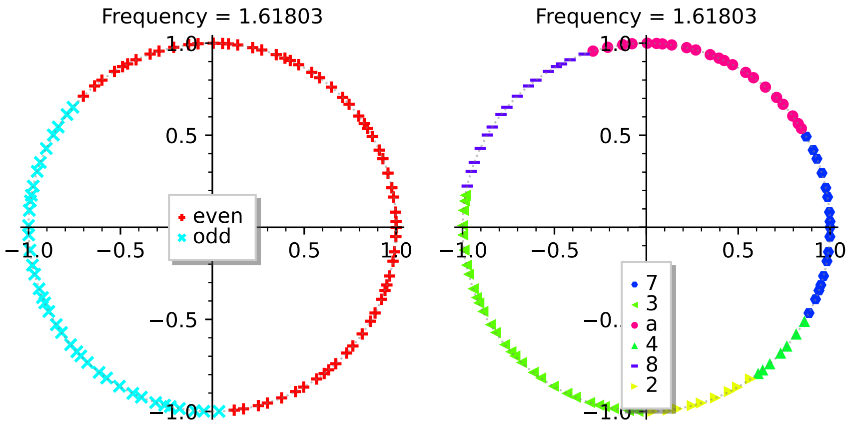


Figure 1.6: Given a finite word $w = w_1 w_2 \cdots w_n \in \Sigma^*$ and a frequency $\alpha \in \mathbb{R}$, we color the point $e^{2\pi i k\alpha}$ according to the value of w_k for every integer $1 \leq k \leq n$. When α is the golden mean and w is some word in the language of the Fibonacci word, we obtain the image on the left (see the details of the experiment in Section 3.10). When α is the golden mean and w is some horizontal row of tiles within some Jeandel-Rao tiling, we obtain the image on the right (see the details of the experiment in Section 11.1). This experiment performed in April 2017, thanks to a finite rectangular patch given to me by Michael Rao, was a motivation to start investigating the relations between Sturmian sequences and aperiodic Wang tilings. The results obtained on this subject over the past years are summarized in Part IV.

projections to the physical space are degenerate and non-injective. This is why Jeandel-Rao tilings are harder to lift to a surface in $\mathbb{R}^4$ compared to lifting Penrose tiling to a surface in $\mathbb{R}^5$ which can be done in a nonambiguous way; see Figure 1.7.

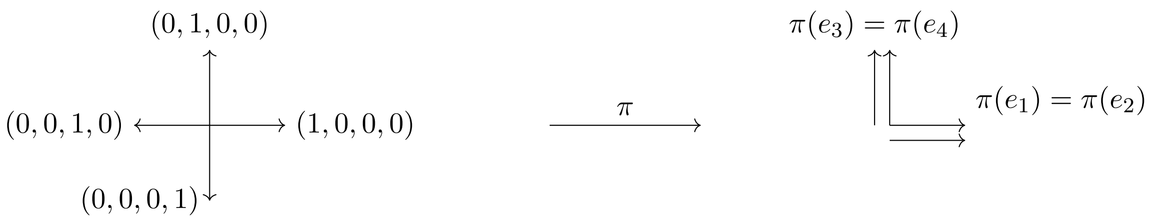


Figure 1.7: Lifting the vertices of Jeandel-Rao tiling to a discrete surface in $\mathbb{R}^4$ is more difficult than with a Penrose tiling because the projection $\pi : \mathbb{R}^4 \to \mathbb{R}^2$ to the physical space is degenerate.

Since 4-to-2 cut and project schemes are essentially a Cartesian product of 2-to-1 cut and project schemes, many of the intuitions known from Sturmian sequences apply. This part also contain the computations of nonexpansive directions performed with Casey Mann and Jennifer McLoud-Mann during their sabbatical year in Bordeaux (2019-2020). It also include the recent work on the family of sets of metallic mean Wang tiles, which is a first attempt at a generalization of Jeandel-Rao tilings. Metallic mean Wang tiles belong to the setup of 4-to-2 cut and project schemes.

Finally, Part V presents the work done in collaboration with Sebastián Barbieri about indistinguishable asymptotic pairs in $\mathbb{Z}^d$. This concept provides a characterization of d-dimensional Sturmian configurations. It belongs to $(d + 1)$-to-d cut and project schemes.

In all of these results, there is always an interplay between combinatorics on one side and dynamics and number theory. Something interesting to observe are the different generalizations of the characterization of Sturmian sequences in higher dimensions and how the combinatorial side change. The most fluid generalization in terms of pattern complexity seems to be within $(d + 1)$-

to-d cut and project schemes (Part V) where we prove that the pattern complexity characterizes multidimensional Sturmian configurations. Within 3-to-1 cut and project schemes (Part III), we describe a nice set of sequences of factor complexity $2n + 1$. But having factor complexity $2n + 1$ is not a characteristic property of these sequences (coding of 3-interval exchange transformations also have complexity $2n+1$). Within 4-to-2 cut and project schemes (Part IV), pattern complexity is not relevant. Instead, it seems that languages and subshifts defined by a finite set of forbidden patterns (subshifts of finite type or Wang shifts) is the important notion. As developed in Part IV, many of the tools (desubstitutions, Rauzy induction, continued fraction expansions, numeration systems) and intuitions that were developed in the one-dimensional setup can be adapted in higher dimensions.

Scope of this thesis

A choice was made to include in this habilitation thesis only the research initiated after obtaining the position at CNRS at LaBRI in 2017. This include a dozen of articles published since 2019. Many of these results are related to cut and project schemes. Thus, it was natural to organize this document according to the dimensions and codimensions of the cut and project schemes involved.

The results published during the direction of the Ph. D. thesis of Jana Lepšová [194] are not included in this habilitation thesis. During the research made on Jeandel-Rao tilings, it became clear that numeration systems were hidden in their description. With Jana, we started the exploration of aperiodic Wang shifts using numeration systems. We proposed a complement version of the Zeckendorf numeration system [18] and of the Dumont-Thomas numeration system [16] allowing to represent every integers (not only the nonnegative ones). This gives an automatic characterization of self-similar subshifts of finite type [17]. Configurations in the subshift are defined by the output of an automata taking as input the representation of any integer coordinates in some well-chosen numeration system. These results will be important for future research on 2-dimensional subshifts defined from 4-to-2 cut and project schemes.

A lot of SageMath/Python code was written to obtain the results described in this thesis (Wang shifts, polyhedron exchange transformations (PETs), polyhedron partitions, Rauzy induction of PETs, 2-dimensional substitutions, cut and project schemes, multidimensional Sturmian configurations and more). Their code is available in the optional package `slabbe` [24] of SageMath [246]. Its documentation is available at

`https://pypi.org/project/slabbe/`

These modules are not described in this thesis.

Batteries included

Computations included in this thesis are based on the open-source mathematical software SageMath [246] and the optional package `slabbe` [24]. All SageMath input/output blocks in this thesis were created using the `sageexample` environment with SageTeX version `2021/10/16 v3.6` and with the following software versions:

```
sage: version()
SageMath version 10.6.beta7, Release Date: 2025-02-21
sage: import importlib.metadata
sage: importlib.metadata.version("slabbe")
0.8.0
```

The fact that these software are open-source means that anyone is free to use, reproduce, verify, adapt for their own needs all of the computations performed therein according to the GNU General Public License (version 2, 1991, `http://www.gnu.org/licenses/gpl.html`).

The contents of all of the sageexample environments from the tex source are gathered in the file demos/hdr_doctest.sage autogenerated by SageTeX when running pdflatex. This file is included in the slabbe package and available at https://gitlab.com/seblabbe/slabbe/. It allows to make sure that future releases of the package do not break the code included in this thesis. It is possible to reproduce all computations present in this thesis and check that all outputs are correct, by *doctesting* this file, that is, by running the command `sage -t demos/hdr_doctest.sage`. It should output `All tests passed!` and `[96 tests, 8.13s wall]` (most probably with a different timing).

About the period 2017-2025

During the period 2017-2025, the author took part in the organization of conferences at *Laboratoire Bordelais de Recherche en Informatique* (LaBRI), Université de Bordeaux:

- GasCOM 2024, The 13th edition of Random Generation of Combinatorial Structures, 24-28 June 2024.
- Journées de combinatoire de Bordeaux (JCB 2021, JCB 2022, JCB 2023, JCB 2024, JCB 2025)
- École des jeunes chercheuses et chercheurs en informatique mathématique (EJCIM 2020), Bordeaux, initially scheduled from April 6 to 10, 2020. Cancelled and finally organized online on BigBlueButton from June 8 to 18, 2020. The book prepared for the school is available from Éditions CNRS [19].
- 17-th Mons Theoretical Computer Science Days, 10-14 septembre 2018.

Participation in the organization of Sage Days (SageMath, a mathematical open-source software workshop) in Maison de la nature du bassin d'Arcachon, Le Teich, France:

- Sage Days 128, 10-14 février 2025.
- Sage Days 125, 29 janvier au 1er février 2024.
- Sage Days 117, 6-10 février 2023.

Ph. D. Students, postdoctorates and long-term visitors during the period:

- Jana Lepšová PhD Student (2020-2024), cosupervised with Lubomíra Dvořáková (Czech Republic)
- Reza Mohammadpour, Postdoctorate (2020-2021), funded by ANR CODYS
- Sebastián Barbieri, Postdoctorate (2019-2020), funded by ANR CODYS
- Jennifer McLoud-Mann, Idex Bordeaux Visiting Scholars positions (2019-2020).
- Casey Mann, Idex Bordeaux Visiting Scholars positions (2019-2020).

Internships: Nicolas Darboux (2018), Eugénie Meryl (2019), Khati–Lefrançois Elias (2019), Steffania Sierre Galvis (2021), Adrian Pino (2023).

Outreach: Collège Cassignol, Bordeaux (2025); Collège Jean Rostand, Casteljaloux (2025); Collège Olympe, Gouges de Vélines (2025); Collège Laure Gatet, Périgueux (2025); Fête de la science, LaBRI (2024, 2025); Lycée Pauillac, Pauillac (2019); Lycée Kastler, Talence (2019, 2022, 2023, 2024); École Gambetta, Bègles (2021, 2023, 2024).

PART I

BACKGROUND

CONTENTS

Chapter 2

Symbolic dynamics

This section follows the preliminary section of the chapter [21] and article [8].

2.1 Dynamical systems

Most of the notions introduced here can be found in [275]. A **dynamical system** is a triple (X, G, T), where X is a topological space, G is a topological group and T is a continuous function $G \times X \to X$ defining a left action of G on X: if $x \in X$, e is the identity element of G and $g, h \in G$, then using additive notation for the operation in G we have $T(e, x) = x$ and $T(g + h, x) = T(g, T(h, x))$. In other words, if one denotes the transformation $x \mapsto T(g, x)$ by T^g, then $T^{g+h} = T^g T^h$. In this work, we consider the Abelian group $G = \mathbb{Z} \times \mathbb{Z}$.

If $Y \subset X$, let $\overline{Y}$ denote the topological closure of Y and let $\overline{Y}^T := \cup_{g \in G} T^g(Y)$ denote the T-closure of Y. A subset $Y \subset X$ is **T-invariant** if $\overline{Y}^T = Y$. A dynamical system (X, G, T) is called **minimal** if X does not contain any nonempty, proper, closed T-invariant subset. The left action of G on X is **free** if $g = e$ whenever there exists $x \in X$ such that $T^g(x) = x$.

Let (X, G, T) and (Y, G, S) be two dynamical systems with the same topological group G. A **homomorphism** $\theta : (X, G, T) \to (Y, G, S)$ is a continuous function $\theta : X \to Y$ satisfying the commuting property that $S^g \circ \theta = \theta \circ T^g$ for every $g \in G$. A homomorphism $\theta : (X, G, T) \to (Y, G, S)$ is called an **embedding** if it is one-to-one, a **factor map** if it is onto, and a **topological conjugacy** if it is both one-to-one and onto and its inverse map is continuous. If $\theta : (X, G, T) \to (Y, G, S)$ is a factor map, then (Y, G, S) is called a **factor** of (X, G, T) and (X, G, T) is called an **extension** of (Y, G, S). Two dynamical systems are **topologically conjugate** if there is a topological conjugacy between them.

A **measure-preserving dynamical system** is defined as a system $(X, G, T, \mu, \mathcal{B})$, where μ is a probability measure defined on the Borel σ-algebra $\mathcal{B}$ of subsets of X, and $T^g : X \to X$ is a measurable map which preserves the measure μ for all $g \in G$, that is, $\mu(T^g(B)) = \mu(B)$ for all $B \in \mathcal{B}$. The measure μ is said to be **T-invariant**. In what follows, when it is clear from the context, we omit the Borel σ-algebra $\mathcal{B}$ of subsets of X and write (X, G, T, μ) to denote a measure-preserving dynamical system.

The set of all T-invariant probability measures of a dynamical system (X, G, T) is denoted by $\mathcal{M}^T(X)$. A T-invariant probability measure on X is called **ergodic** if for every set $B \in \mathcal{B}$ such that $T^g(B) = B$ for all $g \in G$, we have that B has either zero or full measure. A dynamical system (X, G, T) is **uniquely ergodic** if it has only one invariant probability measure, i.e., $|\mathcal{M}^T(X)| = 1$. One can prove that a uniquely ergodic dynamical system is ergodic. A dynamical system (X, G, T) is said **strictly ergodic** if it is uniquely ergodic and minimal.

Let $(X, G, S, \mu, \mathcal{A})$ and $(Y, G, T, \nu, \mathcal{B})$ be two measure-preserving dynamical systems. We say that the two systems are **isomorphic** (mod 0) if there exist measurable sets $X_0 \subset X$ and $Y_0 \subset Y$ of full measure (i.e., $\mu(X_0) = 1$ and $\nu(Y_0) = 1$) with $S^g(X_0) \subset X_0$, $T^g(Y_0) \subset Y_0$ for all $g \in G$ and there exists a bi-measurable bijection $\phi_0 : X_0 \to Y_0$,

- which is measure-preserving, that is, $\mu(\phi_0^{-1}(B)) = \nu(B)$ for all measurable sets $B \subset Y_0$,
- satisfying $\phi_0 \circ S^g(x) = T^g \circ \phi_0(x)$ for all $x \in X_0$ and $g \in G$.

The role of the set X_0 is to make precise the fact that the properties of the isomorphism need to hold only on a set of full measure. In this case, we call ϕ_0 an **isomorphism** (mod 0) with respect to μ and ν. We also refer to an everywhere defined measurable map $\phi : X \to Y$ as an **isomorphism** (mod 0) with respect to μ and ν if $\phi(x) = \phi_0(x)$ with $x \in X$ for some ϕ_0 and X_0 as above. When ϕ is also a factor map, some authors say that ϕ is a **topo-isomorphism** in order to express both its topological and measurable nature [139].

2.2 Maximal equicontinuous factor

A metrizable dynamical system (X, G, T) is called **equicontinuous** if the family of homeomorphisms $\{T^g\}_{g\in G}$ is equicontinuous, i.e., if for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\mathrm{dist}(T^g(x), T^g(y)) < \varepsilon$$

for all $g \in G$ and all $x, y \in X$ with $\mathrm{dist}(x, y) < \delta$. According to a well-known theorem [45, Theorem 3.2], equicontinuous minimal systems defined by the action of an Abelian group are rotations on groups.

We say that $\theta : (X, G, T) \to (Y, G, S)$ is an **equicontinuous factor** if θ is a factor map and (Y, G, S) is equicontinuous. We say that $(X_{\max}, G, T_{\max})$ is the **maximal equicontinuous factor** of (X, G, T) if there exists an equicontinuous factor $\pi_{\max} : (X, G, T) \to (X_{\max}, G, T_{\max})$, such that for any equicontinuous factor $\theta : (X, G, T) \to (Y, G, S)$, there exists a unique factor map $\psi : (X_{\max}, G, T_{\max}) \to (Y, G, S)$ with $\psi \circ \pi_{\max} = \theta$. The maximal equicontinuous factor exists and is unique (up to topological conjugacy), see [45, Theorem 3.8] and [183, Theorem 2.44].

Let $\theta : (X, G, T) \to (Y, G, S)$ be a factor map. We call the preimage set $\theta^{-1}(y)$ of a point $y \in Y$ the **fiber** of θ over y. The cardinality of the fiber $\theta^{-1}(y)$ for some $y \in Y$ has an important role and is related to the definition of other notions, see [45]. In particular, the factor map θ is **almost one-to-one** if $\{y \in Y : \mathrm{card}(\theta^{-1}(y)) = 1\}$ is a G_δ-dense set in Y (that is a countable intersection of open sets which is dense in Y). In that case, (X, G, T) is an **almost one-to-one extension** of (Y, G, S). The **set of fiber cardinalities** of a factor map $\theta : (X, G, T) \to (Y, G, S)$ is the set $\{\mathrm{card}(\theta^{-1}(y)) : y \in Y\} \subset \mathbb{N} \cup \{\infty\}$, see [131]. The set of fiber cardinalities of the maximal equicontinuous factor of a minimal dynamical system is invariant under topological conjugacy, see for instance [8, Lemma 2.2].

2.3 Subshifts and shifts of finite type

In this section, we introduce multidimensional subshifts, a particular type of dynamical systems [200, §13.10], [247, 199, 158]. Let $\mathcal{A}$ be a finite set, $d \geq 1$, and let $\mathcal{A}^{\mathbb{Z}^d}$ be the set of all maps $x : \mathbb{Z}^d \to \mathcal{A}$, equipped with the compact product topology. An element $x \in \mathcal{A}^{\mathbb{Z}^d}$ is called **configuration** and we write it as $x = (x_m) = (x_m : m \in \mathbb{Z}^d)$, where $x_m \in \mathcal{A}$ denotes the value of x at m. The topology on $\mathcal{A}^{\mathbb{Z}^d}$ is compatible with the metric defined for all configurations $x, x' \in \mathcal{A}^{\mathbb{Z}^d}$ by $\mathrm{dist}(x, x') = 2^{-\min\{\|n\| \,:\, x_n \neq x'_n\}}$ where $\|n\| = |n_1| + \cdots + |n_d|$. The **shift action** $\sigma : n \mapsto \sigma^n$ of the additive group $\mathbb{Z}^d$ on $\mathcal{A}^{\mathbb{Z}^d}$ is defined by

$$(\sigma^n(x))_m = x_{m+n} \tag{2.1}$$

for every $x = (x_m) \in \mathcal{A}^{\mathbb{Z}^d}$ and $n \in \mathbb{Z}^d$. If $X \subset \mathcal{A}^{\mathbb{Z}^d}$, let $\overline{X}$ denote the topological closure of X and let $\overline{X}^{\sigma} := \{\sigma^n(x) \mid x \in X, n \in \mathbb{Z}^d\}$ denote the shift-closure of X. A subset $X \subset \mathcal{A}^{\mathbb{Z}^d}$ is

shift-invariant if $\overline{X}^{\sigma} = X$. A closed, shift-invariant subset $X \subset \mathcal{A}^{\mathbb{Z}^d}$ is a **subshift**. If $X \subset \mathcal{A}^{\mathbb{Z}^d}$ is a subshift we write $\sigma = \sigma^X$ for the restriction of the shift action (2.1) to X. When X is a subshift, the triple $(X, \mathbb{Z}^d, \sigma)$ is a dynamical system and the notions presented in the previous section hold.

A configuration $x \in X$ is **periodic** if there is a nonzero vector $n \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$ such that $x = \sigma^n(x)$ and otherwise it is **nonperiodic**. We say that a nonempty subshift X is **aperiodic** if the shift action σ on X is free.

For any subset $S \subset \mathbb{Z}^d$ let $\pi_S : \mathcal{A}^{\mathbb{Z}^d} \to \mathcal{A}^S$ denote the projection map which restricts every $x \in \mathcal{A}^{\mathbb{Z}^d}$ to S. A **pattern** is a function $p \in \mathcal{A}^S$ for some finite subset $S \subset \mathbb{Z}^d$. To every pattern $p \in \mathcal{A}^S$ corresponds a subset $\pi_S^{-1}(p) \subset \mathcal{A}^{\mathbb{Z}^d}$ called **cylinder**. A nonempty set $X \subset \mathcal{A}^{\mathbb{Z}^d}$ is a **subshift** if and only if there exists a set $\mathcal{F}$ of **forbidden** patterns such that

$$X = \{x \in \mathcal{A}^{\mathbb{Z}^d} \mid \pi_S \circ \sigma^n(x) \notin \mathcal{F} \text{ for every } n \in \mathbb{Z}^d \text{ and } S \subset \mathbb{Z}^d\}, \tag{2.2}$$

see [158, Prop. 9.2.4]. A subshift $X \subset \mathcal{A}^{\mathbb{Z}^d}$ is a **subshift of finite type** (SFT) if there exists a finite set $\mathcal{F}$ such that (2.2) holds. In this work, we consider shifts of finite type on $\mathbb{Z} \times \mathbb{Z}$, that is, the case $d = 2$.

2.4 Symbolic representations

In this section, we define the notion of symbolic representation as in the section on Markov partitions from [200]. Intended to formalize the definition of Markov partition for hyperbolic automorphisms on the torus, it is also very convenient to formalize the symbolic representations of $\mathbb{Z}^d$-actions acting by rotations on the torus [8, 2].

Let M be a compact metric space. Consider $\mathbb{Z}^d \overset{R}{\curvearrowright} M$ a continuous $\mathbb{Z}^d$-action on M where $R\colon \mathbb{Z}^d \times M \to M$. For some finite set $\mathcal{A}$, a **topological partition** of M (in the sense of Definition 6.5.3 of [200]) is a collection $\{P_a\}_{a\in\mathcal{A}}$ of disjoint open sets $P_a \subset M$ such that $M = \bigcup_{a\in\mathcal{A}} \overline{P_a}$. If $S \subset \mathbb{Z}^d$ is a finite set, we say that a pattern $w \in \mathcal{A}^S$ is **allowed** for $\mathcal{P}, R$ if

$$\bigcap_{k\in S} R^{-k}(P_{w_k}) \neq \varnothing. \tag{2.3}$$

Let us recall that a $\mathbb{Z}^d$-**subshift** is a set of the form $X \subset \mathcal{A}^{\mathbb{Z}^d}$ which is closed in the prodiscrete topology and invariant under the shift action; and its language is the union of $\mathcal{L}(x)$ for every $x \in X$. Let $\mathcal{L}_{\mathcal{P},R}$ be the collection of all allowed patterns for $\mathcal{P}, R$. The set $\mathcal{L}_{\mathcal{P},R}$ is the language of a subshift $\mathcal{X}_{\mathcal{P},R} \subseteq \mathcal{A}^{\mathbb{Z}^d}$ defined as follows, see [158, Prop. 9.2.4],

$$\mathcal{X}_{\mathcal{P},R} = \{x \in \mathcal{A}^{\mathbb{Z}^d} \mid \sigma^n(x)|_S \in \mathcal{L}_{\mathcal{P},R} \text{ for every } n \in \mathbb{Z}^d \text{ and finite subset } S \subset \mathbb{Z}^d\}.$$

We call $\mathcal{X}_{\mathcal{P},R}$ the **symbolic extension** of $\mathbb{Z}^d \overset{R}{\curvearrowright} M$ determined by the partition $\mathcal{P}$.

For each $x \in \mathcal{X}_{\mathcal{P},R}$ and $m \geq 0$ there is a corresponding nonempty open set

$$D_m(x) = \bigcap_{\|k\|_\infty \leq m} R^{-k}(P_{x_k}) \subset M.$$

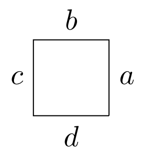
The sequence of compact closures $(\overline{D}_m(x))_{m\in\mathbb{N}}$ of these sets is nested and thus it follows that their intersection is nonempty. Notice that there is no reason why $\operatorname{diam}(\overline{D}_m(x))$ should converge to zero, and thus the intersection could contain more than one point. In order for $\mathcal{X}_{\mathcal{P},R}$ to capture the dynamics of $\mathbb{Z}^d \overset{R}{\curvearrowright} M$, this intersection should contain only one point. This leads to the following definition.

Definition 2.1. *A topological partition* $\mathcal{P}$ *of* M *gives a* ***symbolic representation*** $\mathcal{X}_{\mathcal{P},R}$ *of* $\mathbb{Z}^d \overset{R}{\curvearrowright} M$ *if for every* $x \in \mathcal{X}_{\mathcal{P},R}$ *the intersection* $\bigcap_{m=0}^{\infty} \overline{D}_m(x)$ *consists of exactly one point* $\rho \in M$*. We call* x *a* ***symbolic representation of*** ρ*.*

If $\mathcal{P}$ gives a symbolic representation of the dynamical system $\mathbb{Z}^d \overset{R}{\curvearrowright} M$, then there is a well-defined map $f\colon \mathcal{X}_{\mathcal{P},R} \to M$ which maps a configuration $x \in \mathcal{X}_{\mathcal{P},R} \subset \mathcal{A}^{\mathbb{Z}^d}$ to the unique point $f(x) \in M$ in the intersection $\cap_{n=0}^{\infty}\overline{D}_n(w)$. It is not hard to prove that f is in fact a factor map, that is, such that f is continuous, surjective and $\mathbb{Z}^d$-equivariant ($f(\sigma^k(x)) = R^k(f(x))$ for every $k \in \mathbb{Z}^d$). A proof of this fact for the case $d = 1$ can be found in [200, Prop. 6.5.8]. A proof for $\mathbb{Z}^2$-actions can be found in [8, Prop. 5.1] and a proof for general group actions follows the same arguments.

2.5 Wang shifts

A **Wang tile** is a tuple of four colors $(a, b, c, d) \in I \times J \times I \times J$ where I is a finite set of vertical colors and J is a finite set of horizontal colors, see [276, 243]. A Wang tile is represented as a unit square with colored edges:



For each Wang tile $\tau = (a, b, c, d)$, let $\textsc{Right}(\tau) = a$, $\textsc{Top}(\tau) = b$, $\textsc{Left}(\tau) = c$, $\textsc{Bottom}(\tau) = d$ denote respectively the colors of the right, top, left and bottom edges of τ.

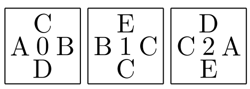


Figure 2.1: The set of 3 Wang tiles introduced in [276] using letters $\{A, B, C, D, E\}$ instead of numbers from the set $\{1, 2, 3, 4, 5\}$ for labeling the edges. Each tile is identified uniquely by an index from the set $\{0, 1, 2\}$ written at the center each tile.

Let $\mathcal{T} = \{t_0, \ldots, t_{m-1}\}$ be a set of Wang tiles as the one shown in Figure 2.1. A configuration $x : \mathbb{Z}^2 \to \{0, \ldots, m-1\}$ is **valid** with respect to $\mathcal{T}$ if it assigns a tile in $\mathcal{T}$ to each position of $\mathbb{Z}^2$ so that contiguous edges of adjacent tiles have the same color, that is,

$$\textsc{Right}(t_{x(n)}) = \textsc{Left}(t_{x(n+e_1)}) \tag{2.4}$$

$$\textsc{Top}(t_{x(n)}) = \textsc{Bottom}(t_{x(n+e_2)}) \tag{2.5}$$

for every $n \in \mathbb{Z}^2$ where $e_1 = (1, 0)$ and $e_2 = (0, 1)$. A finite pattern which is valid with respect to $\mathcal{U}$ is shown in Figure 2.2.

Let $\Omega_{\mathcal{T}} \subset \{0, \ldots, m-1\}^{\mathbb{Z}^2}$ denote the set of all valid configurations with respect to $\mathcal{T}$. Together with the shift action σ of $\mathbb{Z}^2$, $\Omega_{\mathcal{T}}$ is a subshift that we call a **Wang shift**. Furthermore, $\Omega_{\mathcal{T}}$ is a subshift of finite type (SFT) of the form (2.2) since $\Omega_{\mathcal{T}}$ is the subshift defined from the finite set of forbidden patterns made of all horizontal and vertical dominoes of two tiles that do not share an edge of the same color. Reciprocally, every subshift of finite type can be encoded into a Wang shift following a well-known construction (see [217, p. 141-142]).

To a configuration $x \in \Omega_{\mathcal{T}}$ corresponds a tiling of the plane $\mathbb{R}^2$ by the tiles $\mathcal{T}$ where the unit square Wang tile $t_{x(n)}$ is placed at position n for every $n \in \mathbb{Z}^2$, as in Figure 2.2. In this document,

$$\begin{pmatrix} 2 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix} \mapsto$$

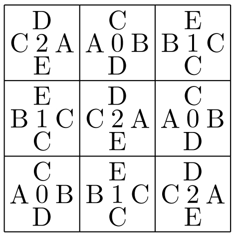


Figure 2.2: A finite 3×3 pattern on the left is valid with respect to the Wang tiles since it respects Equations (2.4) and (2.5). Validity can be verified on the tiling shown on the right.

we consider tilings from the symbolic point of view. In particular, we represent tilings of plane by Wang tiles symbolically by configurations $\mathbb{Z}^2 \to \mathcal{T}$.

A configuration $x \in \Omega_\mathcal{T}$ is **periodic** if there exists $n \in \mathbb{Z}^2 \setminus \{0\}$ such that $x = \sigma^n(x)$. A set of Wang tiles $\mathcal{T}$ is **periodic** if there exists a periodic configuration $x \in \Omega_\mathcal{T}$. Originally, Wang thought that every set of Wang tiles $\mathcal{T}$ is periodic as soon as $\Omega_\mathcal{T}$ is nonempty [276]. This statement is equivalent to the existence of an algorithm solving the *domino problem*, that is, taking as input a set of Wang tiles and returning *yes* or *no* whether there exists a valid configuration with these tiles. Berger, a student of Wang, later proved that the domino problem is undecidable and he also provided a first example of an aperiodic set of Wang tiles [67]. A set of Wang tiles $\mathcal{T}$ is **aperiodic** if the Wang shift $\Omega_\mathcal{T}$ is a nonempty aperiodic subshift. This means that in general one can not decide the emptiness of a Wang shift $\Omega_\mathcal{T}$.

Chapter 3

Sturmian sequences

What Coven and Hedlund proved in [113] based on the initial work of Morse and Hedlund [216] on Sturmian sequences dating from 1940 is that a biinfinite sequence is Sturmian if and only if it is the coding of an irrational rotation. Proving that the coding of an irrational rotation is a Sturmian sequence is the easy part and corresponds to what we did above. The difficult part is to prove that a Sturmian sequence can be obtained as the coding of an irrational rotation for some starting point. The proof is explained nowadays in terms of S-adic development of Sturmian sequences, Rauzy induction of circle rotations, the continued fraction expansion of real numbers and the Ostrowski numeration system [132]. Rauzy discovered that the connection between Sturmian sequences and rotations can be generalized to sequences using three symbols [237] involving a rotation on a 2-dimensional torus $\mathbb{T}^2$. This result was extended recently for almost all rotations on $\mathbb{T}^2$ [83], see also [268].

Sturmian words form a deeply studied class of binary words with lots of equivalent definitions [202]. They are for instance the aperiodic words with minimal factor complexity $\#\mathcal{L}_{\boldsymbol{w}}(n) = n+1$ [113], where $\mathcal{L}_{\boldsymbol{w}}(n)$ denotes the language of words of length n of $\boldsymbol{w} \in A^{\mathbb{N}}$, i.e., $\mathcal{L}_{\boldsymbol{w}}(n) = \{u \in A^n \mid u \text{ occurs in } \boldsymbol{w}\}$. Sturmian words are also the aperiodic 1-balanced binary words [216], where an infinite word $\boldsymbol{w} \in A^{\mathbb{N}}$ is K-balanced if any two finite words of the same length occurring in $\boldsymbol{w}$ have, up to K, the same number of occurrences of each letter. The balance property allows to prove that for any Sturmian word $\boldsymbol{w}$, the frequencies of 1 and 2 exist and are irrational. More than that, any Sturmian word $\boldsymbol{w}$ has uniform word frequencies, that is, for all finite word u occurring in $\boldsymbol{w}$, the ratio $\frac{|w_k w_{k+1} \cdots w_{k+n}|_u}{n+1}$ has a limit f_u when n goes to infinity, uniformly in k.

3.1 The Fibonacci word

An integer is even if and only if the least significant digit of its expansion in base 2 is 0. Using this connection, the concept of even/odd depends on the numeration system which is used. Instead of using the base 2 expansion of a integer, we may consider the Zeckendorf numeration system which expresses any nonnegative integer as a sum of nonconsecutive Fibonacci numbers [193, 118, 101, 279]; see Table 3.1. Note that this numeration system appeared earlier in a more general form in Ostrowski's work [223].

Sums of distinct nonconsecutive distinct Fibonacci numbers are naturally encoded as binary sequences in the monoid $\{\mathtt{0},\mathtt{1}\}^*$ with the most significant digit on the left. For example,

$$\begin{aligned} 11 &= 8 + 3 \\ &= \underline{\mathtt{1}} \cdot 8 + \underline{\mathtt{0}} \cdot 5 + \underline{\mathtt{1}} \cdot 3 + \underline{\mathtt{0}} \cdot 2 + \underline{\mathtt{0}} \cdot 1. \end{aligned}$$

Thus, the integer 11 is represented by the binary string 10100 in the Zeckendorf numeration system. The integer 11 is even in the Zeckendorf numeration system because its least significant digit is 0. In general, every integer n is uniquely represented by a binary string $\mathrm{rep}_F(n) \in \{\mathtt{0},\mathtt{1}\}^*$ not starting

n	$\mathrm{rep}_2(n)$	parity
$0 = 0$	0	even
$1 = 1$	1	odd
$2 = 2$	10	even
$3 = 2 + 1$	11	odd
$4 = 4$	100	even
$5 = 4 + 1$	101	odd
$6 = 4 + 2$	110	even
$7 = 4 + 2 + 1$	111	odd
$8 = 8$	1000	even
$9 = 8 + 1$	1001	odd
$10 = 8 + 2$	1010	even
$11 = 8 + 2 + 1$	1011	odd
$12 = 8 + 4$	1100	even
$13 = 8 + 4 + 1$	1101	odd

n	$\mathrm{rep}_F(n)$	parity
$0 = 0$	0	even
$1 = 1$	1	odd
$2 = 2$	10	even
$3 = 3$	100	even
$4 = 3 + 1$	101	odd
$5 = 5$	1000	even
$6 = 5 + 1$	1001	odd
$7 = 5 + 2$	1010	even
$8 = 8$	10000	even
$9 = 8 + 1$	10001	odd
$10 = 8 + 2$	10010	even
$11 = 8 + 3$	10100	even
$12 = 8 + 3 + 1$	10101	odd
$13 = 13$	100000	even

Table 3.1: Left: parity of nonnegative integers written in base 2. Right: parity of nonnegative numbers expressed as a sum of nonconsecutive Fibonacci numbers.

with 0 and containing no two consecutive 1's. A binary sequence in $\{\mathrm{even}, \mathrm{odd}\}^{\mathbb{N}}$ is deduced from this construction: $(s_F(n))_{n\geq 0} = (\mathrm{parity}(\mathrm{rep}_F(n)))_{n\geq 0}$ where the parity is *even* if the least significant digit is 0, and is *odd* if the least significant digit is 1. A priori, after observing Table 3.1, the sequence s_F seems to contain more even numbers than odd numbers. In fact, we can show that the ratio of frequencies of even vs odd numbers is equal to the golden ratio (in particular, this implies that the sequence s_F is not periodic). The sequence s_F is a Sturmian sequence: it is another definition of the well-known Fibonacci word [68].

With Jana Lepšová, we proposed a complement version of the Zeckendorf numeration system [18] and of the Dumont-Thomas numeration system [16] allowing to represent every integers (not only the nonnegative ones).

3.2 Balanced sequences

In this section, we define balanced sequences which were studied by Morse and Hedlund in 1940 shortly after their seminal article on symbolic dynamics [215].

Let $s \in \Sigma^{\mathbb{Z}}$ be a sequence over a finite set Σ. The language of s is $\mathcal{L}(s) = \{s_k s_{k+1} \cdots s_{k+n-1} \mid k \in \mathbb{Z}, n \geq 0\} \subset \Sigma^*$ is the set of subwords (or factors) occurring in s. The language of subwords of length $n \in \mathbb{Z}_{\geq 0}$ is $\mathcal{L}_n(s) = \mathcal{L}(s) \cap \Sigma^n$.

Definition 3.1 ([216]). *A sequence $s \in \Sigma^{\mathbb{Z}}$ is* balanced *if for every positive integer n, for every $u, v \in \mathcal{L}_n(s)$ and every letter $a \in \Sigma$, the number of a's occurring in u and v differ by at most 1.*

For example, the right-infinite Fibonacci word [68]

$$F = \mathtt{01001010010010100101}\ldots \in \Sigma^{\mathbb{Z}_{\geq 0}}$$

over the alphabet $\Sigma = \{\mathtt{0}, \mathtt{1}\}$, and its left-infinite reversal

$$\widetilde{F} = \ldots \mathtt{10100101001001010010} \in \Sigma^{\mathbb{Z}_{<0}}$$

are such that both

$$\widetilde{F} \cdot \mathtt{01} \cdot F = \ldots \mathtt{10100101001001010010} \cdot \mathtt{01} \cdot \mathtt{01001010010010100101}\ldots$$

and

$$\widetilde{F} \cdot \mathtt{10} \cdot F = \ldots \mathtt{10100101001001010010} \cdot \mathtt{10} \cdot \mathtt{01001010010010100101} \ldots$$

are balanced sequences. This observation is illustrated for factors of length up to six in the following table.

n	$\mathcal{L}_n(\widetilde{F}\mathtt{01}F)$	number of 0's	number of 1's
0	$\{\varepsilon\}$	0	0
1	$\{\mathtt{0}, \mathtt{1}\}$	0 or 1	0 or 1
2	$\{\mathtt{00}, \mathtt{01}, \mathtt{10}\}$	1 or 2	0 or 1
3	$\{\mathtt{001}, \mathtt{010}, \mathtt{100}, \mathtt{101}\}$	1 or 2	1 or 2
4	$\{\mathtt{0010}, \mathtt{0100}, \mathtt{0101}, \mathtt{1001}, \mathtt{1010}\}$	2 or 3	1 or 2
5	$\{\mathtt{00100}, \mathtt{00101}, \mathtt{01001}, \mathtt{01010}, \mathtt{10010}, \mathtt{10100}\}$	3 or 4	1 or 2
6	$\{\mathtt{001001}, \mathtt{001010}, \mathtt{010010}, \mathtt{010100}, \mathtt{100100}, \mathtt{100101}, \mathtt{101001}\}$	3 or 4	2 or 3

From the table, we confirm that the number of 0's and the number of 1's occurring in two factors of the same length differ by at most 1.

Other examples are the two-sided sequences

$$\ldots \mathtt{010101010101010101010101010101} \ldots$$
$$\ldots \mathtt{0000000000000001000000000000000} \ldots$$

which are also balanced. In other words, balanced sequences may be periodic or ultimately periodic.

3.3 Mechanical sequences

Morse and Hedlund proved that balanced sequences can be described by what they called mechanical sequences.

Let $\alpha \in [0,1]$ and $\rho \in \mathbb{R}$ and consider the *lower* and *upper mechanical sequences* $s_{\alpha,\rho}$ and $s'_{\alpha,\rho}$ with *slope* α and *intercept* ρ given respectively by

$$\begin{array}{rrcl} s_{\alpha,\rho}: & \mathbb{Z} & \to & \{\mathtt{0},\mathtt{1}\} \\ & n & \mapsto & \lfloor \alpha(n+1)+\rho \rfloor - \lfloor \alpha n + \rho \rfloor \end{array}$$

and

$$\begin{array}{rrcl} s'_{\alpha,\rho}: & \mathbb{Z} & \to & \{\mathtt{0},\mathtt{1}\} \\ & n & \mapsto & \lceil \alpha(n+1)+\rho \rceil - \lceil \alpha n + \rho \rceil. \end{array}$$

When α is rational, the sequences $s_{\alpha,\rho}$ and $s'_{\alpha,\rho}$ are periodic and their period corresponds to a Christoffel word. Christoffel words are very interesting object on their own and are presented in Chapter 4. When α is irrational, then $s_{\alpha,\rho}$ and $s'_{\alpha,\rho}$ are not periodic. It is clear that if $\rho - \rho'$ is an integer, then $s_{\alpha,\rho} = s_{\alpha,\rho'}$ and $s'_{\alpha,\rho} = s'_{\alpha,\rho'}$. Thus we may always assume $0 \leq \rho < 1$. Moreover, if $\mathbb{Z} \cap \alpha\mathbb{Z} + \rho = \varnothing$, then $s_{\alpha,\rho} = s'_{\alpha,\rho}$.

Theorem 3.2 (Morse, Hedlund [216]). *Let $w \in \{\mathtt{0},\mathtt{1}\}^{\mathbb{Z}}$ be a non-ultimately periodic two-sided sequence. The following conditions are equivalent:*

- *w is balanced,*
- *there exists $\alpha \in [0,1] \setminus \mathbb{Q}$ and $\rho \in [0,1)$ such that $w = s_{\alpha,\rho}$ or $w = s'_{\alpha,\rho}$ is a mechanical sequence.*

Note that Morse and Hedlund's theorem is stronger as they considered all balanced sequences including periodic and skew (ultimately periodic) sequences. As the stronger result is best stated in terms of Christoffel words, we postpone it to Section 4.7.

3.4 Maximal equicontinuous factor

The goal of this section is to reformulate Theorem 3.2 using factor maps within topological and measurable dynamical systems, because this is how we state such results in higher dimensions for Jeandel-Rao tilings and metallic mean Wang shifts in Part IV.

Let $u \in \{\mathtt{0},\mathtt{1}\}^{\mathbb{Z}}$ be a non-ultimately periodic two-sided balanced sequence. Let

$$X_u = \overline{\{\sigma^n(u) \colon n \in \mathbb{Z}\}}$$

be the subshift defined as the topological closure of the orbit of u under the shift map. Since every sequence $w \in X_u$ is balanced, using Theorem 3.2, we may define a map

$$\begin{array}{rccc} \Phi_u : & X_u & \to & \mathbb{T} \\ & w & \mapsto & \rho \end{array}$$

where $\rho \in [0,1)$ is the intercept such that $u = s_{\alpha,\rho}$ or $u = s'_{\alpha,\rho}$ is a mechanical sequence for some $\alpha \in [0,1] \setminus \mathbb{Q}$. The map Φ_u relates the shift action on X_u with the irrational rotation by α on the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$:

$$\begin{array}{rccc} \sigma : & X_u & \to & X_u \\ & w & \mapsto & (w_{i+1})_{i\in\mathbb{Z}} \end{array} \qquad \text{and} \qquad \begin{array}{rccc} R_\alpha : & \mathbb{T} & \to & \mathbb{T} \\ & x & \mapsto & x+\alpha \pmod 1 \end{array}$$

because every sequence $w \in X_u$ is a mechanical sequence of the same slope α.

Theorem 3.3 ([216, 156, 113, 202, 69, 38, 33])**.** *Let $u \in \{\mathtt{0},\mathtt{1}\}^{\mathbb{Z}}$ be a non-ultimately periodic two-sided balanced sequence. The Sturmian subshift X_u has the following properties:*

(i) the map $\Phi_u : X_u \to \mathbb{T}$ is a factor map, that is, it is continuous, onto and satisfies $\Phi_u \circ \sigma = R_\alpha \circ \Phi_u$ for some irrational $\alpha \in [0,1] \setminus \mathbb{Q}$:

$$\begin{array}{ccc} X_u & \xrightarrow{\sigma} & X_u \\ {\scriptstyle\Phi_u}\big\downarrow & & \big\downarrow{\scriptstyle\Phi_u} \\ \mathbb{T} & \xrightarrow{R_\alpha} & \mathbb{T} \end{array}$$

(ii) $\mathbb{Z} \overset{R_\alpha}{\curvearrowright} \mathbb{T}$ is the maximal equicontinuous factor of $\mathbb{Z} \overset{\sigma}{\curvearrowright} X_u$,

(iii) the factor map $f : X_u \to \mathbb{T}$ is almost one-to-one and its set of fiber cardinalities is $\{1,2\}$,

(iv) $\#\Phi_u^{-1}(\rho) = 2$ if and only if $\rho \in \mathbb{Z} + \mathbb{Z}\alpha$ if and only if $s_{\alpha,\rho} \neq s'_{\alpha,\rho}$, in which case $\Phi_u^{-1}(\rho) = \{s_{\alpha,\rho}, s'_{\alpha,\rho}\}$,

(v) the shift-action $\mathbb{Z} \overset{\sigma}{\curvearrowright} X_u$ on the Sturmian subshift is uniquely ergodic,

(vi) the measure-preserving dynamical system $(X_u, \mathbb{Z}, \sigma, \nu)$ is isomorphic to $(\mathbb{T}, \mathbb{Z}, R_\alpha, \lambda)$ where ν is the unique shift-invariant probability measure on X_u and λ is the Haar measure on $\mathbb{T}$.

Theorem 3.3 is an important stepping stone for this habilitation thesis. The factor map Φ_u is not one-to-one precisely on the pairs

$$\Phi_u^{-1}(\rho) = \{s_{\alpha,\rho}, s'_{\alpha,\rho}\}$$

such that $\rho \in \mathbb{Z} + \mathbb{Z}\alpha$. The importance of these asymptotic pairs was already noticed by Hedlund [156]. In Part II of this thesis, we propose a new characterization of Sturmian sequences using the notion of indistinguishability of these asymptotic pairs. In Part III, we extend Theorem 3.3 to sequences over a three-letter alphabet (Theorem 10.1 and Theorem 10.3). In Part IV, we prove higher dimensional versions of Theorem 3.3 about Jeandel-Rao tilings (Theorem 11.2) and metallic mean Wang shifts (Theorem 12.3). Finally, the notion of indistinguishable asymptotic pairs was rich enough to extend nicely in higher dimensions: in Part V, we propose a characterization of multidimensional Sturmian configurations.

3.5 Sequences of complexity $n+1$

In [216], Morse and Hedlund proved that mechanical sequences have complexity $n+1$, but the converse remained open. The factor complexity of a sequence $s \in \Sigma^{\mathbb{Z}}$ is the function $n \mapsto \#\mathcal{L}_n(s)$ that counts the number of factors of length n in its language [104]. While Morse and Hedlund prove that a sequence of complexity $\leq n$ is ultimately periodic [215], the description of sequences of complexity $n+1$ was completed 30 years later by Coven and Hedlund.

For example, the factor complexity of the Fibonacci word is computed in the following table up to words of length 6.

n	$\mathcal{L}_n(\widetilde{F}\mathtt{01}F)$	$\#\mathcal{L}_n(\widetilde{F}\mathtt{01}F)$
0	$\{\varepsilon\}$	1
1	$\{\mathtt{0},\mathtt{1}\}$	2
2	$\{\mathtt{00},\mathtt{01},\mathtt{10}\}$	3
3	$\{\mathtt{001},\mathtt{010},\mathtt{100},\mathtt{101}\}$	4
4	$\{\mathtt{0010},\mathtt{0100},\mathtt{0101},\mathtt{1001},\mathtt{1010}\}$	5
5	$\{\mathtt{00100},\mathtt{00101},\mathtt{01001},\mathtt{01010},\mathtt{10010},\mathtt{10100}\}$	6
6	$\{\mathtt{001001},\mathtt{001010},\mathtt{010010},\mathtt{010100},\mathtt{100100},\mathtt{100101},\mathtt{101001}\}$	7

From the table, we confirm that there are $n+1$ factors of length n.

Other examples of two-sided sequences of complexity $n+1$ are

$$\ldots\mathtt{0000000000000000100000000000000000}\ldots$$
$$\ldots\mathtt{0000000000000000011111111111111}\ldots$$

In other words, two-sided sequences of complexity $n+1$ may be ultimately periodic. Also, notice that the latter one is not balanced.

One way to get around those problematic limit cases is to consider only right-infinite sequences where the limit cases do not occur. This is what is done in the treatment of Sturmian sequences made in [202, Chapter 2] and [33, Chapter 9]. The chapter [132, Chapter 6] considers both one-sided and two-sided sequences.

Theorem 3.4 (Coven, Hedlund [113]). *Let $w \in \{\mathtt{0},\mathtt{1}\}^{\mathbb{Z}}$ be a non-ultimately periodic two-sided sequence. The following conditions are equivalent:*

- *w is balanced,*
- *w has factor complexity $n+1$,*
- *there exists $\alpha \in [0,1] \setminus \mathbb{Q}$ and $\rho \in [0,1)$ such that $w = s_{\alpha,\rho}$ or $w = s'_{\alpha,\rho}$ is a mechanical sequence.*

Note that Coven and Hedlund considered all balanced sequences including periodic and skew (ultimately periodic) sequences [113, Theorem 4.12]. As this result is best stated in terms of Christoffel words, we postpone it to Section 4.7.

3.6 About the proof of Coven-Hedlund theorem

The proof of Theorem 3.4 made in [113] and in books [202, Theorem 2.1.13], [132, Theorem 6.1.8] always follows the same structure:

$$\text{has factor complexity } n+1 \iff \text{is balanced} \iff \text{is a mechanical sequence.}$$

Moreover, proving that mechanical sequences are balanced or has complexity $n+1$ is relatively easy. However, proving that a balanced sequence (or a sequence of complexity $n+1$) is mechanical needs more work as we need to a way to construct the slope α and intercept ρ.

A proof which is very instructive [38] is using the desubstitution of balanced sequences and of mechanical sequences by the substitutions

$$\tau_0 = \begin{cases} 0 \mapsto 0 \\ 1 \mapsto 01 \end{cases} \quad \text{and } \tau_1 = \begin{cases} 0 \mapsto 01 \\ 1 \mapsto 1 \end{cases}.$$

On the one hand, balanced sequences may be desubstituted by τ_0 or τ_1. The following proposition is not explicitly expressed in [38], but it can be deduced from the results in its Section 6.3.

Proposition 3.5 ([38, Section 6.3])**.** *Let $\Sigma \subset \{0,1\}^{\mathbb{Z}}$ be the subset of balanced sequences. Then,*

$$\Sigma = \overline{\tau_0(\Sigma)}^{\sigma} \cup \overline{\tau_1(\Sigma)}^{\sigma}.$$

More precisely, every balanced sequence is, up to a shift, the image of a unique balanced sequence under the substitution τ_0 or τ_1.

On the other hand, mechanical sequences may also be desubstituted by τ_0 or τ_1. The following proposition is not explicitly expressed in [38], but it can be deduced from the results in its Section 6.4.

Proposition 3.6 ([38, Section 6.4])**.** *Let $\Phi \subset \{0,1\}^{\mathbb{Z}}$ be the subset of mechanical sequences. Then,*

$$\Phi = \overline{\tau_0(\Phi)}^{\sigma} \cup \overline{\tau_1(\Phi)}^{\sigma}.$$

More precisely, every mechanical sequence is, up to a shift, the image of a unique mechanical sequence under the substitution τ_0 or τ_1.

From Proposition 3.5 and Proposition 3.6, we can deduce that $\Sigma = \Phi$. But, it can also be used to prove in a constructive way that every balanced sequence is a mechanical sequence.

Indeed, following Proposition 3.5, any balanced sequence $x \in \Sigma$ can be written as

$$x = \lim_{n\to\infty} \sigma^{-b_0}\tau_0^{a_0} \cdot \sigma^{-b_1}\tau_1^{a_1} \cdot \ldots \cdot \sigma^{-b_{2n}}\tau_0^{a_{2n}} \cdot \sigma^{-b_{2n+1}}\tau_1^{a_{2n+1}} (1 \cdot 0)$$

where

- $0 \leq a_0$ and $0 < a_n$ if $n > 0$,
- $b_n \leq a_n$ for every $n \geq 0$,
- if $b_{n+1} = a_{n+1}$, then $b_n = 0$ [38, Theorem 6.3.33].

The sequence (a_n, b_n) is called *the second multiplicative coding* of x ([38, Definition 6.3.34]). From [38, Theorem 6.4.21], the sequence $(a_n)_n$ is the continued fraction expansion of the slope of the mechanical sequence and the sequence $(b_n)_n$ is the Ostrowski expansion of the intercept ρ of the mechanical sequence with respect to the sequence $(a_n)_n$.

The proof of Proposition 3.5 is combinatorial, for instance see [38, Lemma 6.3.5]. However, the proof of Proposition 3.6 has a dynamical system and number theory flavor. In particular, it involves Rauzy induction (first return maps) and continued fraction expansion of real numbers.

Any attempts at generalizing Coven-Hedlund theorem in higher dimensions can not ignore these key concepts.

3.7 Continued fraction expansion

A theorem of Dirichlet says that every positive irrational number α has infinitely many rational approximations $\frac{p}{q} \in \mathbb{Q}$ such that $|\alpha - \frac{p}{q}| < \frac{1}{q^2}$ [153]. Such approximations can be computed from the continued fraction expansion of α

$$\alpha = [a_0; a_1, a_2, \dots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots}}}$$

where $a_0 \in \mathbb{N}$ and $a_1, a_2, \dots \in \mathbb{N} \setminus \{0\}$. Indeed, for all $n \in \mathbb{N}$, the truncation $\frac{p_n}{q_n} = [a_0; a_1, \dots, a_n]$ provides a sequence $(p_n/q_n)_{n\in\mathbb{N}}$ of rational approximations of α called convergents satisfying Dirichlet's theorem. Equivalently, the convergents p_n/q_n can be computed from a product of the matrices $A_\mathtt{0} = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $A_\mathtt{1} = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ involving the above sequence of partial quotients:

$$\begin{pmatrix} p_{2n+1} & p_{2n} \\ q_{2n+1} & q_{2n} \end{pmatrix} = A_\mathtt{0}^{a_0} A_\mathtt{1}^{a_1} A_\mathtt{0}^{a_2} \cdots A_\mathtt{1}^{a_{2n+1}}.$$

The convergence of p_n/q_n to α then implies that

$$\begin{pmatrix} \alpha \\ 1 \end{pmatrix} \mathbb{R}_{\geq 0} = \bigcap_{k\geq 0} A_{i_0} A_{i_1} \cdots A_{i_k} \mathbb{R}^2_{\geq 0} \tag{3.1}$$

where the sequence $(i_n)_{n\in\mathbb{N}} \in \{\mathtt{0}, \mathtt{1}\}^{\mathbb{N}}$ is $\mathtt{0}^{a_0}\mathtt{1}^{a_1}\mathtt{0}^{a_2}\cdots\mathtt{0}^{a_{2k}}\mathtt{1}^{a_{2k+1}}\cdots$. Equation (3.1) holds even if $\mathtt{0}$ and $\mathtt{1}$ do not both occur infinitely many times in $(i_n)_{n\in\mathbb{N}}$, in which case α is rational. If $\Delta = \{(x, y) \in \mathbb{R}^2_{\geq 0} \mid x + y = 1\}$ denotes the projection of the positive cone $\mathbb{R}^2_{\geq 0}$ under the map $\boldsymbol{x} \mapsto \boldsymbol{x}/\|\boldsymbol{x}\|_1$, Equation (3.1) defines a continuous and onto map

$$\pi : \{\mathtt{0}, \mathtt{1}\}^{\mathbb{N}} \to \Delta.$$

This map is almost one-to-one and its (almost everywhere) inverse is obtained by iterating the normalized Euclid algorithm. More precisely, Euclid's algorithm is the map F_E defined on $\mathbb{R}^2_{\geq 0}$ by, for $\boldsymbol{x} = (x, y)$,

$$F_E(\boldsymbol{x}) = \begin{cases} (x - y, y) = A_\mathtt{0}^{-1}\boldsymbol{x}, & \text{if } x \geq y; \\ (x, y - x) = A_\mathtt{1}^{-1}\boldsymbol{x}, & \text{if } x < y. \end{cases}$$

The map F_E induces a map $f_E : \Delta \to \Delta$ defined by

$$f_E(\boldsymbol{x}) = \frac{F_E(\boldsymbol{x})}{\|F_E(\boldsymbol{x})\|_1}.$$

which subtract the smallest entry to the largest and renormalize the vector so that the entries sum to one.

Thus the shift map on $\{\mathtt{0}, \mathtt{1}\}^{\mathbb{N}}$ defines a symbolic representation of the dynamical system (Δ, f_E). Setting $\Delta_\mathtt{0} = \{(x, y) \in \Delta \mid x \geq y\}$ and $\Delta_\mathtt{1} = \{(x, y) \in \Delta \mid x < y\}$, it induces a map $\delta : \Delta \to \{\mathtt{0}, \mathtt{1}\}^{\mathbb{N}}$ defined by $\delta(\boldsymbol{x}) = (i_n)_{n\in\mathbb{N}}$, where $f_E^n(\boldsymbol{x}) \in \Delta_{i_n}$ and this map satisfies

$$\pi \circ \delta = \mathrm{Id}_\Delta \quad \text{and} \quad \delta \circ f_C = \sigma \circ \delta.$$

where σ is the shift-map on $\{\mathtt{0}, \mathtt{1}\}^{\mathbb{N}}$.

3.8 Continued fraction expansion and Sturmian sequences

Sturmian words give a combinatorial flavor to Equation (3.1). With the matrices A_0 and A_1 are respectively associated the substitutions

$$\tau_0 = \begin{cases} 0 \mapsto 0 \\ 1 \mapsto 01 \end{cases} \quad \text{and } \tau_1 = \begin{cases} 0 \mapsto 01 \\ 1 \mapsto 1 \end{cases} .$$

Recall that the incidence matrix of a substitution $\sigma : A^* \to A^*$ is the matrix $M_\sigma = (|\sigma(a)|_b)_{b,a\in A}$, where $|u|_v$ stands for the number of occurrences of a word v in a word u. It is easily seen that for any word $w \in A^*$, $M_\sigma(|w|_a)_{a\in A} = (|\sigma(w)|_a)_{a\in A}$. Here, the incidence matrix of the substitution τ_0 (resp. τ_1) is the matrix A_0 (resp. A_1).

With the directive sequence $(i_n)_{n\in\mathbb{N}} \in \{0, 1\}^{\mathbb{N}}$ is then associated the $\{\tau_0, \tau_1\}$-adic word $\boldsymbol{w} \in \{0, 1\}^{\mathbb{N}}$:

$$\boldsymbol{w} = \lim_{n\to\infty} \tau_{i_0}\tau_{i_1} \cdots \tau_{i_n}(0^\omega) \tag{3.2}$$

which is a Sturmian word [38] if both letters 0 and 1 appear infinitely often in the directive sequence. Since A_j is the incidence matrix of the substitution τ_i for $i \in \{0, 1\}$, Equation (3.1) ensures that the vector of frequencies of letters in $\boldsymbol{w}$ exists and is equal to $\pi((i_n)_{n\in\mathbb{N}}) = \frac{1}{1+\alpha}(\alpha, 1)$.

The following statement can be deduced from the results in Section 6.4 of [38].

Theorem 3.7. *[38, § 6.4] Let $\alpha \in \mathbb{R}_{>0} \setminus \mathbb{Q}$. Consider the partition of the circle $\mathbb{R}/(1 + \alpha)\mathbb{Z}$ into $I_1 = [-1, 0)$ and $I_0 = [0, \alpha)$. Let $w : \mathbb{Z} \to \{0, 1\}$ be the sequence such that*

$$w_n = \begin{cases} 0 & \text{if } n \in I_0 \pmod{1 + \alpha}, \\ 1 & \text{if } n \in I_1 \pmod{1 + \alpha}. \end{cases}$$

Then the substitutive structure of w is

$$w = \lim_{n\to\infty} s_0^{a_0} s_1^{a_1} \dots s_{2n}^{a_{2n}} s_{2n+1}^{a_{2n+1}} (1 \cdot 0)$$

where

$$s_{2n} = \tau_0 = \begin{cases} 0 \mapsto 0 \\ 1 \mapsto 01 \end{cases} \quad \text{and } s_{2n+1} = \tau_1 = \begin{cases} 0 \mapsto 01 \\ 1 \mapsto 1 \end{cases} \tag{3.3}$$

for all $n \geq 0$ and $\alpha = [a_0; a_1, a_2, \dots]$ is the continuous fraction expansion of α.

For general starting point on the circle, the walking sequence has the same substitutive structure as in the theorem but one must add a certain amount $b_n \in \mathbb{N}$ of shifts after each substitution $\tau_n^{a_n}$ with $0 \leq b_n \leq a_n$ and the sequence $(b_n)_{n\in\mathbb{N}}$ of shifts is given by the expansion of the origin of the walk in the Ostrowski numeration system [38, Theorem 6.4.21].

The proof of Theorem 3.7 is based on induced transformations (first return maps) of irrational rotations on the circle. In general, the first return map of a rotation to an interval is a 3-interval exchange transformation (IET) but for some particular values for the length of the interval, it may be a 2-IET which corresponds to a rotation [235]. This allows to desubstitute a mechanical sequence and write it as the image of another mechanical sequence using a procedure known as **Rauzy induction** for general k-IETs [236]. Here $k = 2$ is always satisfied, hence the induced transformation is always a rotation on a circle.

3.9 Rauzy induction of a rotation computed with SageMath

The goal of this section is to illustrate Rauzy induction in the case of rotations on a torus on a simple example. This is a key concept in the proof of Theorem 3.7. It turned out to also be a key concept to understand Jeandel-Rao tilings. To study Jeandel-Rao tilings, we extended the notion of Rauzy induction to polygonal partitions and to $\mathbb{Z}^2$-actions acting by rotations on a 2-dimensional torus [9]. The algorithm computes the first return map and the induced substitution of any polyhedron exchange transformation whose domain in $\mathbb{R}^d$ is restricted to some half space.

Using rectangles in $\mathbb{R}^2$, we illustrate how to use this algorithm is the easier setup of a single one-dimensional irrational rotation. Let $\alpha = \frac{3}{110}\sqrt{5} + \frac{75}{22} \approx 3.47007$ be one of the root of the quadratic polynomial $p(x) = 55x^2 - 375x + 639$ and whose continued fraction expansion is

$$\alpha = [3; 2, 7, (1,5)^*] = 3 + \frac{1}{[2; 7, (1,5)^*]}.$$

Our goal is to understand the orbits under the rotation by $x \mapsto x + \alpha$ on the circle $\mathbb{R}/(1+\alpha)\mathbb{Z}$.

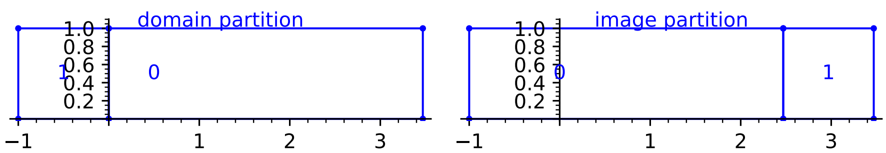
We construct α in SageMath as a number field element.

```
sage: K.<sqrt5> = NumberField(x^2-5, embedding=2.2)
sage: alpha = sqrt5*3/110 + 75/22
sage: continued_fraction(alpha)
[3; 2, 7, (1, 5)*]
sage: alpha.n()
3.47007458120454
```

Polygon exchange transformations were implemented in the `slabbe` optional package. We may use them to define interval exchange transformation by ignoring one of the coordinates.

We define the horizontal translation by α on the 2-torus $\mathbb{R}/(1+\alpha)\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$. We represent it as a polygon exchange transformation on the fundamental domain $[-1, \alpha) \times [0, 1)$. It is essentially a horizontal interval exchange transformation.

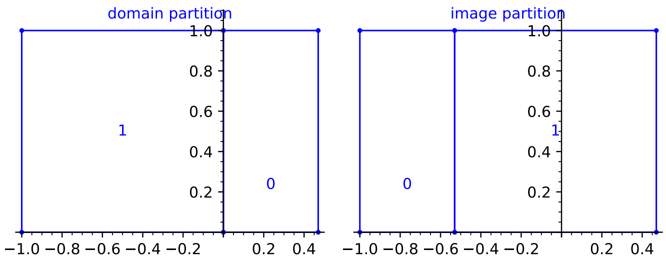
```
sage: from slabbe import PolyhedronExchangeTransformation as PET
sage: base = diagonal_matrix((1+alpha,1))
sage: translation = vector((alpha, 0))
sage: fundamental_domain = polytopes.hypercube(2, intervals=[(-1,alpha), (0,1)])
sage: T = PET.toral_translation(base, translation, fundamental_domain)
sage: T
Polyhedron Exchange Transformation of
Polyhedron partition of 2 atoms with 2 letters
with translations {0: (-1, 0), 1: (3/110*sqrt5 + 75/22, 0)}
```



We compute the induced transformation, also known as the first return map of T when restricted to the half-space defined by the inequality $\alpha - 3 - x \geq 0$, that is, $x \leq \alpha - 3$. The original symbolic dynamical system is the image of the induced symbolic dynamical system under a substitution computed below.

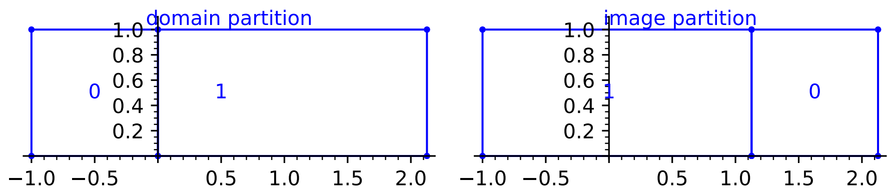
```
sage: Tind,s = T.induced_transformation(ieq=[alpha-3,-1,0])
sage: Tind
Polyhedron Exchange Transformation of
Polyhedron partition of 2 atoms with 2 letters
with translations {0: (-1, 0), 1: (3/110*sqrt5 + 9/22, 0)}
sage: s
```

```
{0: [0], 1: [1, 0, 0, 0]}
```



We renormalize the transformation in order for the smallest interval to have size 1. It is better to renormalize by a negative value, that is, by the factor $\frac{-1}{\alpha-3}$, so to have the same left end-point of the domain as at the beginning: -1.

```
sage: D = diagonal_matrix((-1/(alpha-3),1))
sage: Tindzoom = D * Tind
sage: Tindzoom
Polyhedron Exchange Transformation of
Polyhedron partition of 2 atoms with 2 letters
with translations {0: (-1/6*sqrt5 + 5/2, 0), 1: (-1, 0)}
```



The new horizontal translation is

```
sage: beta = Tindzoom.translations()[0][0]
sage: continued_fraction(beta)
[2; 7, (1, 5)*]
```

We observe that the continued fraction expansion of β is a shift of that of α. This means that the orbits under the rotation α on $\mathbb{R}/(1+\alpha)\mathbb{Z}$ coded by the partition are the images under the substitution (computed above) $0 \mapsto 0, 1 \mapsto 1000$ of the orbits under the rotation β on $\mathbb{R}/(1+\beta)\mathbb{Z}$.

We may repeat this process indefinitely. In this example, the process loops because α is a quadratic algebraic number and its continued fraction expansion is ultimately periodic. It gives a substitutive description of the orbits under the rotation α.

3.10 Guessing that a sequence is the coding of a rotation in SageMath

It is worthwhile to recall here in this introduction the intuitions associated to Sturmian sequences. And the best way to achieve this is by performing a computer experiment.

There is a very nice and easy computer experiment allowing to guess that a sequence $s : \mathbb{N} \to \mathcal{A}$ is the coding of a rotation. For every $a \in \mathcal{A}$, let

$$P_a(\gamma) = e^{2\pi i\, \gamma\, s^{-1}(a)} = \{e^{2\pi i\, \gamma\, n} \mid n \in s^{-1}(a)\}$$

be a set of points on the unit circle where $\gamma \in [0,1)$ is some chosen frequency. To every $a \in \mathcal{A}$, we associate a unique color and we draw every points in the set $P_a(\gamma)$ using this color. The goal of the experiment is to find a frequency γ such that the closure of the sets $P_a(\gamma)$, $a \in \mathcal{A}$, are disjoint intervals on the unit circle.

The above experiment can be performed in SageMath/Python using the following two functions. The reader not knowledgeable in the Python language can skip reading their code with no problem. If painful looking at code here in the introduction, their presence allows the easy reproducibility of the following results by the reader. The first Python function computes the sets of preimages $\{s^{-1}(a)\}_{a\in\mathcal{A}}$ of a sequence s.

```
sage: def preimage(sequence):
....:     from collections import defaultdict
....:     d = defaultdict(list)
....:     for (n,a) in enumerate(sequence):
....:         d[a].append(n)
....:     return dict(d)
```

Then, we define a function which draws the sets of points $P_a(\gamma)$ with a different color for every $a \in \mathcal{A}$ (using the `rainbow` function in SageMath):

```
sage: def draw_sequence_on_circle(sequence, frequency, keys=None):
....:     d = preimage(sequence)
....:     keys = sorted(d.keys()) if keys is None else keys
....:     c_dict = dict(zip(keys, rainbow(len(keys))))
....:     markers = "+x><^vDH_|os*,dh12345678"
....:     m_dict = dict(zip(keys, markers))
....:     G = Graphics()
....:     for a in d:
....:         L = [e^(2*pi*I*frequency*n) for n in d[a]]
....:         G += points(L, color=c_dict[a], legend_label=a, marker=m_dict[a], pointsize=30)
....:     title = "Frequency␣=␣{:.5f}".format(frequency.n())
....:     G += circle((0,0), 1, linestyle="dotted", alpha=.5, color="gray", title=title)
....:     return G
```

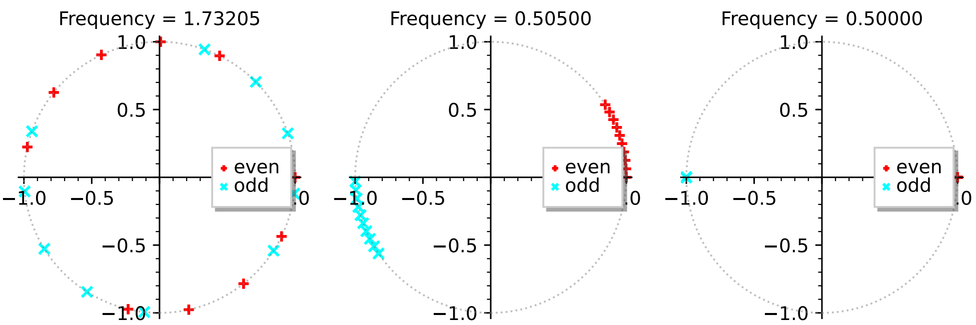
We first illustrate the usage of the above two functions on a simple periodic example.

Example 3.8. *Consider the periodic sequence* (even, odd, even, odd, even, odd, even, odd, . . .) *of period 2. The first function returns a Python dictionary giving the even and odd positions:*

```
sage: L = ["even","odd"] * 6
sage: preimage(L)
{'even': [0, 2, 4, 6, 8, 10], 'odd': [1, 3, 5, 7, 9, 11]}
```

The second function allows to confirm that the frequency $\gamma = \frac{1}{2}$ *allows to separate the two sets* $P_{odd}(\gamma)$ *and* $P_{even}(\gamma)$*. Testing also a frequency* $\gamma = \frac{1}{2} + \varepsilon$ *for some small value of* $\varepsilon > 0$ *allows to confirm that no two points of different color are drawn at the same position when* $\gamma = \frac{1}{2}$*.*

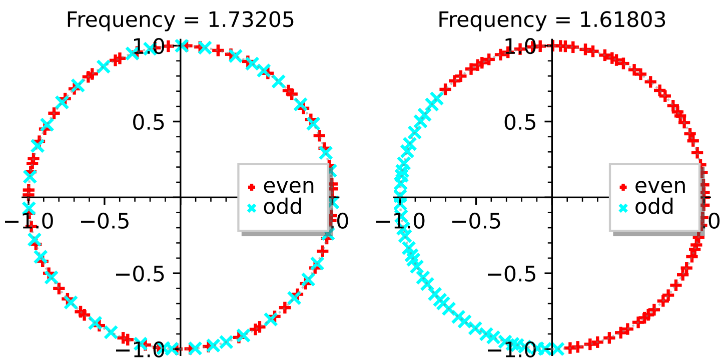
```
sage: G1 = draw_sequence_on_circle(["even","odd"]*10, sqrt(3))
sage: G2 = draw_sequence_on_circle(["even","odd"]*10, 1/2+1/200)
sage: G3 = draw_sequence_on_circle(["even","odd"]*10, 1/2)
sage: G = graphics_array([G1,G2,G3])
```



Concluding that the frequency of even integers is $\frac{1}{2}$ is quite reassuring. Let us now consider a more subtle example based on the Fibonacci word.

Example 3.9. *Below we draw the colored set of points $P_{odd}(\gamma)$ and $P_{even}(\gamma)$ on the circle when $\gamma = \sqrt{3}$ on the left and $\gamma = \frac{1+\sqrt{5}}{2}$ on the right.*

```
sage: F = words.FibonacciWord(['even','odd'])
sage: F[:15]
even,odd,even,even,odd,even,odd,even,even,odd,even,even,odd,even,odd
sage: GFibo1 = draw_sequence_on_circle(F[:100], frequency=sqrt(3))
sage: GFibo2 = draw_sequence_on_circle(F[:100], frequency=(1+sqrt(5))/2)
sage: GFibo = graphics_array([GFibo1,GFibo2])
```



We observe that when the frequency is $\gamma = \frac{1+\sqrt{5}}{2}$, the two sets of colored points belong to two disjoint intervals. This constitutes an experimental proof that the Fibonacci word is the coding of a rotation. As a consequence, to know if a positive integer n is even, it is sufficient to check if $\frac{1+\sqrt{5}}{2} \cdot n \pmod 1$ belongs to a certain interval in the unit circle. We say that the Fibonacci word is the coding of a irrational rotation by two intervals because the n-th symbol of the sequence is obtain by coding the n-th image of a rotation by angle $2\pi\alpha$ according to which of the two intervals it belongs. In this example, the ratio of the length of the two intervals is the golden ratio. In general, every Sturmian sequence is the coding of an irrational rotation where the ratio of the lengths of the two intervals is some positive irrational number.

The same experiment was very useful to study the Jeandel-Rao tilings, see Section 11.1.

Chapter 4

Christoffel words and Markoff theory

In this chapter, we briefly present the notion of Christoffel words, their relation with the convergents of continued fraction expansions and the theory developed by Markoff during his study of quadratic forms. For a deeper reading of this very interesting topic, we suggest the very nice books [71, 240].

4.1 Stern-Brocot tree

The Stern-Brocot tree is a binary tree of rational numbers defined as follows. Every rational number $q \in \mathbb{Q}_{>0}$ has exactly two children in the Stern-Brocot tree: if

$$q = [a_0; a_1, a_2, \dots, a_k] = [a_0; a_1, a_2, \dots, a_k - 1, 1]$$

are the two possible continued fraction expansion of the rational number q, then one child is the number represented by the continued fraction

$$[a_0; a_1, a_2, \dots, a_k + 1]$$

while the other child is represented by the continued fraction

$$[a_0; a_1, a_2, \dots, a_k - 1, 2].$$

One of these children is less than q and this is the left child; the other is greater than q and it is the right child, see Figure 4.1.

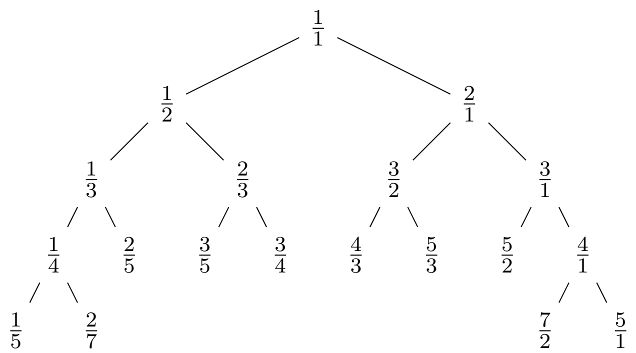


Figure 4.1: The Stern-Brocot tree.

The Stern-Brocot tree is such that every rational number appears exactly once [146, §4.5].

4.2 Christoffel words

Christoffel words were introduced by E. B. Christoffel in his 1875 article [110]. They are words over the alphabet $\{\mathtt{0},\mathtt{1}\}$ that can be defined recursively as follows: $\mathtt{0}$, $\mathtt{1}$ and $\mathtt{01}$ are Christoffel words and if $u, v, uv \in \{\mathtt{0},\mathtt{1}\}^*$ are Christoffel words then uuv and uvv are Christoffel words [71]. This gives the set of Christoffel words the structure of a binary tree, see Figure 4.2. The shortest Christoffel words are:

$$\mathtt{0}, \mathtt{1}, \mathtt{01}, \mathtt{001}, \mathtt{011}, \mathtt{0001}, \mathtt{00101}, \mathtt{01011}, \mathtt{0111}, \mathtt{00001}, \mathtt{0001001}, \mathtt{00100101}, \mathtt{0010101}, \ldots$$

Note that these are also named *lower* Christoffel words.

Every Christoffel word w can be associated to a rational number by the map $w \mapsto \frac{|w|_\mathtt{1}}{|w|_\mathtt{0}}$ counting the ratio between the number of occurrences of the letters $\mathtt{1}$ and $\mathtt{0}$ in the word w. It is known that this maps sends bijectively the binary tree of Christoffel words onto the Stern-Brocot tree of rational numbers [71, Proposition 7.6].

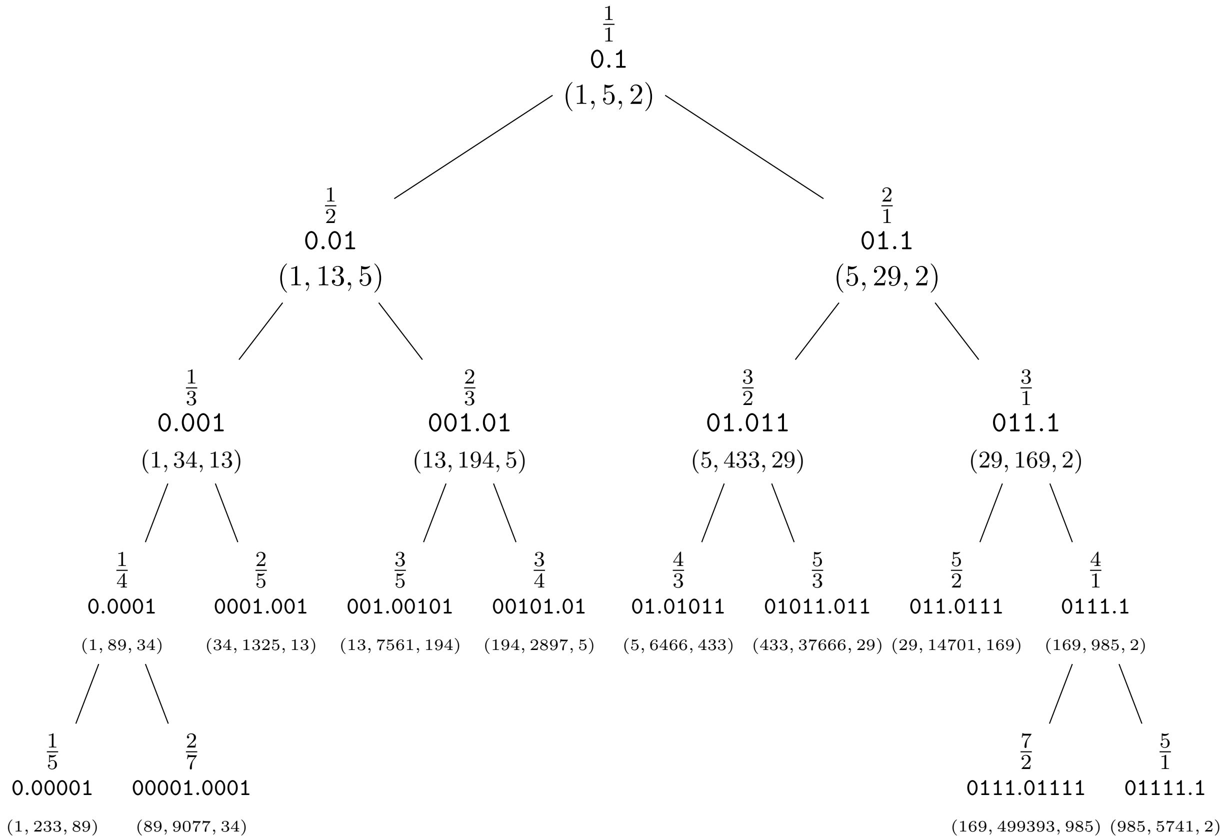


Figure 4.2: The Stern-Brocot tree, the tree of proper Christoffel words and tree of proper Markoff triples merged into a single infinite binary tree.

4.3 Christoffel sequences within periodic Sturmian sequences

Let $\alpha \in [0,1]$ and consider the lower and upper sequences c_α and c'_α given respectively by

$$\begin{array}{rccl} c_\alpha : & \mathbb{Z} & \to & \{\mathtt{0},\mathtt{1}\} \\ & n & \mapsto & \lfloor \alpha(n+1) \rfloor - \lfloor \alpha n \rfloor \end{array} \quad \text{and} \quad \begin{array}{rccl} c'_\alpha : & \mathbb{Z} & \to & \{\mathtt{0},\mathtt{1}\} \\ & n & \mapsto & \lceil \alpha(n+1) \rceil - \lceil \alpha n \rceil. \end{array}$$

When α is rational, the sequences c_α and c'_α are periodic and their period corresponds to Christoffel words [71], see Figure 4.3. More precisely, the shortest periodic pattern and smallest for the

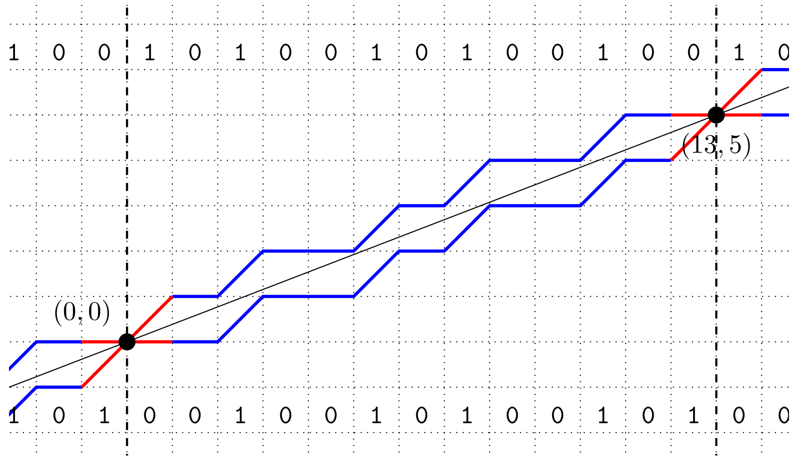


Figure 4.3: The lower and upper sequences c_α and c'_α when $\alpha = 5/13$ are periodic.

lexicographic order of c_α is the lower Christoffel word of slope p/q where p and q are nonnegative coprime integers such that $\alpha = p/(p+q)$.

For example, when $\alpha = 5/13$, the lower sequence c_α has period 0010010100101 which is the lower Christoffel word of slope 5/8 and the upper sequence c'_α has period 1010010100100 which is the upper Christoffel word of slope 5/8. When α is irrational, then c_α and c'_α are not periodic. The restrictions of c_α and c'_α to $\mathbb{Z}_{\geq 1}$ are equal and correspond to the well-known one-sided characteristic Sturmian sequence of slope α [216]. In this work, we consider biinfinite sequences as opposed to one-sided sequences. Over the domain $\mathbb{Z}$, we say that c_α and c'_α are respectively the **lower** and **upper characteristic Sturmian sequences of slope** α whenever α is irrational.

Sturmian sequences have many equivalent definitions, for example, in terms of aperiodic balanced sequences [216], irrational rotations [38, 202], factor complexity [113] or return words [274]. On the other hand, Christoffel words also have many equivalent definitions, including 14 characterizations listed in [70], see also [72, 71]. A recent book [240] gathers exhaustively the combinatorial properties of Christoffel words and uses them to prove two important theorems of Markoff for Diophantine approximations and quadratic forms [207].

4.4 Pirillo's theorem

A theorem of Pirillo [230] provides a nice characterization of Christoffel words of slope p/q where p and q are positive coprime integers. If p and q are nonzero, the lower Christoffel word of slope p/q starts with letter 0 and ends with letter 1, so it can be written as $0m1$ for some finite word $m \in \{0,1\}^*$ and the corresponding upper Christoffel word is $1m0$. Pirillo gave the following elegant characterization of Christoffel words. Recall that two words $w, w' \in \{0,1\}^*$ are **conjugate** if there exists $u, v \in \{0,1\}^*$ such that $w = uv$ and $w' = vu$.

Theorem 4.1 (Pirillo's theorem [230]). *The word $0m1 \in \{0,1\}^*$ is a lower Christoffel word if and only if $0m1$ and $1m0$ are conjugate.*

Pirillo's theorem is illustrated in Figure 4.4. We observe that the conjugacy of $0m1$ into $1m0$ is done via their factorization into a product of two palindromes: $0m1 = 00100 \cdot 10100101$ and

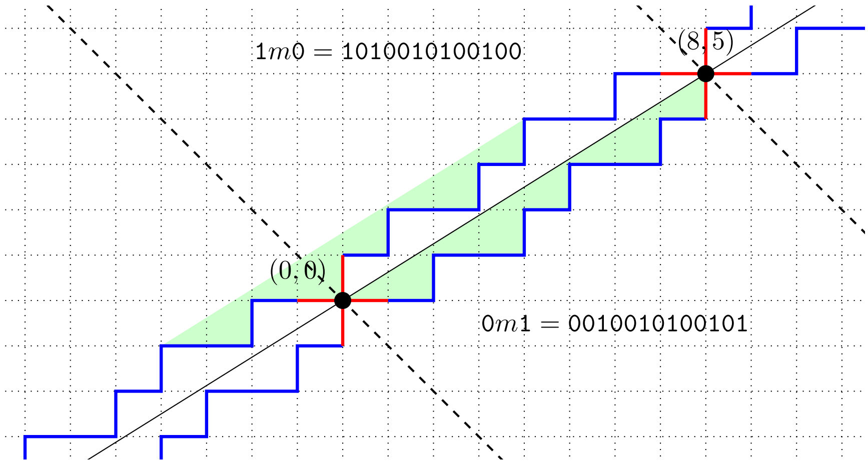


Figure 4.4: Pirillo's theorem characterizes Christoffel words: the lower Christoffel word $\mathtt{0}m\mathtt{1} \in \{\mathtt{0},\mathtt{1}\}^*$ is conjugate to the upper Christoffel word $\mathtt{1}m\mathtt{0}$.

$\mathtt{1}m\mathtt{0} = \mathtt{10100101} \cdot \mathtt{00100}$. The factorization of $\mathtt{0}m\mathtt{1}$ as a product of two palindromes and the fact that the central word m is a palindrome [71, Prop. 4.2] is also a characterization of Christoffel words, see [204] and [240, Theorem 12.2.10].

With Christophe Reutenauer, we used Theorem 4.1 to provide a d-dimensional extension of Christoffel words [14]. Since then, I was wondering if Pirillo's theorem could be extended to characterize Sturmian sequences, but it was unclear which notion would replace conjugacy. An open question asked by Sebastian Barbieri during his postdoctorate with us in Bordeaux (funded by our ANR CODYS) coming from an earlier work of his in Vancouver with Brian Marcus on Gibbs theory turned out to be the notion which was needed: indistinguishable asymptotic pairs. It provides a new characterization of Sturmian sequences which works in higher dimensions as well. This is expanded in Chapter 8 and Chapter 13.

4.5 Markoff numbers and the Markoff injectivity conjecture

A Markoff triple is a positive solution of the Diophantine equation $x^2 + y^2 + z^2 = 3xyz$ [207, 206]. Markoff triples can be defined recursively as follows: $(1,1,1)$, $(1,2,1)$ and $(1,5,2)$ are Markoff triples and if (x,y,z) is a Markoff triple with $y \geq x$ and $y \geq z$, then $(x, 3xy - z, y)$ and $(y, 3yz - x, z)$ are Markoff triples. This gives to the set of Markoff triple, the structure of a infinite binary tree, see Figure 4.2. A list of small Markoff numbers (elements of a Markoff triple) is

$$1, 2, 5, 13, 29, 34, 89, 169, 194, 233, 433, 610, 985, 1325, 1597, 2897, 4181, \ldots$$

referenced as sequence A002559 in OEIS [221].

It is known that each Markoff number can be expressed in terms of a Christoffel word. More precisely, let μ be the monoid homomorphism $\{\mathtt{0},\mathtt{1}\}^* \to \mathrm{GL}_2(\mathbb{Z})$ defined by

$$\mu(\mathtt{0}) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \mu(\mathtt{1}) = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}.$$

Each Markoff number is equal to $\mu(w)_{12}$ for some Christoffel word w [239], where M_{12} denotes the element above the diagonal in a matrix $M = \left(\begin{smallmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z})$.

For example, the Markoff number 194 is associated with the Christoffel word $\mathtt{00101}$ as it is the entry at position $(1,2)$ in the matrix

$$\mu(\mathtt{00101}) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 463 & 194 \\ 284 & 119 \end{pmatrix}.$$

Whether the map $w \mapsto \mu(w)_{12}$ provides a bijection between Christoffel words and Markoff numbers is a question (also known as the uniqueness conjecture and stated differently in [138]) that has remained open for more than 100 years [27]. The conjecture can be expressed in terms of the injectivity of the map $w \mapsto \mu(w)_{12}$ [240, §3.3].

Conjecture 4.2 (Markoff Injectivity Conjecture)**.** *The map $w \mapsto \mu(w)_{12}$ is injective on the set of Christoffel words.*

4.6 The Markoff property

It is worth recalling that balanced sequences appeared in the work of Markoff himself [206, 208] under an equivalent condition, called Markoff property (M) in [238].

Definition 4.3. *[238] We say that a biinfinite word $s \in \{\mathtt{0},\mathtt{1}\}^{\mathbb{Z}}$ satisfies the* Markoff property *if for any factorization $s = uxyv$, where $\{x, y\} = \{\mathtt{0},\mathtt{1}\}$, one has*

- *either $\widetilde{u} = v$,*
- *or there is a factorization $u = u'ym$, $v = \widetilde{m}xv'$.*

The Markoff property is related to the Markoff spectrum. Let $U = (a_i)_{i\in\mathbb{Z}}$ be a biinfinite sequence such that a_i are positive integers. For $i \in \mathbb{Z}$, let

$$\lambda_i(U) = a_i + [0; a_{i+1}, a_{i+2}, \dots] + [0; a_{i-1}, a_{i-2}, \dots].$$

The *Markoff supremum of U* is

$$M(U) = \sup_{i\in\mathbb{Z}} \lambda_i(U).$$

Two results of Markoff can be stated in terms of Christoffel words and balanced sequences as follows where σ is the substitution from $\{\mathtt{0},\mathtt{1}\}^*$ to $\{1,2\}^*$ defined by $\mathtt{0} \mapsto 11$ and $\mathtt{1} \mapsto 22$. It provides an equivalence between sequences satisfying the Markoff property and sequences of positive integers such that the Markoff supremum is at most 3. The equivalence between sequences satisfying the Markoff property and balanced sequences was not proved by Markoff himself: it was stated without proof in [116] and a proof was provided in [238].

Theorem 4.4 (Markoff)**.** *[238, Theorem 3.1 and 7.1] Let $s \in \{\mathtt{0},\mathtt{1}\}^{\mathbb{Z}}$ be a biinfinite word. The following conditions are equivalent:*

- *s satisfies the Markoff property,*
- *s is balanced,*
- *$M(\sigma(s)) \leq 3$.*

The Markoff supremum of a purely periodic balanced sequence can be computed from the Markoff number associated to the Christoffel word which is a period of the sequence.

Theorem 4.5 (Markoff)**.** *[240, Theorem 6.2.1] Let w be some lower Christoffel word associated with Markoff number $m = \mu(w)_{12}$. Let s be the biinfinite sequence ${}^{\infty}\sigma(w)^{\infty}$. Then $M(s) = \sqrt{9 - \frac{4}{m^2}}$.*

4.7 Four classes of balanced sequences

Biinfinite balanced sequences can be split into four different types of sequences. Reutenauer proposed the following refinement of the Markoff property [238] which was restated in [145] as follows. If a biinfinite sequence $u \in \{\mathtt{0},\mathtt{1}\}^{\mathbb{Z}}$ satisfies the Markoff property, then it falls into exactly one of the following classes:

(M_1) u cannot be written as $u = \widetilde{p}xyp$ where $\{x,y\} = \{\mathtt{0},\mathtt{1}\}$ and the lengths of the Christoffel words occurring in u are bounded;

(M_2) u cannot be written as $u = \widetilde{p}xyp$ where $\{x,y\} = \{\mathtt{0},\mathtt{1}\}$ and the lengths of the Christoffel words occurring in u are unbounded;

(M_3) u has a unique factorization $u = \widetilde{p}xyp$ where $\{x,y\} = \{\mathtt{0},\mathtt{1}\}$;

(M_4) u has at least two factorizations $u = \widetilde{p}xyp$ where $\{x,y\} = \{\mathtt{0},\mathtt{1}\}$.

Morse and Hedlund gave a classification of balanced biinfinite sequences into three classes (periodic, Sturmian, skew) [216]. Since the Sturmian case naturally splits into two, Reutenauer proposed the following four classes $(MH_i)_{i\in\{1,2,3,4\}}$ and proved their equivalence with the (M_i).

Theorem 4.6. *[238, Theorem 6.1] Let $u \in \{\mathtt{0},\mathtt{1}\}^{\mathbb{Z}}$ be a balanced sequence. For every $i \in \{1,2,3,4\}$, u satisfies (M_i) if and only if u satisfies (MH_i) where*

(MH_1) *u is a purely periodic word ${}^{\infty}w^{\infty}$ for some Christoffel word w,*

(MH_2) *u is a generic aperiodic Sturmian word, i.e., $u = s_{\alpha,\rho} = s'_{\alpha,\rho}$ for some $\alpha \in [0,1]\setminus\mathbb{Q}$ and $\rho \in \mathbb{R}$ such that $\mathbb{Z} \cap \alpha\mathbb{Z} + \rho = \varnothing$.*

(MH_3) *u is a characteristic aperiodic Sturmian word, i.e., $u = s_{\alpha,\rho}$ or $u = s'_{\alpha,\rho}$ for some $\alpha \in [0,1]\setminus\mathbb{Q}$ and $\rho \in \mathbb{R}$ such that $\mathbb{Z} \cap \alpha\mathbb{Z} + \rho \neq \varnothing$.*

(MH_4) *u is an ultimately periodic word but not purely periodic, i.e., $u = \cdots xxyxx \cdots$ or $u = \cdots(ymx)(ymx)(ymy)(xmy)(xmy)\cdots$ where $\{x,y\} = \{\mathtt{0},\mathtt{1}\}$ and $\mathtt{0}m\mathtt{1}$ is a Christoffel word.*

Chapter 5

Some Sturmian extensions

5.1 Tribonacci example

A generalization of the result of Morse and Hedlund was provided by Rauzy for a single example [237]. Based on the right-infinite sequence often called the Tribonacci word

$$T = 12131211213121213121121312131211213121 21...$$

which is fixed by $1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 1$, Rauzy proved that the system (X_T, σ), where σ is the shift action, is measurably conjugate to the toral rotation $(\mathbb{T}^2, x \mapsto x + (\beta^{-1}, \beta^{-2}))$ where β is the real root of $x^3 - x^2 - x - 1$, the characteristic polynomial of the incidence matrix of the substitution. The coding of the toral translation is made through the partition into three parts of a fundamental domain of $\mathbb{T}^2$ known as the Rauzy fractal [132, §7.4]. Proving that this holds for all Pisot substitution is known as the Pisot Conjecture [28], an important and still open question.

Finding further generalizations was coined the term of *Rauzy program* in [81], a survey divided into three parts: the *good* coding of k-interval exchange transformations (IETs); the *bad* coding of a rotation on $\mathbb{T}^k$; and the *ugly* coding of two rotations on $\mathbb{T}^k$ for $k = 1$. The IETs are the good part since they behave well with induced transformations and admit continued fraction algorithms [236, 271, 277, 47]. The bad part was much improved since then with various recent results using multidimensional continued fraction algorithms including Brun's algorithm [99] which provides measurable-theoretic conjugacy with symbolic systems for almost every toral rotations on $\mathbb{T}^2$ [83, 268]. As the authors wrote in [81], the term ugly "*refers to some esthetic difficulties in building two-dimensional sequences by iteration of patterns*". Indeed, digital planes [40, 39, 81, 74, 76] are typical objects that are described by the coding of two rotations on $\mathbb{T}^1$ and they are not built by rectangular shaped substitutions.

5.2 Markov partitions for automorphisms of the torus

While Morse-Hedlund's theorem deals with the coding of irrational rotations, other kinds of dynamical systems admit a symbolic representation. Hyperbolic automorphisms of the torus are one example [200, 177]. Suppose that one starts at some position $v \in \mathbb{R}^2$ and moves according to the successive images under the application of the map $v \mapsto Mv$ with $M = \left(\begin{smallmatrix}1 & 1\\ 1 & 0\end{smallmatrix}\right)$ as shown in Figure 5.1. The map $v \mapsto Mv$ is an automorphism of $\mathbb{R}^2/\mathbb{Z}^2$ which is *hyperbolic* since M has no eigenvalue of modulus 1. It allows one to code the orbit $(M^k v)_{k\in\mathbb{Z}}$ as a sequence in $\{A, B, C\}^{\mathbb{Z}}$ according to a well-chosen partition $\mathcal{P}$ of a fundamental domain of $\mathbb{R}^2/\mathbb{Z}^2$ into three rectangles indexed by letters in the set $\{A, B, C\}$. In Figure 5.1, the positive orbit $(M^k v)_{k\geq 0}$ of the starting point $v = \left(\frac{-7}{10}, \frac{14}{10}\right)^T$ is coded by the sequence $CBABABCB\ldots$ which avoids the patterns in $\mathcal{F} = \{AA, BB, CC, CA\}$. We denote the set of obtained sequences as $\mathcal{X}_{\mathcal{P},M}$. The partition of $\mathbb{R}^2/\mathbb{Z}^2$ is a Markov partition for the automorphism because it has two important properties [200, §6.5]:

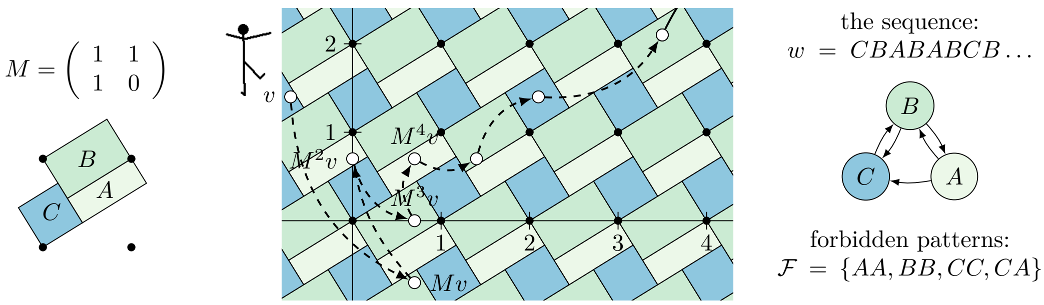


Figure 5.1: The automorphism of $\mathbb{R}^2/\mathbb{Z}^2$ defined as $v \mapsto Mv$ admits a Markov partition.

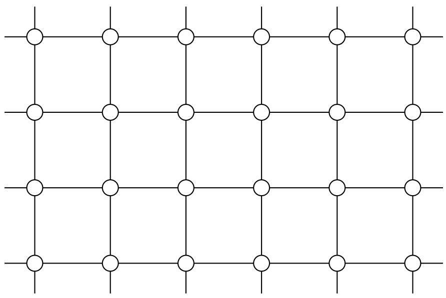
(C1) every sequence in $\mathcal{X}_{\mathcal{P},M}$ is obtained from a unique starting point in $\mathbb{R}^2/\mathbb{Z}^2$,

(C2') the set $\mathcal{X}_{\mathcal{P},M}$ is a shift of finite type (SFT), i.e., there exists a finite set $\mathcal{F}$ of patterns such that $\mathcal{X}_{\mathcal{P},M}$ is the set of sequences in $\{A,B,C\}^{\mathbb{Z}}$ which avoids the patterns in $\mathcal{F}$.

Such Markov Partitions exist for all hyperbolic automorphisms of the torus [26, 258, 92] and various kinds of diffeomorphisms [91], see also [176, 25, 175]. Surprisingly, it turns out that Markov partitions also exist for toral $\mathbb{Z}^2$-rotations R and 2-dimensional subshifts $\mathcal{X}_{\mathcal{P},R}$.

5.3 From two-sided sequences to 2-dimensional configurations

One of our goal is to extend the behavior of Sturmian sequences beyond the 1-dimensional case by considering d-dimensional configurations. We say that a *configuration* is an assignment of colored beads from a finite set $\mathcal{A}$ to every coordinate of the lattice $\mathbb{Z}^d$. Are there rules describing how to place colored beads in a configuration in such a way that it encodes rotations on a higher dimensional torus?



This is related to a question of Adler: "*how and to what extent can a dynamical system be represented by a symbolic one*" [25]. The kind of dynamical system we consider are toral $\mathbb{Z}^d$-rotations, that is, $\mathbb{Z}^d$-actions by rotations on a torus.

When $d = 1$, the answer is given in terms of Sturmian sequences and factor complexity. While Berthé and Vuillon [87] considered the coding of $\mathbb{Z}^2$-rotations on the 1-dimensional torus, we consider $\mathbb{Z}^d$-rotations on the d-dimensional torus.

Can an answer to the question of Adler when $d = 2$ be made in terms of sets of configurations avoiding a finite set of forbidden patterns known as *subshifts of finite type* and more precisely in terms of aperiodic tilings by Wang tiles?

Such a possibility contrasts with the one-dimensional case, since Sturmian sequences can not be described by a finite set of forbidden patterns (a one-dimensional shift of finite type is nonempty if and only if it has a periodic point [200, §13.10]).

We explore this in more details in Part IV.

Chapter 6

Aperiodic tilings

6.1 Aperiodic sets of Wang tiles

The study of aperiodic order [148, 51] gained a lot of interest since the discovery in 1982 of quasicrystals by Shechtman [255] for which he was awarded the Nobel Prize in Chemistry in 2011. The first known aperiodic structure was based on the notion of Wang tiles. *Wang tiles* can be represented as unit square with colored edges, see Figure 6.4.

Given a finite set of Wang tiles $\mathcal{T}$, we consider *tilings* of the Euclidean plane using arbitrarily many translated (but not rotated) copies of the tiles in $\mathcal{T}$. Tiles are placed on the integer lattice points of the plane with their edges oriented horizontally and vertically. The tiling is *valid* if every pair of contiguous edges have the same color. Deciding if a set of Wang tiles admits a valid tiling of the plane is a difficult question known as the *domino problem*. Answering a question of Wang [276], Berger proved that the domino problem is undecidable [67] using a reduction to the halting problem of Turing machines. As noticed by Wang, if every set of Wang tiles that admits a valid tiling of the plane would also admit a periodic tiling, then the domino problem would be decidable. As a consequence, there exist aperiodic sets of Wang tiles. A set $\mathcal{T}$ of Wang tiles is called *aperiodic* if there exists a valid tiling of the plane with the tiles from $\mathcal{T}$ and none of the valid tilings of the plane with the tiles from $\mathcal{T}$ is invariant under a nonzero translation.

Berger constructed an aperiodic set made of 20426 Wang tiles [67], later reduced to 104 by himself [66] and further reduced by others [180, 243]. Small aperiodic sets of Wang tiles include Ammann's 16 tiles [148, p. 595], Kari's 14 tiles [167] and Culik's 13 tiles [114]. The search for the smallest aperiodic set of Wang tiles continued until Jeandel and Rao proved the existence of an aperiodic set $\mathbb{T}_0$ of 11 Wang tiles and that no set of Wang tiles of cardinality ≤ 10 is aperiodic [163]. Thus their set, shown in Figure 6.4, is a smallest possible set of aperiodic Wang tiles.

6.2 The Ammann set of 16 Wang tiles

Noteworthy examples of aperiodic tilings were discovered by Ammann, a mathematician's hobbyist, in the 1970's and 1980's. Although Ammann had a great influence on the early developments made in the theory of aperiodic tilings, he published a single article in his life [34]. He learned about Penrose tilings while reading Martin Gardner in popular science journals exchanges letters with him [253]. From that, he went on to discover new aperiodic shapes and how to put markings on them (now called Ammann bars) in a very elegant way. As we learn from the nice article of Marjorie Senechal [253], Robert Ammann was having symptoms similar to autism which partly explains the mystery surrounding him and his discoveries. It is a chance that some of his great intuitions were shared to us as they may have not delivered all their secrets.

One of shapes discovered by Ammann is called the Ammann A2 L-shaped tiles [34] (also studied in [29, 126]); see Figure 6.1.

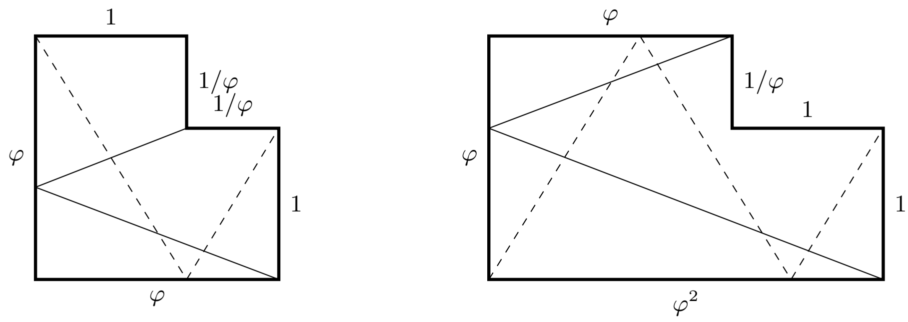


Figure 6.1: Two shapes belonging to the Ammann A2 family. The matching conditions are given by what are called Ammann bars appearing as dashed and solid lines in the interior of the tiles and which must continue straight across the edges of the tiling. This is a reproduction of Figure 10.4.1 from [148]. See also Figure 12 from [29].

In a tiling of the plane by the two shapes shown in Figure 6.1 respecting the matching condition, there appear what are called Ammann bars. In this case, the slopes of the Ammann bars takes four different values: two slope values for the dashed Ammann bars and two slope values for the solid Ammann bars. As explained in [148, p.594–598], the solid bars can be regarded as the edges of a new tiling by rhombs and parallelograms, for which the dashed bars can be regarded as markings on the tiles specifying the matching conditions. Sixteen parallelogram tiles arise from this construction which can be recoded as 16 Wang tiles. The Ammann set of 16 Wang tiles are shown in Figure 6.2. The tiling of a rectangle with the tiles is shown in Figure 6.3.

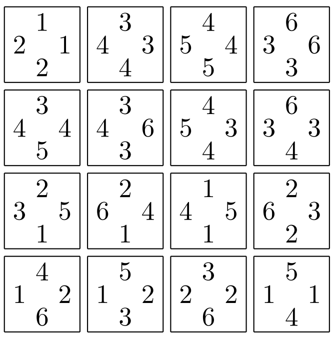
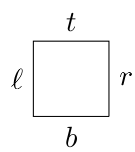


Figure 6.2: The Ammann aperiodic set of 16 Wang tiles [148, p.595, Figure 11.1.13].

6.3 The Kari–Culik sets of Wang tiles

The smallest sets of aperiodic Wang tiles until 2015 were discovered by Kari and Culik in 1996. Kari [167] proved that a well-chosen set of 14 Wang tiles admits tilings of the plane, and that none of them is periodic. The proof that they are not periodic is cleverly short. It is based on an arithmetic interpretation of the edge labels of the Wang tiles. The tiles have labels $r, t, \ell, b \in \mathbb{Q}$ satisfying an equation



$$qt + \ell = b + r \tag{6.1}$$

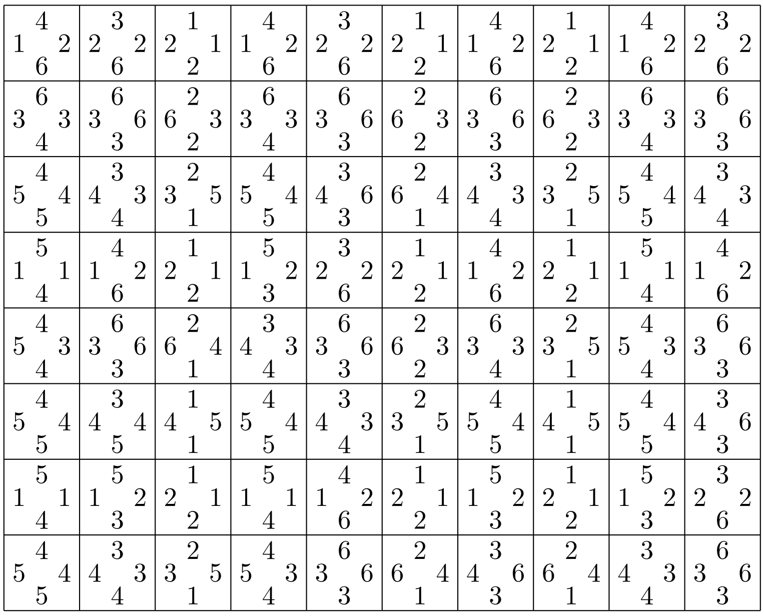


Figure 6.3: Valid tiling of a rectangle with Ammann tiles.

for some $q \in \mathbb{Q}$. We may interpret the Wang tile as a computation (the multiplication by q) with value t as an input and b as an output. The value ℓ is a carry input on the left and r is a carry output on the right. Kari [167] proposed a set of four tiles satisfying (6.1) with $q = 2$ and ten tiles with $q = \frac{2}{3}$. The proof of the non-existence of a periodic tiling with those 14 tiles uses the fact that the equation $2^m 3^n = 1$ has only one solution over the integers ($m = n = 0$). Based on the same idea, Culik [114] proposed a smaller aperiodic set of 13 tiles (four tiles satisfying (6.1) with $q = 3$ and nine tiles with $q = \frac{1}{2}$). Note that generalizations of Kari–Culik tilings exist [128] and that further results were obtained about their entropy [125] and on a minimal subsystem [256].

Among aperiodic tilings of the plane by Wang tiles, Kari and Culik sets seem like outliers. The aperiodicity of Penrose tiles [228], Berger tiles [67], Robinson tiles [243], Knuth tiles [180], Ammann tiles [148, 34] can be explained by the hierarchical decomposition of their tilings. Often, aperiodic tilings have a self-similar structure [264, 263, 233, 232, 31] and this is the case for recently discovered aperiodic geometrical tiles [262, 260, 259]. However, Kari and Culik tilings have positive entropy. Thus, they are not self-similar and do not possess a hierarchical decomposition [125]. Note that the absence of hierarchical decomposition also follows from a cylindricity argument proposed by Thierry Monteil and explained in [125, §4.2]. Moreover, except some extensions of Kari and Culik sets [128, §6], no other known aperiodic sets of tiles satisfy equations explaining their non-periodicity.

6.4 Jeandel-Rao Wang tiles

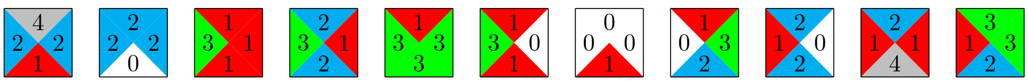


Figure 6.4: The aperiodic set $\mathcal{T}_0$ of 11 Wang tiles discovered by Jeandel and Rao in 2015 [163].

For almost twenty years, the Kari and Culik sets of Wang tiles were the smallest known aperiodic sets of Wang tiles. In 2015, Jeandel and Rao performed an exhaustive search on all sets of Wang tiles of cardinality up to 11 [163] and proved that sets of Wang tiles of cardinality at most 10 either do not tile the plane or tile the plane and one of the valid tilings is periodic. Moreover, they provided a list of 33 sets of 11 Wang tiles considered to be candidates for being aperiodic. One of candidates was intriguing because Fibonacci numbers appeared in the structure of the transducers involved in the computation of valid tilings. Jeandel and Rao focused on the intriguing candidate, shown in Figure 6.4, and they proved it to be aperiodic. The set of valid configurations over these 11 tiles forms a subshift that we call the Jeandel–Rao Wang shift. An equivalent geometric representation of their set of 11 tiles is shown in Figure 6.5. A rectangular pattern with Jeandel-Rao tiles is shown in Figure 6.6.

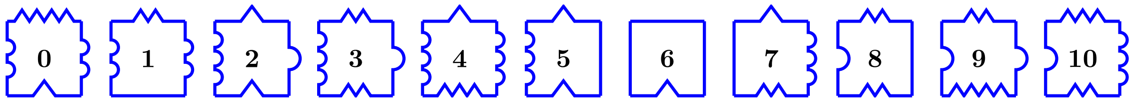


Figure 6.5: Jeandel-Rao tiles can be encoded into a set of equivalent geometrical shapes in the sense that every tiling using Jeandel-Rao tiles can be transformed into a unique tiling with the corresponding geometrical shapes and vice versa.

Recall that we consider Wang tilings from the point of view of symbolic dynamics [242]. While a tiling by a set of Wang tiles $\mathcal{T}$ is a tiling of the plane $\mathbb{R}^2$ whose validity is preserved by translations of $\mathbb{R}^2$ (leading to the notion of *hull*, see [51]), we prefer to consider maps $\mathbb{Z}^2 \to \mathcal{T}$, that we call *configurations*, whose validity is preserved by translations of $\mathbb{Z}^2$. The set $\Omega_\mathcal{T}$ of all valid configurations $\mathbb{Z}^2 \to \mathcal{T}$ is called a *Wang shift* as it is closed under the shift σ by integer translates. The passage from Wang shifts ($\mathbb{Z}^2$-actions) to Wang tiling dynamical systems ($\mathbb{R}^2$-action) can be made with the 2-dimensional suspension of the former as in the classical construction of a "flow under a function" in Ergodic Theory, see [241].

The aperiodicity of the Jeandel-Rao set of 11 Wang tiles follows from the decomposition of tilings as horizontal strips of height 4 or 5 (see Figure 6.6). Using the representation of Wang tiles by transducers, Jeandel and Rao proved that the language of sequences describing the heights of consecutive horizontal strips in the decomposition is exactly the language of the Fibonacci word on the alphabet $\{4, 5\}$ [163]. This proves the absence of any vertical period in every tiling with Jeandel-Rao tiles. This is enough to conclude aperiodicity in all directions, see [51, Prop. 5.9]. The presence of the Fibonacci word in the vertical structure of Jeandel-Rao tilings was a first hint that Jeandel-Rao tilings are related to irrational rotations on a torus.

6.5 Other aperiodic sets of tiles

There are many other aperiodic sets of tiles often discovered by amateur mathematicians like an aperiodic hexagonal tile by Joan Taylor [262] or the recent hat monotile discovered by David Smith [260].

Smith and coauthors presented a single shape , a 13-edge polygon called the hat, whose isometric copies tile the plane but never periodically; see Figure 6.7. The hat monotile attracted a lot of attention [261, 50, 30]. Two months later the same authors discovered another aperiodic tile called Spectre which does not need its mirror image to tile the plane [259]. Tilings by the Spectre are not all combinatorially equivalent to tilings by the hat: some are periodic (if the reflected tile is allowed). But every tiling by the hat tile is combinatorially equivalent to some Spectre tiling.

```
sage: from slabbe.arXiv_1903_06137 import jeandel_rao_tiles
sage: from slabbe.arXiv_1903_06137 import geometric_edges_shapes
sage: T0 = jeandel_rao_tiles()
sage: tiling = T0.solver(20,20).solve(solver="glucose")
sage: draw_H, draw_V = geometric_edges_shapes()
sage: tikz = tiling.tikz(draw_H=draw_H,draw_V=draw_V, id=True, label=False,
....:                    scale="1,very␣thick", font=r"\bfseries")
```

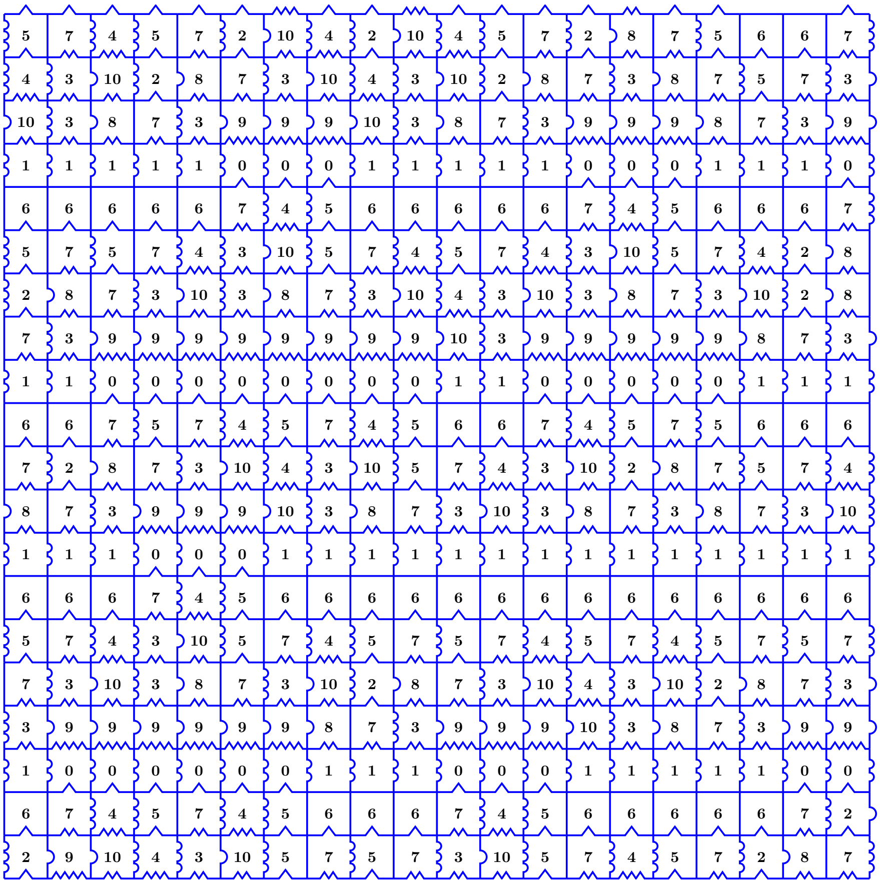


Figure 6.6: A 20×20 Jeandel-Rao tiling found using Glucose SAT solver. There are two types of rows: rows that contain tile 0 or tile 1 and rows that do not contain tile 0 or 1. It was proved by Jeandel and Rao that the distance between consecutive rows of 0/1 is either 4 or 5.

Figure 6.7: Freshly laser-cut copies of the hat, made at the EirLab of Bordeaux INP, January 2025.

Chapter 7

Cut and project schemes

Since the contribution of N. G. de Bruijn [98], we know that Penrose tilings are obtained as the projection of discrete surfaces in a 5-dimensional space onto a 2-dimensional plane. This major discovery was the first time that some aperiodic tilings were expressed as the projection of a higher dimensional cubic structure. However, this construction was formalized earlier in a more general setup by Meyer [211] as noticed later by Lagarias [186] and Moody [213]. This construction is known as model set (or cut and project sets) within cut and project schemes [51].

In one dimension, the fact that Sturmian sequences are codings of rotations implies that they can be seen as model sets of cut and project schemes, see [58, 51]. Other typical examples include Ammann-Beenker tilings [55] and Taylor-Socolar aperiodic hexagonal tilings for which Lee and Moody gave a description in terms of model sets [192]. More recently, it was proved that the aperiodic tilings generated by the Hat and Spectre monotile [260, 259] are described by 4-to-2 cut and project schemes [50, 49, 48].

In this chapter, we introduce cut and project schemes and model sets following the notation used in the book [51]. Classical cut and project schemes turned out to be too restrictive in order to describe properly the Jeandel-Rao tilings by Wang tiles (which can be seen as configurations over $\mathbb{Z}^2$) and multidimensional Sturmian configurations over $\mathbb{Z}^d$. In these works, we needed to introduced a relaxed definition of cut and project schemes allowing degenerate cases.

The first section introduce the classical definition where the projection into the physical space is injective when restricted to the lattice. The second section introduces a notion of degenerate cut and project scheme where the injectivity condition is relaxed, but still allowing the star map to be defined. In both sections, we illustrates the notions in the case of Sturmian sequences. Degenerate cut and project schemes are used in this thesis to describe Jeandel-Rao tilings, see Section 11.3, and multidimensional Sturmian configurations (Section 13.3).

7.1 Cut and project schemes

We recall from [51, §7.2] the definition of cut and project scheme and we reuse their notation.

Definition 7.1. *A* ***cut and project scheme*** *(CPS) is a triple* $(\mathbb{R}^d, H, \mathcal{L})$ *with a (compactly generated) locally compact Abelian group (LCAG)* H*, a lattice* $\mathcal{L}$ *in* $\mathbb{R}^d \times H$ *and the two natural projections* $\pi : \mathbb{R}^d \times H \to \mathbb{R}^d$ *and* $\pi_{int} : \mathbb{R}^d \times H \to H$*, subject to the conditions that* $\pi|_{\mathcal{L}}$ *is injective and that* $\pi_{int}(\mathcal{L})$ *is dense in* H*.*

A CPS is called **Euclidean** when $H = \mathbb{R}^m$ for some $m \in \mathbb{N}$. Such a Euclidean CPS is said to be $(m + d)$**-to-**d.

A general CPS is summarized in the following diagram.

$$\begin{array}{ccccc} \mathbb{R}^d & \xleftarrow{\;\pi\;} & \mathbb{R}^d \times H & \xrightarrow{\;\pi_{\text{int}}\;} & H \\ \cup & & \cup & & \cup \text{ dense} \\ \pi(\mathcal{L}) & \xleftarrow{\;1-1\;} & \mathcal{L} & \longrightarrow & \pi_{\text{int}}(\mathcal{L}) \\ \| & & & & \| \\ L & & \xrightarrow{\qquad\star\qquad} & & L^\star \end{array}$$

The image is denoted $L = \pi(\mathcal{L})$. Since for a given CPS, π is a bijection between $\mathcal{L}$ and L, there is a well-defined mapping $\star : L \to H$ given by

$$x \mapsto x^\star := \pi_{\text{int}} \Big((\pi|_{\mathcal{L}})^{-1}(x) \Big)$$

where $(\pi|_{\mathcal{L}})^{-1}(x)$ is the unique point in the set $\mathcal{L} \cap \pi^{-1}(x)$. This mapping is called the **star map** of the CPS. The $\star$-image of L is denoted by $L^\star$. The set $\mathcal{L}$ can be viewed as a diagonal embedding of L as

$$\mathcal{L} = \{(x, x^\star) \mid x \in L\}.$$

For a given CPS $(\mathbb{R}^d, H, \mathcal{L})$ and a (general) set $A \subset H$,

$$\curlywedge(A) := \{x \in L \mid x^\star \in A\}$$

is the projection set within the CPS. The set A is called its **acceptance set**, **window** or **coding set**.

Definition 7.2. *If $A \subset H$ is a relatively compact set with non-empty interior, the projection set $\curlywedge(A)$, or any translate $t + \curlywedge(A)$ with $t \in \mathbb{R}^d$, is called a **model set**.*

A model set is termed **regular** when $\mu_H(\partial A) = 0$, where μ_H is the Haar measure of H. If $L^\star \cap \partial A = \varnothing$, the model set is called **generic**. If the window is not in a generic position (meaning that $L^\star \cap \partial A \neq \varnothing$), the corresponding model set is called **singular**.

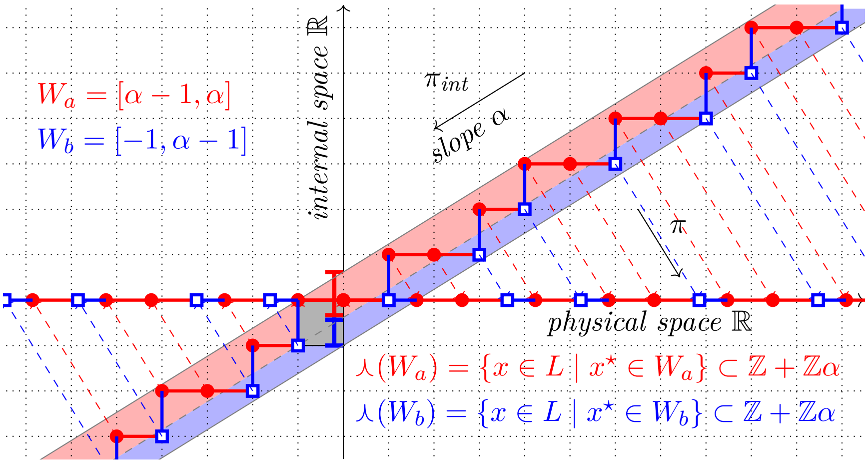
The shape of the acceptance set A is important and implies structure on the model set $\Lambda = t + \curlywedge(A)$. For example, if A is relatively compact, Λ has finite local complexity and thus also is uniformly discrete; if $A^\circ \neq \varnothing$, Λ is relatively dense. If Λ is a model set, it is also a Meyer set, [51, Prop. 7.5]. For regular model set $\Lambda = \curlywedge(A)$ with a compact window $A = \overline{A^\circ}$, it is known [51, Theorem 7.2] that the points $\{x^\star \mid x \in \Lambda\}$ are uniformly distributed in A.

Linear repetitivity of model sets is an important notion. Recall that a Delone set $Y \subseteq \mathbb{R}^d$ is called **linearly repetitive** if there exists a constant $C > 0$ such that, for any $r \geq 1$, every patch of size r in Y occurs in every ball of diameter Cr in $\mathbb{R}^d$. It was shown by Lagarias and Pleasants in [187, Theorem 6.1] that linear repetitivity of a Delone set implies the existence of strict uniform patch frequencies, equivalently the associated dynamical system on the hull of the point set is strictly ergodic (minimal and uniquely ergodic). As a consequence [187, Cor. 6.1], a linearly repetitive Delone set X in $\mathbb{R}^n$ has a unique autocorrelation measure γ_X. This measure γ_X is a pure discrete measure supported on $X - X$. In particular X is diffractive. A characterization of linearly repetitive model sets $\curlywedge(A)$ for cubical acceptance set A was recently proved by Haynes, Koivusalo and Walton [155].

Example 7.3. *In this example, we illustrate the cut and project scheme associated to a Sturmian sequence of slope α with $\alpha > 0$. The cut and project scheme is given by the 5-uple $(H, K, \mathcal{L}, \pi, \pi_{int})$ where $H = \mathbb{R}$, $K = \mathbb{R}$, $\mathcal{L} = \mathbb{Z}^2$ and*

$$\begin{array}{cccc} \pi : & \mathbb{R}^2 & \to & \mathbb{R} \\ & (x_1, x_2) & \mapsto & x_1 + \alpha x_2 \end{array} \quad \textit{and} \quad \begin{array}{cccc} \pi_{\text{int}} : & \mathbb{R}^2 & \to & \mathbb{R} \\ & (x_1, x_2) & \mapsto & -\alpha x_1 + x_2 \end{array}$$

The window $W = [-1, \alpha)$ defines the model set $\curlywedge(W) \subset \mathbb{Z} + \mathbb{Z}\alpha$. Moreover, the window W is naturally partitioned into the two sets $W_a = [\alpha - 1, \alpha)$ and $W_b = [-1, \alpha - 1)$. They define the subsets $\curlywedge(W_a)$ and $\curlywedge(W_b)$ represented with red dots and blue squares on the physical space in the image below.



7.2 Degenerate cut and project schemes

When considering symbolic dynamics where sequences are defined over $\mathbb{Z}$ and configurations are defined over $\mathbb{Z}^d$, the condition that $\pi_\mathcal{L}$ is injective is problematic. To describe Jeandel-Rao tilings over $\mathbb{Z}^2$ and multidimensional Sturmian configurations over $\mathbb{Z}^d$, it is convenient to consider a relaxed version of cut and project schemes where the projection in the physical space might not be injective when restricted to the lattice.

We try to formalize this degenerate extension in this section together with an example in the case of Sturmian sequences.

Let H be a locally compact abelian group and $K = \mathbb{R}^d$ for some $d \geq 1$. We call K the **physical space** and H the **internal space**. Let $m \geq 1$ be an integer. Let $\pi \colon \mathbb{R}^{m+d} \to K$ be a projection on the physical space and $\pi_{\text{int}} \colon \mathbb{R}^{m+d} \to H$ be a projection on the internal space:

$$K \xleftarrow{\ \pi\ } \mathbb{R}^{m+d} \xrightarrow{\ \pi_{\text{int}}\ } H$$

Then, let $\mathcal{L} \subset \mathbb{R}^d$ be a co-compact lattice with the property that $\pi_{\text{int}}(\mathcal{L})$ is dense in H. But, we no longer require that $\pi|_\mathcal{L}$ be injective.

$$\begin{array}{ccccc} K & \xleftarrow{\ \pi\ } & \mathbb{R}^{m+d} & \xrightarrow{\ \pi_{\text{int}}\ } & H \\ \cup & & \cup & & \text{dense} \cup \\ \pi(\mathcal{L}) & \longleftarrow & \mathcal{L} & \longrightarrow & \pi_{\text{int}}(\mathcal{L}) \end{array}$$

The image of the lattice in the physical space is denoted $L = \pi(\mathcal{L})$.

Definition 7.4. *A **relaxed** $(m + d)$**-to-**d **cut and project scheme** is a 5-uple $(H, K, \mathcal{L}, \pi, \pi_{int})$ satisfying*

$$\operatorname{Ker} \pi \cap \mathcal{L} \subseteq \operatorname{Ker} \pi_{int}. \tag{7.1}$$

*A relaxed cut and project scheme such that $\pi|_\mathcal{L}$ is not injective is called a **degenerate** $(m+d)$**-to-**d **cut and project scheme**.*

We may observe that classical cut and project schemes are relaxed cut and project schemes because $\mathrm{Ker}(\pi) \cap \mathcal{L} = \{0\}$ when $\pi|_{\mathcal{L}}$ is injective. Thus condition (7.1) is satisfied. For relaxed cut and project schemes, there is also a well-defined mapping $\star : L \to H$ given by

$$x \mapsto x^\star \coloneqq \pi_{\mathrm{int}}\left(\mathcal{L} \cap \pi^{-1}(x)\right).$$

where for every $x \in L$, $x^\star$ is the only element of $\pi_{\mathrm{int}}(\mathcal{L})$ obtained by applying the map π_{int} to an element of $\pi^{-1}(\{x\}) \cap \mathcal{L}$. This mapping is called the **star map** of the CPS. The $\star$-image of L is denoted by $L^\star$.

A relaxed CPS is summarized in the following diagram.

$$\begin{array}{ccccc}
K & \xleftarrow{\ \pi\ } & \mathbb{R}^{m+d} & \xrightarrow{\ \pi_{\mathrm{int}}\ } & H \\
\cup & & \cup & & \cup \text{ dense} \\
\pi(\mathcal{L}) & \longleftarrow & \mathcal{L} & \longrightarrow & \pi_{\mathrm{int}}(\mathcal{L}) \\
\| & & & & \| \\
L & & \xrightarrow{\quad\quad \star \quad\quad} & & L^\star
\end{array}$$

For a given **window** $W \subset H$ in the internal space,

$$\curlywedge(W) \coloneqq \{x \in L \mid x^\star \in W\}$$

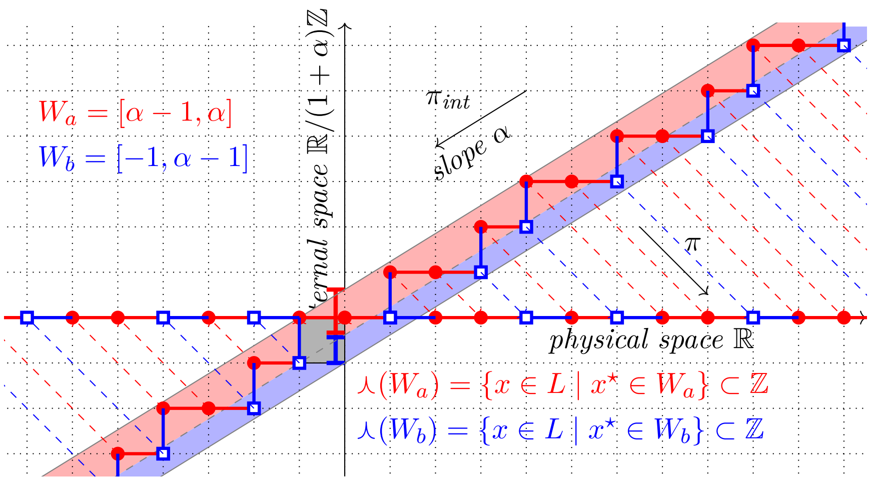
is the projection set within the cut and project scheme.

Let us now consider some examples. We may change the projection in the physical space used in Example 7.3 so that the distance between each point in $\curlywedge(W)$ is uniform. This gives a description of the positions of the letters in a characteristic Sturmian sequence using a degenerate cut and project scheme. But this can not be done without removing the condition that $\pi|_{\mathcal{L}}$ is injective.

Example 7.5. *In this example, we illustrate the degenerate cut and project scheme associated to a Sturmian sequence of slope α with $\alpha > 0$. The cut and project scheme is given by the 5-uple $(H, K, \mathcal{L}, \pi, \pi_{int})$ where the internal space is $H = \mathbb{R}/(1+\alpha)\mathbb{Z}$, the physical space is $K = \mathbb{R}$, $\mathcal{L} = \mathbb{Z}^2$ and*

$$\pi : \begin{array}{ccc} \mathbb{R}^2 & \to & \mathbb{R} \\ (x_1, x_2) & \mapsto & x_1 + x_2 \end{array} \quad \text{and} \quad \pi_{\mathrm{int}} : \begin{array}{ccc} \mathbb{R}^2 & \to & \mathbb{R}/(1+\alpha)\mathbb{Z} \\ (x_1, x_2) & \mapsto & -\alpha x_1 + x_2. \end{array}$$

Notice that if $(x_1, x_2) \in \mathbb{Z}^2$, then $\pi_{\mathrm{int}}(x_1, x_2) = -\alpha x_1 + x_2 \equiv x_1 + x_2$. The window $W = [-1, \alpha)$ defines the model set $\curlywedge(W) = \mathbb{Z}$. The window W is naturally partitioned into the two sets $W_a = [\alpha - 1, \alpha)$ and $W_b = [-1, \alpha - 1)$. They define the subsets $\curlywedge(W_a)$ and $\curlywedge(W_b)$ represented with red dots and blue squares at integer positions on the physical space in the image below.

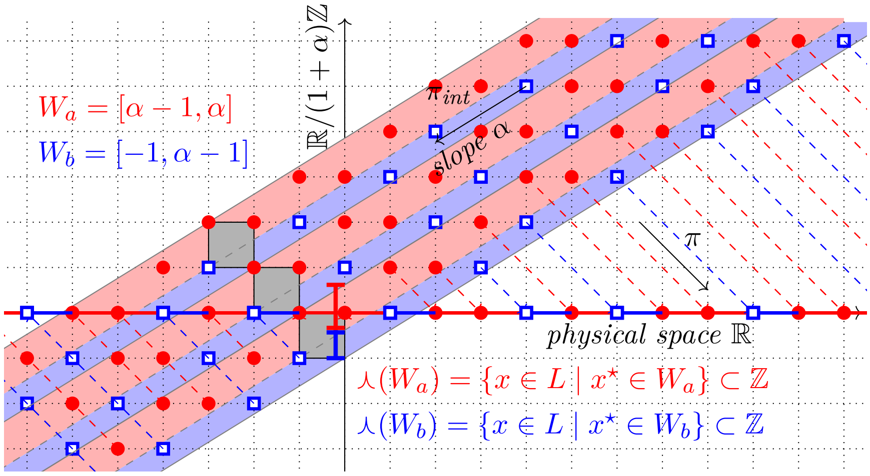


The subsets $\curlywedge(W_a)$ and $\curlywedge(W_b)$ are the positions of the letters in a characteristic Sturmian sequence.

The projection $\pi|_{\mathcal{L}} : \mathcal{L} \to \mathbb{R}$ is no longer injective, but for every $x \in \pi(\mathcal{L})$, the lattice points in the fiber $\pi^{-1}(x) \cap \mathcal{L}$ are mapped on the same point in the internal space $\mathbb{R}/(1+\alpha)\mathbb{Z}$ under π_{int}. More precisely, condition (7.1) is satisfied:

$$\operatorname{Ker}(\pi) \cap \mathcal{L} = \{(a, -a) : a \in \mathbb{Z}\} \subset \operatorname{Ker}(\pi_{int}).$$

This is illustrated in the image below.



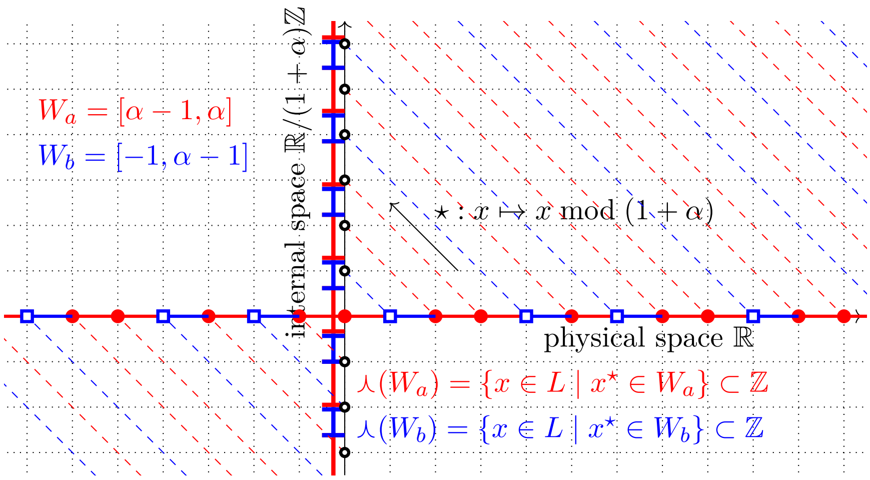
Finally, if $(x_1, x_2) \in \mathcal{L} = \mathbb{Z}^2$, then we have

$$\begin{aligned} \pi(x_1, x_2) &= x_1 + x_2, \\ \pi_{\text{int}}(x_1, x_2) &= -\alpha x_1 + x_2 = x_1 + x_2 \pmod{1+\alpha}. \end{aligned}$$

Thus, the star map is the natural projection $\mathbb{Z} \to \mathbb{R}/(1+\alpha)\mathbb{Z}$:

$$\star : x \mapsto x \pmod{1+\alpha}.$$

The fact that the $\star$ map is the identity modulo a lattice is what allows to identify the physical space with the internal space as in the image below.



Note that even if $\pi|_{\mathcal{L}}$ is not injective, the data of the cut and project scheme can still be deduced from the model set $\curlywedge(W)$ (unless $\curlywedge(W)$ is a lattice itself). For example, using projection $\pi(x_1, x_2) = x_1 + 2x_2$ instead in Example 7.3 would still allow to deduce that the point set $\curlywedge(W)$ is a 2-to-1 cut and project scheme. Each red point is followed by a short interval, each blue point is followed by a long interval.

However, when using projection $\pi(x_1, x_2) = x_1 + x_2$ as in Example 7.5, the distance between consecutive points is uniform. It becomes impossible to guess the 2-to-1 cut and project scheme only from the point set $\curlywedge(W)$ because $\curlywedge(W)$ is $\mathbb{Z}$, a uniformly distributed point set!

Remark 7.6. *Within a degenerate cut and project scheme, if the model set $\curlywedge(W)$ is a lattice, then we need symbolic dynamics to encode how the points were projected. In this limit and degenerate case, the model set $\curlywedge(W)$ is more appropriately described by a configuration $\curlywedge(W) \to \mathcal{A}$ for some set $\mathcal{A}$ of symbols and a topological partition $W = \cup_{a\in\mathcal{A}} \overline{W_a}$ of the window.*

The construction of Jeandel-Rao configurations $\mathbb{Z}^2 \to \{0, 1, \ldots, 10\}$ from 4-to-2 cut and project schemes follows this idea, see Section 11.3. Moreover, extending Sturmian sequences to multidimensional configurations $\mathbb{Z}^d \to \{\mathtt{0}, \mathtt{1}, \ldots, d\}$ needs the same kind of degenerate $(d+1)$-to-d cut and project schemes, see Section 13.3.

PART II

CONTRIBUTIONS WITHIN 2-TO-1 CUT AND PROJECT SCHEMES

CONTENTS

Chapter 8

A new characterization of Sturmian sequences

8.1 Indistinguishable asymptotic pairs

In this section, we present a surprising connection between Sturmian sequences and asymptotic pairs satisfying a natural combinatorial property which originates in thermodynamical formalism. This property characterizes asymptotic pairs which induce the trivial linear functional on a space of continuous and shift-invariant cocycles on the asymptotic relation of the full $\mathbb{Z}$-shift. See Section 3 of [60] for further details.

The characterization of indistinguishable asymptotic pairs was presented to Štěpán Starosta and me by Sebastián Barbieri during a short stay in Prague in Fall 2019 funded by bilateral PHC Barrande project. The hypothesis lead to so many overlaps between factors of the sequences that it was expected to deduce periodicity of the solution using Fine and Wilf theorem. It turned out that nonperiodic sequences are possible (including Sturmian sequences) and that was a nice surprise. We split our work into two articles. In this section, we present the characterization of indistinguishable asymptotic pairs over $\mathbb{Z}$ [3]. The characterization of indistinguishable asymptotic pairs over $\mathbb{Z}^d$ was more time-consuming [2]: it is presented in Chapter 13.

Concretely, given a finite set Σ, we consider the space of sequences $\Sigma^{\mathbb{Z}} = \{x\colon \mathbb{Z} \to \Sigma\}$ endowed with the prodiscrete topology and the shift action $\mathbb{Z} \overset{\sigma}{\curvearrowright} \Sigma^{\mathbb{Z}}$. In this setting, two sequences $x, y \in \Sigma^{\mathbb{Z}}$ are **asymptotic** if x and y differ in finitely many positions of $\mathbb{Z}$. The finite set $F = \{n \in \mathbb{Z} : x_n \neq y_n\}$ is called the **difference set** of (x, y).

Given two asymptotic sequences $x, y \in \Sigma^{\mathbb{Z}}$ with the difference set F, we want to compare the number of occurrences of a fixed pattern. As x and y are asymptotic, occurrences of patterns whose support do not intersect F are the same, so we only need to consider the occurrences of patterns that appear intersecting F. As an example, we can take a fixed symbol $a \in \Sigma$ and define $\Delta_a(x, y)$ as the number of positions $n \in F$ such that $y_n = a$ minus the number of positions $n \in F$ such that $x_n = a$. As F is finite, this value is well defined. More generally, for any given pattern $p\colon S \to \Sigma$ where S is a finite subset of $\mathbb{Z}$, we can consider the difference $\Delta_p(x, y)$ of the number of occurrences of p in y intersecting F minus the number of occurrences of p in x intersecting F.

We say that (x, y) is an **indistinguishable asymptotic pair** if (x, y) is asymptotic and $\Delta_p(x, y) = 0$ for every pattern p. A trivial example of an indistinguishable asymptotic pair is (x, x) for any $x \in \Sigma^{\mathbb{Z}}$. Another simple example is $x, y \in \{\mathtt{0}, \mathtt{1}\}^{\mathbb{Z}}$ where x is equal to $\mathtt{1}$ at the origin, and $\mathtt{0}$ everywhere else, and y is equal to $\mathtt{1}$ at some nonzero $n \in \mathbb{Z}$ and $\mathtt{0}$ everywhere else. Note that in both of these examples x and y lie on the same orbit of $\mathbb{Z} \overset{\sigma}{\curvearrowright} \Sigma^{\mathbb{Z}}$.

In [60] the authors define the following norm on asymptotic sequences of $\Sigma^{\mathbb{Z}}$

$$\|(x, y)\|^*_{\mathsf{NS}} = \sup_{\substack{S \subseteq \mathbb{Z} \\ S \text{ finite}}} \frac{1}{|S|} \sum_{p \in \Sigma^S} |\Delta_p(x, y)|.$$

Every asymptotic pair induces an evaluation map on the space of continuous cocycles on the equivalence relation of asymptotic pairs. The authors show that this norm coincides with the dual norm in the space of linear functionals on the space of continuous cocycles. In other words, the asymptotic pairs which induce the trivial linear functional are precisely the indistinguishable pairs. In this work, we provide a full characterization of which asymptotic pairs induce the trivial linear functional.

Using the notion of indistinguishability, we provide a characterization of the lower and the upper characteristic Sturmian sequences. Recall that the lower and upper sequences c_α and c'_α are given respectively by

$$\begin{array}{rccl} c_\alpha : & \mathbb{Z} & \to & \{\mathtt{0},\mathtt{1}\} \\ & n & \mapsto & \lfloor \alpha(n+1)\rfloor - \lfloor \alpha n \rfloor \end{array} \quad \text{and} \quad \begin{array}{rccl} c'_\alpha : & \mathbb{Z} & \to & \{\mathtt{0},\mathtt{1}\} \\ & n & \mapsto & \lceil \alpha(n+1)\rceil - \lceil \alpha n \rceil . \end{array}$$

where $\alpha \in [0,1]$. Note that the fact that c_α and c'_α form an asymptotic pair as biinfinite sequences was observed by Hedlund [156]. We proved the following result.

Theorem 8.1 ([3])**.** *Let $x, y \in \{\mathtt{0},\mathtt{1}\}^{\mathbb{Z}}$ and assume that x is recurrent. The pair (x, y) is an indistinguishable asymptotic pair with difference set $F = \{-1, 0\}$ such that $x_{-1}x_0 = \mathtt{10}$ and $y_{-1}y_0 = \mathtt{01}$ if and only if there exists $\alpha \in [0,1] \setminus \mathbb{Q}$ such that $x = c_\alpha$ and $y = c'_\alpha$ are the lower and upper characteristic Sturmian sequences of slope α.*

In particular, for all $n \in \mathbb{N}$ the words $x_{-n}x_{-n+1}\cdots x_{n-1}$ and $y_{-n}y_{-n+1}\cdots y_{n-1}$ of length $2n$ are optimal representations of the language of c_α as both contain exactly one occurrence of every factor of length n.

Corollary 8.2 ([3])**.** *If $x, y \in \Sigma^{\mathbb{Z}}$ is a non-trivial indistinguishable asymptotic pair with difference set $F = \{-1, 0\}$, then each of the words $x_{-n}x_{-n+1}\cdots x_{n-1}$ and $y_{-n}y_{-n+1}\cdots y_{n-1}$ contain exactly one occurrence of each word in $\mathcal{L}_n(x)$.*

Removing the hypothesis that x is recurrent, we obtain a unifying description of the lower and upper characteristic Sturmian sequences and their limits as their slope tends towards a rational value.

Theorem 8.3 ([3])**.** *Let $x, y \in \{\mathtt{0},\mathtt{1}\}^{\mathbb{Z}}$. The pair (x, y) is an indistinguishable asymptotic pair with difference set $F = \{-1, 0\}$ such that $x_{-1}x_0 = \mathtt{10}$ and $y_{-1}y_0 = \mathtt{01}$ if and only if there exists $(\alpha_n)_{n\in\mathbb{N}}$ with $\alpha_n \in [0,1] \setminus \mathbb{Q}$ such that*

$$x = \lim_{n\to\infty} c_{\alpha_n} \quad \text{and} \quad y = \lim_{n\to\infty} c'_{\alpha_n}.$$

In the case where x is not recurrent, then x and y lie on the same orbit and there exist coprime integers $p, q \in \mathbb{Z}_{\geq 0}$ such that (x, y) is the limit of asymptotic pairs formed by the lower and upper characteristic Sturmian sequences of slope α_n as α_n converges toward the rational slope $p/(p+q) \in [0,1] \cap \mathbb{Q}$ either from above or from below, see [3, Theorem 4.5]. Limits of the lower and upper characteristic Sturmian sequences as their slope tends to a rational number are expressed in terms of Christoffel words, see see [3, Lemma 4.2].

Theorem 8.1 and Theorem 8.3 are related to the famous theorem of Pirillo [230] (Theorem 4.1) which provides a characterization of Christoffel words of slope p/q where p and q are positive coprime integers. Pirillo's theorem can be restated for biinfinite sequences as follows: *c_α is the lower sequence associated to the rational slope $\alpha = p/(p+q)$ for some coprime nonnegative integers p, q if and only if c_α is a shift of c'_α.*

It is natural to ask if there is an analogous statement which holds as we take the limit $\frac{p}{p+q} \to \alpha$ for some irrational $\alpha \in [0,1] \setminus \mathbb{Q}$. In this light, Theorem 8.3 can be considered as an extension of Pirillo's theorem to aperiodic biinfinite sequences where the notion of conjugacy of words is replaced by the notion of indistinguishability of an asymptotic pair. This seems to be the correct approach since other alternatives (e.g., having the same language, see [3, Remark 3.8]) fail.

The next result provides a full characterization of non-trivial indistinguishable asymptotic pairs for $\mathbb{Z}$ which does not depend upon the form of the difference set or the alphabet. More precisely, we show that every indistinguishable asymptotic pair can be obtained from limits of pairs of lower and upper characteristic Sturmian sequences by means of shifts and substitutions.

Given finite sets Σ, Γ, a map $\varphi\colon \Sigma \to \Gamma^+$ which replaces symbols of Σ by nonempty words on Γ is called a substitution. This map is naturally extended by concatenation to a continuous map $\varphi\colon \Sigma^{\mathbb{Z}} \to \Gamma^{\mathbb{Z}}$.

Theorem 8.4 ([3]). *Let Σ be a finite alphabet and $x, y \in \Sigma^{\mathbb{Z}}$ a non-trivial asymptotic pair. Then x, y is indistinguishable if and only if either*

- *x is recurrent and there exists $\alpha \in [0,1] \setminus \mathbb{Q}$, a substitution $\varphi\colon \{\mathtt{0},\mathtt{1}\} \to \Sigma^+$ and an integer $m \in \mathbb{Z}$ such that*
$$\{x, y\} = \{\sigma^m \varphi(\sigma(c_\alpha)), \sigma^m \varphi(\sigma(c'_\alpha))\},$$
- *x is not recurrent and there exists a substitution $\varphi\colon \{\mathtt{0},\mathtt{1}\} \to \Sigma^+$ and an integer $m \in \mathbb{Z}$ such that*
$$\{x, y\} = \{\sigma^m \varphi({}^\infty\mathtt{0.10}^\infty), \sigma^m \varphi({}^\infty\mathtt{0.010}^\infty)\}.$$

This means that every indistinguishable asymptotic pair in $\mathbb{Z}$ consists either of (1) two sequences in the same orbit, which are shifts of a sequence of the form ${}^\infty v.uv^\infty$ for some $u, v \in \Sigma^+$, or (2) two sequences which, up to translation, can be obtained through a substitution from a pair of lower and upper characteristic Sturmian sequences. In simpler terms, all non-trivial examples of one-dimensional indistinguishable asymptotic pairs arise from irrational circle rotations. The proof of Theorem 8.4 is based on the well-known notions of return words and derived sequences [127].

8.2 Indistinguishable asymptotic pairs and the Markoff property

In this section, we give equivalent conditions for balanced sequences satisfying cases (M_3) or (M_4) of the Markoff property. Cases (M_3) and (M_4) can be expressed in terms of limits of mechanical words toward an irrational or rational slope from above or from below which were shown to be equivalent to sequences that belong to an indistinguishable asymptotic pair [3].

In the statement that follows, we denote the position of the origin of a biinfinite sequence $s = \cdots s_{-2}s_{-1}.s_0s_1s_2\cdots \in \Sigma^{\mathbb{Z}}$ with a dot (.) between positions -1 and 0.

Theorem 8.5 ([11]). *Let $s \in \{\mathtt{0},\mathtt{1}\}^{\mathbb{Z}}$ and $n_0 \in \mathbb{Z}$. The following are equivalent conditions describing balanced sequences satisfying Markoff property (M_3) or (M_4):*

1. *the sequence s has a factorization $\sigma^{n_0}s = \widetilde{p}x.yp$ where $\{x, y\} = \{\mathtt{0},\mathtt{1}\}$;*
2. *there exists a sequence $(\alpha_k)_{k\in\mathbb{Z}_{\geq 0}}$ with $\alpha_k \in [0,1] \setminus \mathbb{Q}$ such that $\sigma^{n_0}s = \lim_{k\to\infty} s_{\alpha_k,0}$ or $\sigma^{n_0}s = \lim_{k\to\infty} s'_{\alpha_k,0}$;*
3. *there exists a sequence $t \in \{\mathtt{0},\mathtt{1}\}^{\mathbb{Z}}$ such that (s, t) is an indistinguishable asymptotic pair with difference set $\{n_0 - 1, n_0\}$.*

8.3 Open questions

Among the natural question is whether indistinguishable asymptotic pairs leads to interesting generalizations of Sturmian sequences in higher dimension. We have made progress on this question for the generic case. These results are summarized in Chapter 13.

A recent work considered the notion of attractors in the context of biinfinite sequences. They obtained an independent characterization of balanced sequences, including skew ones [54]. For a biinfinite sequence $x \in \Sigma^{\mathbb{Z}}$, an attractor is a finite subset $\Gamma \subset \mathbb{Z}$ such that every word in the language of x has at least one occurrence whose support intersect the set Γ. When (x, y) is an indistinguishable asymptotic pair, it is an easy consequence that x and y have an attractor: it is the difference set of x and y.

What Béaur, Gheeraert and de Menibus proved is that biinfinite sequences having a finite attractor are balanced sequences and their images under a substitution. A corollary of their result and ours is the following.

Corollary 8.6. *Let $x \in \Sigma^{\mathbb{Z}}$. The sequence x has a finite attractor if and only if there exists $y \in \Sigma^{\mathbb{Z}}$ such that (x, y) is an indistinguishable asymptotic pair.*

However, it is remains unclear how to directly construct y from x. In other words, we can ask the following question:

Question 8.7. *Does there exist a simpler direct proof of Corollary 8.6?*

Chapter 9

Contributions to the Markoff injectivity conjecture

9.1 The Markoff injectivity conjecture on the language of a balanced sequence

In [214], a q-analog of rational numbers and of continued fractions were introduced. This was the inspiration for several advances [210, 224, 112, 62, 166] and among them a q-analog of Markoff triples [181]. A q-analog of the matrices $\mu(\mathtt{0})$ and $\mu(\mathtt{1})$ was proposed in [191], which in terms of

$$L_q = \begin{pmatrix} q & 0 \\ q & 1 \end{pmatrix} \qquad \text{and} \qquad R_q = \begin{pmatrix} q & 1 \\ 0 & 1 \end{pmatrix},$$

can be written as

$$\mu_q(\mathtt{0}) = R_q L_q = \begin{pmatrix} q+q^2 & 1 \\ q & 1 \end{pmatrix},$$
$$\mu_q(\mathtt{1}) = R_q R_q L_q L_q = \begin{pmatrix} q+2q^2+q^3+q^4 & 1+q \\ q+q^2 & 1 \end{pmatrix}.$$

It extends to a morphism of monoids $\mu_q : \{\mathtt{0},\mathtt{1}\}^* \to \mathrm{GL}_2(\mathbb{Z}[q^{\pm 1}])$. This q-analog satisfies that $\mu_1(w) = \mu(w)$ for every $w \in \{\mathtt{0},\mathtt{1}\}^*$. Thus if w is a Christoffel word, then the entry above the diagonal $\mu_q(w)_{12}$ is a polynomial of indeterminate q with nonnegative integer coefficients such that it is a Markoff number when evaluated at $q = 1$. For example,

$$\mu_q(\mathtt{00101})_{12} = 1 + 4q + 10q^2 + 18q^3 + 27q^4 + 33q^5 + 33q^6 + 29q^7 + 21q^8 + 12q^9 + 5q^{10} + q^{11}$$

which, when evaluated at $q = 1$, is equal to

$$\mu_1(\mathtt{00101})_{12} = 1 + 4 + 10 + 18 + 27 + 33 + 33 + 29 + 21 + 12 + 5 + 1 = 194.$$

With Mélodie Lapointe, we proved that the map $w \mapsto \mu_q(w)_{12}$ is strictly increasing with respect to the radix order on the language of a fixed balanced sequence [11]. Recall that the radix order is defined for every $u, v \in \{\mathtt{0},\mathtt{1}\}^*$ as

$$u <_{radix} v \qquad \text{if} \qquad \begin{cases} |u| < |v| & \text{or} \\ |u| = |v| & \text{and} \quad u <_{lex} v. \end{cases}$$

Also we define a partial order $\prec$ on $\mathbb{Z}[q]$ as

$$f \prec g \quad \text{if and only if} \quad f \neq g \text{ and } g - f \in \mathbb{Z}_{\geq 0}[q].$$

Theorem 9.1 ([11]). *Let $s \in \{0,1\}^{\mathbb{Z}}$ be a balanced sequence and $u, v \in \mathcal{L}(s)$ be two factors in the language of s. If $u <_{radix} v$, then $\mu_q(u)_{12} \prec \mu_q(v)_{12}$, i.e., $\mu_q(v)_{12} - \mu_q(u)_{12}$ is a nonzero polynomial of indeterminate q with nonnegative integer coefficients.*

The proof of Theorem 9.1 is based on Theorem 8.5 and Corollary 8.2 and follows closely the proof for the case when $q = 1$ made earlier by Lapointe and Reutenauer [190].

As a consequence of Theorem 9.1, we prove injectivity of $w \mapsto \mu_q(w)_{12}$ when restricted to the language of a given balanced sequence.

Corollary 9.2 ([11]). *Let $s \in \{0,1\}^{\mathbb{Z}}$ be a balanced sequence. The map $w \mapsto \mu_q(w)_{12}$ is injective over the language $\mathcal{L}(s)$.*

9.2 A q-analog of the Markoff injectivity conjecture

In a follow-up work with Mélodie Lapointe and Wolfgang Steiner, we went one step further and prove a q-analog of the Markoff Injectivity Conjecture.

Theorem 9.3 ([12]). *The map $w \mapsto \mu_q(w)_{12}$ is injective on the set of Christoffel words.*

The proof Theorem 9.3 is short and is based on the evaluation at $q = \exp(i\pi/3)$. Other roots of unity provide some information on the original problem, which corresponds to the case $q = 1$.

9.3 Open questions

The Markoff injectivity conjecture remains open.

Question 9.4. *Is Theorem 9.3 helpful to prove Conjecture 4.2?*

PART III

CONTRIBUTIONS WITHIN 3-TO-1 CUT AND PROJECT SCHEMES

CONTENTS

Chapter 10

Almost everywhere balanced sequences of complexity $2n+1$

There exist many other generalizations of Sturmian words over larger alphabets, each focusing on particular properties satisfied by Sturmian words. Among the properties is the factor complexity. Given a word $\boldsymbol{w} \in A^{\mathbb{N}}$, we let $\mathcal{L}_{\boldsymbol{w}}$ denote the set of factors of $\boldsymbol{w}$, i.e.,

$$\mathcal{L}_{\boldsymbol{w}} = \{\boldsymbol{w}_i \cdots \boldsymbol{w}_{i+n-1} \in A^* \mid i, n \in \mathbb{N}\}.$$

The *factor complexity* of $\boldsymbol{w}$ is the function

$$p_{\boldsymbol{w}} : \mathbb{N} \to \mathbb{N}, n \mapsto \#(\mathcal{L}_{\boldsymbol{w}} \cap A^n).$$

Words of complexity $2n+1$ were for instance considered by Arnoux and Rauzy [42] with the condition that, like Sturmian words, there is exactly one left and one right special factor of each length; these words are now called Arnoux-Rauzy words. It is known that the frequencies of any Arnoux-Rauzy word are well defined and belong to the Rauzy gasket [43], a fractal set of Lebesgue measure zero. Thus the above condition on the number of special factors is very restrictive for the possible letter frequencies.

Words of complexity $p(n) \leq 2n+1$ include Arnoux-Rauzy words, codings of interval exchange transformations and more [195]. For any given letter frequencies one can construct words of factor complexity $2n+1$ by the coding of a 3-interval exchange transformation. It is however known that these words are almost always unbalanced [280].

In recent years, multidimensional continued fraction algorithms were used to obtain ternary balanced words with low factor complexity for any given vector of letter frequencies. Indeed the Brun algorithm leads to balanced words [119] and it was shown that the Arnoux-Rauzy-Poincaré algorithm leads to words of factor complexity $p(n) \leq \frac{5}{2}n + 1$ [4].

In 2017, the existence of a class of words over an alphabet of size d which simultaneously generalize the three Sturmian properties was still open: having factor complexity $(d-1)n+1$, achieving any possible vector of rationally independent letter frequencies and being almost always balanced. With Julien Cassaigne and Julien Leroy, we proved that the substitutions proposed by Cassaigne [102] satisfy these three properties [5, 6]. We present these results in this chapter. The question remains open for alphabets of size $d \geq 4$.

10.1 Cassaigne Substitutions and $\mathcal{S}$-adic words

Let A be an alphabet, i.e., a finite set. By *substitution* over A, we mean an endomorphism σ of the free monoid A^* which is non-erasing, i.e. $\sigma(a) \neq \varepsilon$ for all a, where ε is the empty word. If $\mathcal{S}$ is a set of substitutions over A^*, a word $\boldsymbol{w} \in A^{\mathbb{N}}$ is said to be *$\mathcal{S}$-adic* if there is a sequence $\boldsymbol{\sigma} = (\sigma_n)_{n\in\mathbb{N}} \in \mathcal{S}^{\mathbb{N}}$ and a sequence $\boldsymbol{a} = (a_n)_{n\in\mathbb{N}} \in A^{\mathbb{N}}$ such that the limit $\lim_{n\to+\infty} \sigma_0\sigma_1 \cdots \sigma_{n-1}(a_n)$ exists and is equal to $\boldsymbol{w}$. The 2-tuple $(\boldsymbol{\sigma}, \boldsymbol{a})$ is called an *$\mathcal{S}$-adic representation* of $\boldsymbol{w}$ and the sequence $\boldsymbol{\sigma}$ a *directive sequence* of $\boldsymbol{w}$. A directive sequence $\boldsymbol{\sigma}$ is *primitive* if for every $n \in \mathbb{N}$, there exists $m > n$ such that for every letters $a, b \in A$, the letter a occurs in $\sigma_{i_n}\sigma_{i_{n+1}} \cdots \sigma_{i_m}(b)$. See [132, 80] for more details about $\mathcal{S}$-adic sequences.

A sequence of substitutions $(\sigma_n)_{n\in\mathbb{N}} \in \mathcal{S}^{\mathbb{N}}$ is said to be *everywhere growing* if $\min_{a\in A} |\sigma_{[0,n)}(a)|$ goes to infinity as n goes to infinity. Also, recall that with an substitution $\sigma : A^* \to A^*$, we associate its *incidence matrix* $M_\sigma \in \mathbb{N}^{A\times A}$ defined by $(M_\sigma)_{a,b} = |\sigma(b)|_a$. Thus, for any word $w \in A^*$, we have $\overrightarrow{\sigma(w)} = M_\sigma \overrightarrow{w}$, where $\overrightarrow{w} \in \mathbb{N}^A$ is defined by $\overrightarrow{w}_a = |w|_a$.

The two substitutions over the alphabet $\mathcal{A} = \{1, 2, 3\}$ proposed by Cassaigne [102] are

$$c_1 : \begin{cases} 1 \mapsto 1 \\ 2 \mapsto 13 \\ 3 \mapsto 2 \end{cases} \qquad \text{and} \qquad c_2 : \begin{cases} 1 \mapsto 2 \\ 2 \mapsto 13 \\ 3 \mapsto 3 \end{cases} .$$

These substitutions lead to $\mathcal{C}$-adic words from directive sequences over the set $\mathcal{C} = \{c_1, c_2\}$. Note that the choice of the above substitutions c_1 and c_2 is less trivial than one may first think. Indeed, not all choices for the image of the letter 2 allow the complexity to remain $2n+1$. In particular, changing c_1 to be $1 \mapsto 1, 2 \mapsto 31, 3 \mapsto 2$ may seem interesting since it makes both c_1 and c_2 left-marked (the first letter of the images are all distinct), but this choice does not work as it increases the factor complexity for the associated $\mathcal{C}$-adic words.

With Cassaigne and Leroy, we proved that primitive $\mathcal{C}$-adic words are equivalently characterized by their factor complexity and the vector of their letter frequencies, generalizing the Sturmian case.

Theorem 10.1 ([6])**.** *Let $\boldsymbol{w}$ be a $\mathcal{C}$-adic word with directive sequence $(i_n)_{n\in\mathbb{N}}$, with $i_n \in \{1, 2\}$ for every $n \in \mathbb{N}$. The following conditions are equivalent:*

(i) $\boldsymbol{w}$ has factor complexity $p(n) = 2n + 1$ for all $n \in \mathbb{N}$;

(ii) the frequencies of letters in $\boldsymbol{w}$ are rationally independent;

(iii) $(c_{i_n})_{n\in\mathbb{N}}$ is primitive;

(iv) $\boldsymbol{w}$ is a uniformly recurrent dendric word.

We also show that the primitive $\mathcal{C}$-adic words are exactly the $\mathcal{C}$-adic words that are dendric, a property recently introduced under the name of "tree sets" [79].

Dendric words are such that the extension graph constructed from the bilateral extensions of the bispecial factors is always a tree. Dendric words are also a natural extension of Sturmian sequences. There have been a flourishing sequence of results about them in recent years [142, 143, 144, 123, 122, 121, 229, 73].

10.2 A bidimensional continued fraction algorithm

In this section, we show that Theorem 10.1 is constructive. Given a vector of rationally independent letter frequencies, it is possible to construct a $\mathcal{C}$-adic word with these letter frequencies. There is an algorithm that constructs the directive sequence from the vector.

On the positive cone $\mathbb{R}^3_{\geq 0}$, the bidimensional continued fraction algorithm introduced by Cassaigne [102] is

$$F_C(x_1, x_2, x_3) = \begin{cases} (x_1 - x_3, x_3, x_2), & \text{if } x_1 \geq x_3; \\ (x_2, x_1, x_3 - x_1), & \text{if } x_1 < x_3. \end{cases}$$

More information on multidimensional continued fraction algorithms can be found in [95, 250].

Alternatively, the map F_C can be defined by associating nonnegative matrices to each part of a partition of $\mathbb{R}^3_{\geq 0}$ into $\Lambda_1 \cup \Lambda_2$ where

$$\Lambda_1 = \{(x_1, x_2, x_3) \in \mathbb{R}^3_{\geq 0} \mid x_1 \geq x_3\},$$
$$\Lambda_2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3_{\geq 0} \mid x_1 < x_3\}.$$

The matrices are given by the rule $\mathsf{M}(\boldsymbol{x}) = C_i$ if and only if $\boldsymbol{x} \in \Lambda_i$ where

$$C_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \text{and} \qquad C_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

The map F_C on $\mathbb{R}^3_{\geq 0}$ and the projective map f_C on $\Delta = \{\boldsymbol{x} \in \mathbb{R}^3_{\geq 0} \mid \|\boldsymbol{x}\|_1 = 1\}$ are then defined as:

$$F_C(\boldsymbol{x}) = \mathsf{M}(\boldsymbol{x})^{-1}\boldsymbol{x} \qquad \text{and} \qquad f_C(\boldsymbol{x}) = \frac{F_C(\boldsymbol{x})}{\|F_C(\boldsymbol{x})\|_1}.$$

Thus, we have

$$f_C(x_1, x_2, x_3) = \begin{cases} (\frac{x_1 - x_3}{1 - x_3}, \frac{x_3}{1 - x_3}, \frac{x_2}{1 - x_3}), & \text{if } x_1 \geq x_3; \\ (\frac{x_2}{1 - x_1}, \frac{x_1}{1 - x_1}, \frac{x_3 - x_1}{1 - x_1}), & \text{if } x_1 < x_3. \end{cases}$$

Many of their properties can be found in the Cheat Sheets made by the author about 3-dimensional continued fraction algorithms [20]. These experiments made us realize that the map f_C is conjugate [5] with a semi-sorted version of another MCFA, the Selmer algorithm [251, 250]. Also note that Selmer algorithm is conjugate on the absorbing simplex to Mönkemeyer's algorithm [212] (see [226]). Also we proved with Pierre Arnoux that f_C has an invariant measure on the 2-simplex whose density function is absolutely continuous with respect to Lebesgue [1].

Since $\{\Lambda_1, \Lambda_2\}$ is a partition of $\mathbb{R}^3_{\geq 0}$, any vector $\boldsymbol{x} \in \mathbb{R}^3_{\geq 0}$ defines a sequence of matrices $(C_{i_n})_{n \in \mathbb{N}}$ by $C_{i_n} = \mathsf{M}(F_C^n(\boldsymbol{x}))$ and we have

$$\boldsymbol{x} \in \bigcap_{n \geq 0} C_{i_0} C_{i_1} \cdots C_{i_n} \mathbb{R}^3_{\geq 0}. \tag{10.1}$$

The n-cylinders induced by f_C on Δ are illustrated in Figure 10.1.

```
sage: from slabbe.matrix_cocycle import cocycles
sage: c = cocycles.Cassaigne()
sage: ctikz1 = c.tikz_n_cylinders(1, labels=True, scale=3)
sage: ctikz2 = c.tikz_n_cylinders(2, labels=True, scale=3)
sage: ctikz3 = c.tikz_n_cylinders(3, labels=True, scale=3)
sage: ctikz4 = c.tikz_n_cylinders(4, labels=False, scale=3)
sage: ctikz5 = c.tikz_n_cylinders(5, labels=False, scale=3)
sage: ctikz6 = c.tikz_n_cylinders(6, labels=False, scale=3)
```

One may check that C_i is the incidence matrix of c_i for $i = 1, 2$.

Like for matrices, any vector $\boldsymbol{x} \in \mathbb{R}^3_{\geq 0}$ defines a directive sequence of substitutions $(c_{i_n})_{n \in \mathbb{N}}$, where $c_{i_n} = \mathsf{c}(F_C^n(\boldsymbol{x}))$ and $\mathsf{c}(\boldsymbol{y}) = c_i$ if and only if $\boldsymbol{y} \in \Lambda_i$.

For example, using vector $\boldsymbol{x} = (1, e, \pi)$, we have

$$\mathsf{c}(\boldsymbol{x})\mathsf{c}(F_C\boldsymbol{x})\mathsf{c}(F_C^2\boldsymbol{x})\mathsf{c}(F_C^3\boldsymbol{x})\mathsf{c}(F_C^4\boldsymbol{x}) = c_2 c_1 c_2 c_1 c_1 = \begin{cases} 1 \mapsto 23 \\ 2 \mapsto 23213 \\ 3 \mapsto 2313 \end{cases}$$

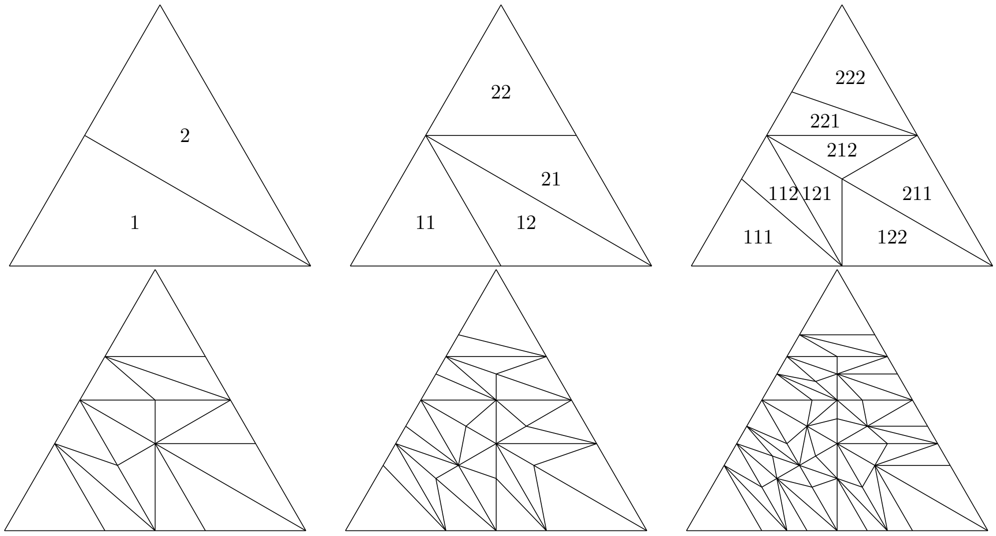


Figure 10.1: The n-cylinders of f_C on Δ for each $n \in \{1,2,3,4,5,6\}$. Any n-cylinder is represented by a word $u_0u_1\cdots u_{n-1}$ over $\{1,2\}^*$ and is the set of points $x \in \Delta$ such that for all $k \in \{0,1,\ldots,n-1\}$, $\mathsf{M}(f_C^k(x)) = C_{u_k}$.

whose incidence matrix is

$$C_2C_1C_2C_1C_1 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 2 \end{pmatrix}.$$

We may check that the vector $\boldsymbol{x} = (1, e, \pi)$ is in the interior of the cone spanned by the columns of the matrix. This holds for every prefix of the directive sequence.

The construction of the directive sequence and $\mathcal{C}$-adic word for a vector was implemented in the `slabbe` optional package [24] of SageMath [246].

```
sage: from slabbe.mult_cont_frac import Cassaigne
sage: c = Cassaigne()
sage: c
Cassaigne 3-dimensional continued fraction algorithm
sage: C = c.substitutions()
sage: C
{1: WordMorphism: 1->1, 2->13, 3->2, 2: WordMorphism: 1->2, 2->13, 3->3}
sage: it = c.coding_iterator((1,e,pi))
sage: directive_sequence = [next(it) for _ in range(20)]
sage: directive_sequence
[2, 1, 2, 1, 1, 1, 1, 2, 1, 1, 2, 1, 2, 2, 1, 2, 2, 2, 2, 2]
sage: w = c.s_adic_word((1,e,pi))
sage: w
232321323232313232321323232132323132323 2...
```

The $\mathcal{C}$-adic word generated from the directive sequence has the good vector of letter frequency because the algorithm is convergent. This is the subject of the next section.

10.3 Weak-convergence

In [6], we also show that for every sequence $(i_n)_{n\in\mathbb{N}} \in \{1,2\}^{\mathbb{N}}$, the set $\bigcap_{n\geq 0} C_{i_0}C_{i_1}\cdots C_{i_n}\mathbb{R}^3_{\geq 0}$ is one-dimensional. This property, sometimes called *weak convergence* [59, 184], is not satisfied by all choices of 3×3 matrices. For instance, Nogueira proved that the Poincaré MCFA is not convergent [220].

Theorem 10.2 ([6]). *The Cassaigne bidimensional continued fraction algorithm is weakly convergent. More precisely, for every sequence $m = (M_n)_{n\in\mathbb{N}} \in \{C_1, C_2\}^{\mathbb{N}}$, there exists a vector $u \in \mathbb{R}^3_{\geq 0} \setminus \{0\}$ satisfying*

$$\bigcap_{n\geq 0} M_0 M_1 \cdots M_{n-1} \mathbb{R}^3_{\geq 0} = \mathbb{R}_{\geq 0} u. \tag{10.2}$$

In our case, convergence allows to define a continuous map $\pi : \{1,2\}^{\mathbb{N}} \to \Delta = \{\boldsymbol{x} \in \mathbb{R}^3_{\geq 0} \mid \|\boldsymbol{x}\|_1 = 1\}$ by

$$\pi((i_n)_{n\in\mathbb{N}})\, \mathbb{R}_{\geq 0} = \bigcap_{n\geq 0} C_{i_0} C_{i_1} \cdots C_{i_n} \mathbb{R}^3_{\geq 0}. \tag{10.3}$$

This map is not injective, as for example $\pi(1222\dots) = (0,1,0) = \pi(2111\dots)$, but it is onto.

We also show that π is injective exactly on the set $\mathcal{P}$ of primitive sequences, i.e., sequences $(C_{i_n})_{n\in\mathbb{N}}$ such that for all m and all large enough $n > m$, $C_{i_m} \cdots C_{i_n}$ has only positive entries. Furthermore, the image $\pi(\mathcal{P})$ is the set $\mathcal{I}$ of normalized vectors with rationally independent entries. The inverse of $\pi : \mathcal{P} \to \mathcal{I}$ is given by the multidimensional continued fraction algorithm introduced by Cassaigne [102] that consists in iterating the map f_C on $\boldsymbol{x} \in \mathcal{I}$.

The class of $\mathcal{C}$-adic words with $\mathcal{C} = \{c_1, c_2\}$ provides a nice generalization of Sturmian words over a three-letter alphabet. From Theorem 10.1 and Theorem 10.2, we have the following interpretations of the previous discussion:

- by weak convergence, the frequencies of letters exist in every $\mathcal{C}$-adic word;
- by surjectivity of π, every $\boldsymbol{x} \in \Delta$ is the vector of letter frequencies of a $\mathcal{C}$-adic word;
- the bijection $\pi : \mathcal{P} \to \mathcal{I}$ induces a bijection between primitive $\mathcal{C}$-adic words and vectors of letter frequencies with rationally independent entries.

10.4 Measure-preserving isomorphisms

The map π also provides measure-preserving isomorphisms between the shift-space $\{1,2\}^{\mathbb{N}}$ and the simplex Δ.

Theorem 10.3 ([6]). *The following holds:*

- *for any shift-invariant Borel probability measure μ on $\{1,2\}^{\mathbb{N}}$ such that $\mu(\mathcal{P}) = 1$, the map $\pi : (\{1,2\}^{\mathbb{N}}, S, \mu) \to (\Delta, f_C, \pi_*\mu)$ is a measure-preserving isomorphism;*
- *for any f_C-invariant Borel probability measure ν on Δ such that $\nu(\mathcal{I}) = 1$, the map $\pi : (\{1,2\}^{\mathbb{N}}, S, \pi_*^{-1}\nu) \to (\Delta, f_C, \nu)$ is a measure-preserving isomorphism.*

This result in particular applies to any positive Bernoulli measure β on $\{1,2\}^{\mathbb{N}}$ and to the f_C-invariant probability measure ξ defined by the density function $6/(\pi^2(1-x_1)(1-x_3))$ [1]. Observe that any Bernoulli measure on $\{1,2\}^{\mathbb{N}}$ is ergodic and that the measure ξ is also ergodic [134]. Thus the pointwise ergodic theorem may be applied to obtain properties for Bernoulli-almost every directive sequence $(i_n)_{n\in\mathbb{N}}$ or for Lebesgue-almost every vector $\boldsymbol{x}$. Theorem 10.4 below is an example of such a result.

10.5 Almost always balanced sequences

The last property of Sturmian words that we consider is their balancedness.

The word $\boldsymbol{w}$ is said to be *finitely balanced* if there exists a constant $C > 0$ such that for any pair u, v of factors of the same length of $\boldsymbol{w}$, and for any letter $i \in A$, $||u|_i - |v|_i| \leq C$.

Not all primitive $\mathcal{C}$-adic words are balanced [36], but we prove that almost all of them are (for many measures). Our proof is based on the method proposed by Avila and Delecroix [46] for Brun and Fully Subtractive MCFA. It consists in applying the pointwise ergodic theorem to show that some fixed contracting matrix appears sufficiently often in almost every sequence $(C_{i_n})_{n\in\mathbb{N}}$. The same method allows to show that the second Lyapunov exponent is negative. The definition of Lyapunov exponents can be found in [6, Section 9]. For general references on Lyapunov exponents, we refer to [222] and [140].

Theorem 10.4 ([6])**.** *Let μ be a shift-invariant ergodic Borel probability measure on $\{1,2\}^{\mathbb{N}}$. If*

$$\mu([12121212]) > 0,$$

then for μ-almost every directive sequence $(i_n)_{n\in\mathbb{N}} \in \{1,2\}^{\mathbb{N}}$, the word $\boldsymbol{w} = \lim_{n\to\infty} c_{i_0} \dots c_{i_n}(1^\omega)$ is balanced and the second Lyapunov exponent θ_2^μ of the cocycle with matrices $\{C_1, C_2\}$ is negative.

This result in particular applies to any positive Bernoulli measure and to the measure $\pi_*^{-1}(\xi)$. Thus it extends a result of Berthé, Steiner and Thuswaldner [84] who proved that the second Lyapunov exponent is negative for the measure $\pi_*^{-1}(\xi)$. Observe that $\pi_*^{-1}(\xi)$ is not a Bernoulli measure so all these measures are pairwise mutually singular.

An application of multidimensional continued fraction algorithms is to provide simultaneous Diophantine approximation of a vector of real numbers [248]. The quality of the approximations can be evaluated in terms of the first two Lyapunov exponents of the MCFA [59, 185]. In particular, if the second Lyapunov exponent is negative, this implies that the algorithm is strongly convergent [150, 151, 152].

10.6 3-to-1 cut and project schemes: Example and applications

Consider the periodic sequence $121212\cdots$. We have that

$$\pi(121212\cdots) = \frac{1}{\beta^2+1}\begin{pmatrix} \beta \\ \beta^2-\beta \\ 1 \end{pmatrix} \approx \begin{pmatrix} 0.4302 \\ 0.3247 \\ 0.2451 \end{pmatrix}$$

is a positive right eigenvector of the primitive matrix C_1C_2 associated with the Perron-Frobenius eigenvalue $\beta \approx 1.7548$ of C_1C_2. It is the positive root of the characteristic polynomial x^3-2x^2+x-1 of C_1C_2. The infinite word on the alphabet $\{1,2,3\}$ obtained by applying our MCFA to the above vector is the $\mathcal{C}$-adic word which is the unique fixed point of the substitution $c_1c_2 : 1 \mapsto 13, 2 \mapsto 12, 3 \mapsto 2$:

$$\boldsymbol{w} = (w_n)_{n\geq 0} = \lim_{k\to\infty} (c_1c_2)^k(1) = 13212131213213121321213213121321213121 32\cdots$$

whose set of factors of lengths 0, 1, 2, 3 and 4 are listed in the following table:

n	$2n+1$ factors of length n
0	$\{\varepsilon\}$
1	$\{1,2,3\}$
2	$\{12,13,21,31,32\}$
3	$\{121,131,132,212,213,312,321\}$
4	$\{1213,1312,1321,2121,2131,2132,3121,3212,3213\}$

The left eigenvector of C_1C_2 associated with the dominant eigenvalue β is $u = (1, \beta^2 - \beta, \beta - 1)$. We define the map $h : \{1, 2, 3\} \to \mathbb{C}$ by $h(1) = 1$, $h(2) = \beta^{*2} - \beta^*$ and $h(3) = \beta^* - 1$ where $\beta^* \approx 0.12256 + 0.74486i$ is one of the two complex Galois conjugates of β. Observe that the vector $u^* = (h(1), h(2), h(3))$ is the image of u under the automorphism of the field $\mathbb{Q}(\beta)$ defined by $\beta \mapsto \beta^*$. The scalar product of u^* with $\pi(121212\cdots)$ is zero. Thus, as w is balanced, the partial sums $S^h(N) = \sum_{i=0}^{N-1} h(w_i)$ are bounded. The set $\{S^h(N) : N \in \mathbb{N}\}$, shown in Figure 10.2, is a well-known construction of the Rauzy fractal associated with a substitution [237, 124, 257, 82]. Theorem 10.4 implies that the Rauzy fractal is bounded for almost every $\mathcal{C}$-adic word. As shown recently, this is not true for all $\mathcal{C}$-adic words [36].

Figure 10.2 can be reproduced in SageMath in few lines:

```
sage: c1 = WordMorphism("1->1,2->13,3->2")
sage: c2 = WordMorphism("1->2,2->13,3->3")
sage: c12 = c1*c2
sage: c12_left_image = c12.rauzy_fractal_plot(n=1000, point_size=80)
sage: c12_right_image = c12.rauzy_fractal_plot(n=1000, point_size=80, exchange=True)
```

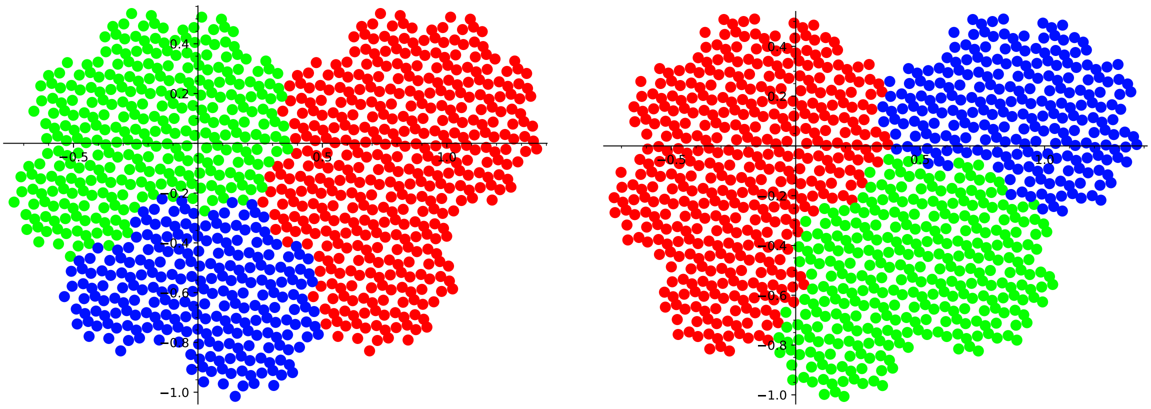


Figure 10.2: The Rauzy fractal associated with the fixed point $\boldsymbol{w}$ of c_1c_2. On the left (right resp.) the color of the point $S^h(N) \in \mathbb{C}$ is chosen according to the letter w_N (w_{N-1} resp.).

In Figure 10.2, we observe that the fractal can be decomposed into three parts in two distinct ways, which defines an exchange of pieces inside the fractal.

10.7 Impact

After the publication of [5, 6], I put my work on multidimensional continued fraction algorithms aside and I started to focus on the newly discovered Jeandel-Rao aperiodic tilings and more generally on 4-to-2 cut and project schemes. Meanwhile, the progresses in this subject did not stop [53, 37, 115, 107, 134, 77, 84, 78, 179].

Recent progresses [133, 85] proved that the exchange of pieces shown in Figure 10.2 is almost surely equivalent to a rotation on a two-dimensional torus, and more importantly that almost every rotation on the 2-dimensional torus admits a coding of complexity $2n + 1$ through such a fractal partition of the 2-torus.

A key point in the proof of balancedness for $\mathcal{C}$-adic sequences is using the semi-norm $\|\cdot\|_D : \mathbb{R}^d \to \mathbb{R}$ defined as $\|v\|_D = \max(v) - \min(v)$ which turned out to be quite effective for this purpose. The same norm and the same approach was recently used to prove that a class of interval translation mappings introduced by Bruin and Troubetzkoy has almost always the Pisot property [44].

10.8 Open questions

$\mathcal{C}$-adic sequences are a very nice generalization of Sturmian sequence to ternary sequences: factor complexity bounded above by $2n+1$, generated by product of simple 2 substitutions, almost always balanced and the subshift they generate is almost surely equivalent to a translation on a 2-torus. Nevertheless, we don't know if such a nice low complexity extension can be made over larger alphabet of size ≥ 4.

Question 10.5. *Let $d \geq 3$, does there exists a family of words of complexity bounded above by $(d-1)n+1$, which are almost always balanced and which can achieve all linearly independent vectors of letter frequencies?*

Recent results by Berthé, Steiner and Thuswaldner seem to indicate that §-adic sequences generated by typical multidimensional continued fraction algorithms will not be balanced if the dimension gets larger than 10 [84].

PART IV

CONTRIBUTIONS WITHIN 4-TO-2 CUT AND PROJECT SCHEMES

CONTENTS

Chapter 11

Jeandel-Rao tilings

"Do cool things"

— A slogan posted on a billboard in the atrium of LaBRI, Spring 2017.

In this chapter, we summarize our main results about Jeandel-Rao tilings split into 5 papers [7, 10, 8, 9, 13] It is the longest chapter of this thesis, because their study turned out to be very rich. However, the objective of this chapter is to remain at a high level. The reader who wants more details on the combinatorial and geometrical algorithms that are developed in this study is suggested to read the articles directly or read the chapter [21]. This other chapter offers a transversal reading of the work made on Jeandel-Rao tilings. It presents the desubstitution of Wang shifts on one-side and the Rauzy induction of $\mathbb{Z}^2$-action and polygonal partitions on the other side while focusing on the simpler self-similar Wang shift hidden in the Jeandel-Rao tilings.

Our work on Jeandel-Rao tilings allowed to make new connections between notions that are a priori disjoint:

- Sturmian sequences obtained as symbolic codings of *zero* entropy irrational rotations (having no periodic points),
- subshifts of finite type obtained as the symbolic codings of *positive* entropy hyperbolic automorphisms of the torus Markov partitions (having lots of periodic points).

Behaviors in one dimension, are quite different in two dimensions and higher. In particular, we know from Berger [67] that there exists two-dimensional aperiodic shift of finite type. Therefore what is generally accepted as incompatible for one-dimensional dynamical systems might become possible in dimension 2. This is exemplified by the existence of a Markov Partition for a $\mathbb{Z}^2$-action acting by rotation (or should we say translation) on a 2-dimensional torus.

11.1 How it all started: computer explorations

After the talk made by Emmanuel Jeandel at the France annual GDR-IM meeting in January 2016, at LIPN, Paris[1], I was quite curious about the fact that Fibonacci numbers were appearing in the description of the new aperiodic tilings allowed with the new smallest aperiodic set of 11 Wang tiles discovered by Emmanuel Jeandel and Michael Rao [163]. At that time, I was too busy with my current postdoctoral projects and job applications. Also, it was too risky to start investigating questions on aperiodic tilings at that time, since I did not work on aperiodic tilings, Wang tilings or multidimensional subshifts before.

One year later, the situation was different (I started my position at CNRS on January 1st, 2017). The presentation of Michael Rao on April 6th 2017 at the Centre de recherche mathématiques (CRM) in Montréal about the new aperiodic set of 11 Wang tiles finished with a slide saying

[1] `https://lipn.univ-paris13.fr/GDR-IM-2016/talks.php`

Open question 2 : proof from the book ?
If we look at densities of 1 on each line on an infinite tiling, one transducer add $\varphi - 1$ and the other add $\varphi - 2$,
$\rightarrow$ additive-Kari-type ?

Since

$$\varphi - 1 \equiv \varphi - 2 \equiv \varphi \pmod 1,$$

what Michael was saying is that there was a uniform *vertical* rotation $\pmod 1$ involving the golden ratio.

Two days before, on April 4th, 2017, Jarkko Kari made a talk at the same conference entitled *Piecewise affine functions, Sturmian sequences and aperiodic tilings* where he presented generalizations of the Kari-Culik aperiodic sets of Wang tiles [171]. The proof of existence of aperiodic tilings with these tiles was involving *horizontal* Sturmian-like rotations. A natural question came up at that point.

Question 11.1. *What if Jeandel-Rao tilings are coded by vertical and horizontal rotations?*

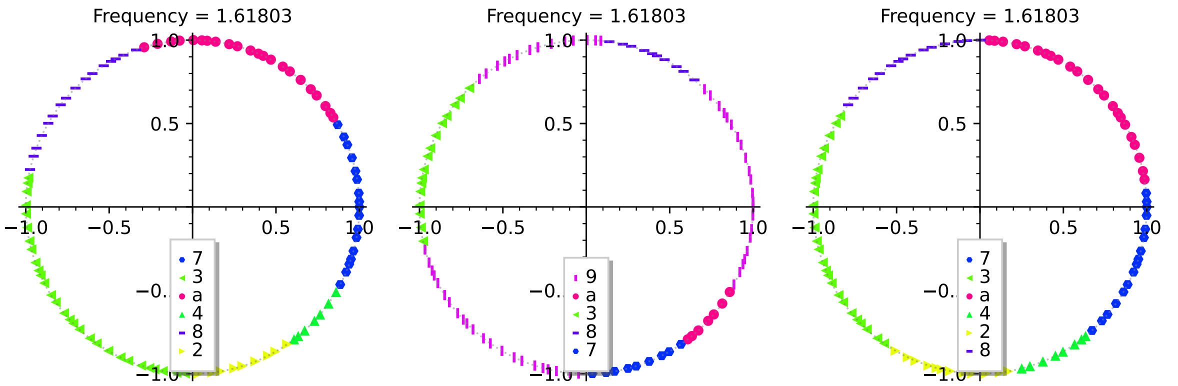
Knowing that we can easily check if a sequence is obtained by a coding of a rotation (for instance using the Python/SageMath function `draw_sequence_on_circle(sequence, frequency)` defined in Section 3.10), I asked Michael to provide me with one big enough tiling with Jeandel-Rao tiles.

He provided me with a 986-line file called `patch` each line made of 2583 characters in the set $\{\mathtt{a}, \mathtt{b}\}$. Already, we could confirm that Fibonacci numbers were involved both horizontal and vertically since 987 and 2584 are Fibonacci numbers. The binary alphabet $\{\mathtt{a}, \mathtt{b}\}$ is related to the decomposition of Jeandel-Rao tilings into horizontal strips of height 4 or 5 as explained in their article. The bottom and top labels of these strips are biinfinite binary sequences. This is what the file Michael gave me was describing.

Here, we may reproduce this day-1 computer experiment using a file present on the Emmanuel Jeandel website describing a 100×100 valid rectangular tiling with Jeandel-Rao tiles[2]. Of course, the first frequency that we try is the golden ratio. Here is what we get.

```
sage: import urllib
sage: url = "https://members.loria.fr/EJeandel/research/100.txt"
sage: content = urllib.request.urlopen(url).read()
sage: Jeandel100 = [line.decode() for line in content.splitlines()]
sage: Jeandel100[35][:80]
73a43a3873a2873a43a3873a2873873a2873a43a3873a2873a43a2873a2873873a2873a43a3873a2
sage: keys = "0123456789a"
sage: GJR35 = draw_sequence_on_circle(Jeandel100[35], frequency=(1+sqrt(5))/2, keys=keys)
sage: GJR36 = draw_sequence_on_circle(Jeandel100[36], frequency=(1+sqrt(5))/2, keys=keys)
sage: GJR58 = draw_sequence_on_circle(Jeandel100[58], frequency=(1+sqrt(5))/2, keys=keys)
sage: GJR = graphics_array([GJR35, GJR36, GJR58])
```

[2]https://members.loria.fr/EJeandel/research/wang.html



On the left, in the middle and on the right, we see intervals appearing for the 35-th, 36-th and 58-th row of the 100×100 patch present on Emmanuel website. Each other row also produces intervals when drawn using the golden ratio frequency. The answer to Question 11.1 was therefore positive.

What was more problematic was that no two rows was producing the same intervals. Sometimes, they looked similar (compare the image for the 35th-row and 58th-row), but never the intervals were of the same lengths. The fact that vertical rotations were involved as deduced from the observation of Michael Rao suggested to search for coding of horizontal and vertical rotations on a 2-dimensional torus. It is then natural to extend the function `draw_sequence_on_circle` defined in Section 3.10 to the 2-dimensional setup. This is what we do in the following Python/SageMath functions.

```
sage: from collections import defaultdict
sage: def preimage2d(rectangular_pattern):
....:     "Input␣is␣using␣matrix␣coordinates,␣output␣using␣Euclidean␣coordinates"
....:     d = defaultdict(list)
....:     nlines = len(rectangular_pattern)
....:     for (j, line) in enumerate(rectangular_pattern):
....:         for (i, a) in enumerate(line):
....:             d[a].append((i,nlines - 1 - j))
....:     return dict(d)
sage: from sage.functions.other import floor
sage: def frac(x):
....:     return x - floor(x)
sage: def draw_pattern_on_torus(pattern, M):
....:     d = preimage2d(pattern)
....:     c_dict = dict(zip(d.keys(), rainbow(len(d))))
....:     markers = "+x><^vDH_|os*,dh12345678"
....:     m_dict = dict(zip(d.keys(), markers))
....:     G = Graphics()
....:     for a in d:
....:         L = [M*vector((i,j)) for (i,j) in d[a]]
....:         fracL = [(frac(x),frac(y)) for (x,y) in L]
....:         G += points(fracL, color=c_dict[a], legend_label=a, marker=m_dict[a])
....:     return G
```

For example, here are some usage of the above functions:

```
sage: preimage2d([[1,2,3],[5,5,5]])
{1: [(0, 1)], 2: [(1, 1)], 3: [(2, 1)], 5: [(0, 0), (1, 0), (2, 0)]}
sage: frac(pi)
pi - 3
```

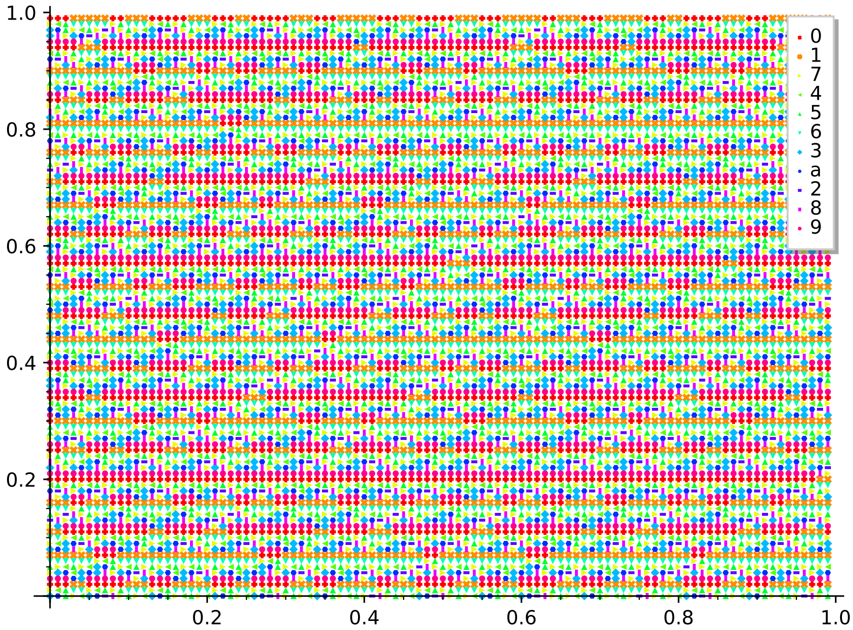
In SageMath, we define the golden ratio as a number field element, because computations are exact and faster this way:

```
sage: z = polygen(QQ, "z")
sage: K.<phi> = NumberField(z^2-z-1, embedding=RR(1.6))
sage: phi.n(digits=80)
1.6180339887498948482045868343656381177203091798057628621354486227052604628189024
```

It took some months to figure out the structure of Jeandel-Rao tilings. But it can be explained in four easy experimentations using the above SageMath functions. In each of the four steps, we draw the Jeandel 100×100 patch by associating a colored point to each tile.

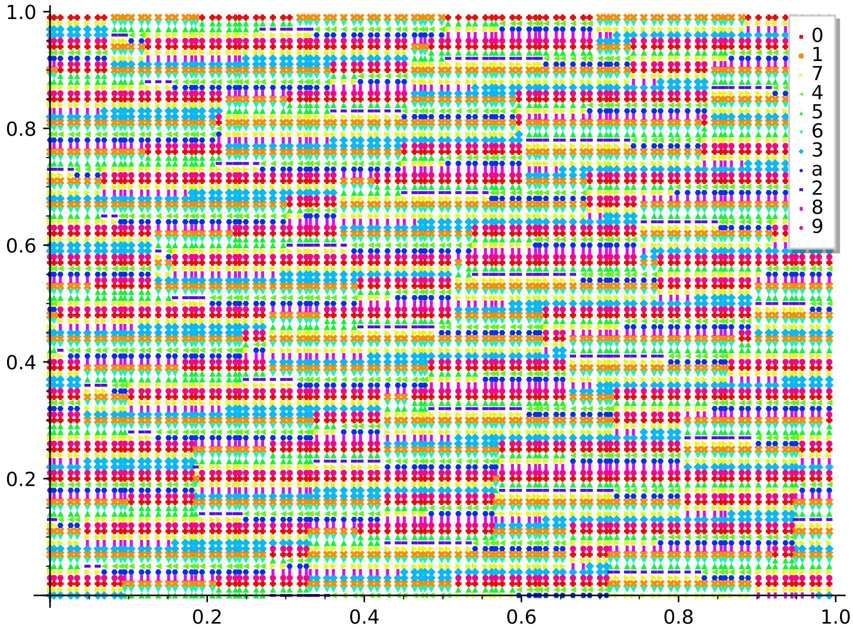
Firstly, we start by using the matrix $\left(\begin{smallmatrix}100 & 0\\ 0 & 100\end{smallmatrix}\right)^{-1}$. Using the frequency $\frac{1}{100}$ horizontally and vertically is a trick to make the points represent the tiling itself: the points at the top correspond to tiles at the top of the patch, the points on the right correspond to tiles on the right of the patch, etc.

```
sage: M1 = matrix.column([(100,0), (0,100)])
sage: G1 = draw_pattern_on_torus(Jeandel100, M1.inverse())
```



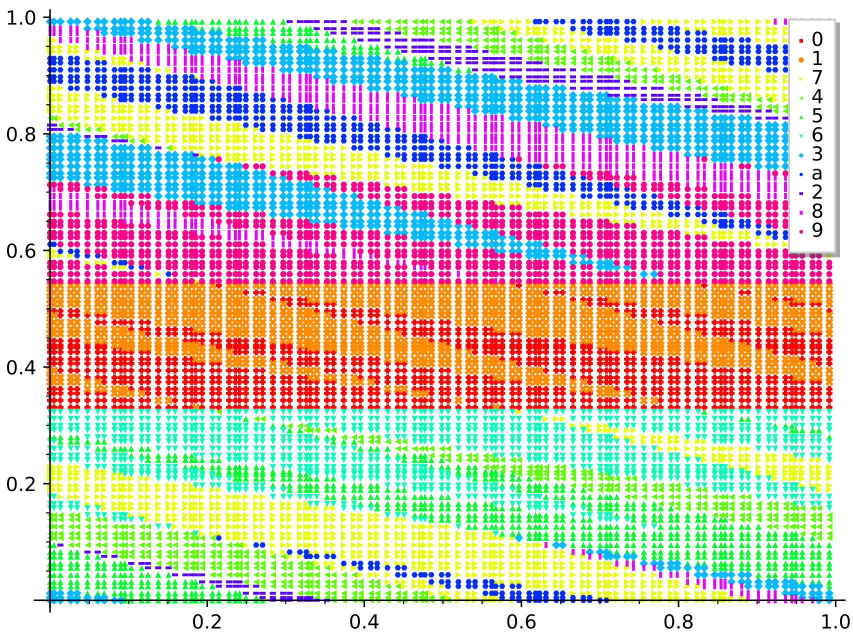
Secondly, we use the matrix $\begin{pmatrix}\varphi & 0\\ 0 & 100\end{pmatrix}^{-1}$. This makes each row in the patch to wrap around a circle (shown horizontally on the image below) with golden mean frequency. The intervals we have seen earlier on a circle now appear horizontally on each row. Using the frequency $\frac{1}{100}$ vertically is a trick to make the points of the j-th row correspond to tiles on the j-th row of the patch.

```
sage: M2 = matrix.column([(phi,0), (0,100)])
sage: G2 = draw_pattern_on_torus(Jeandel100, M2.inverse())
```



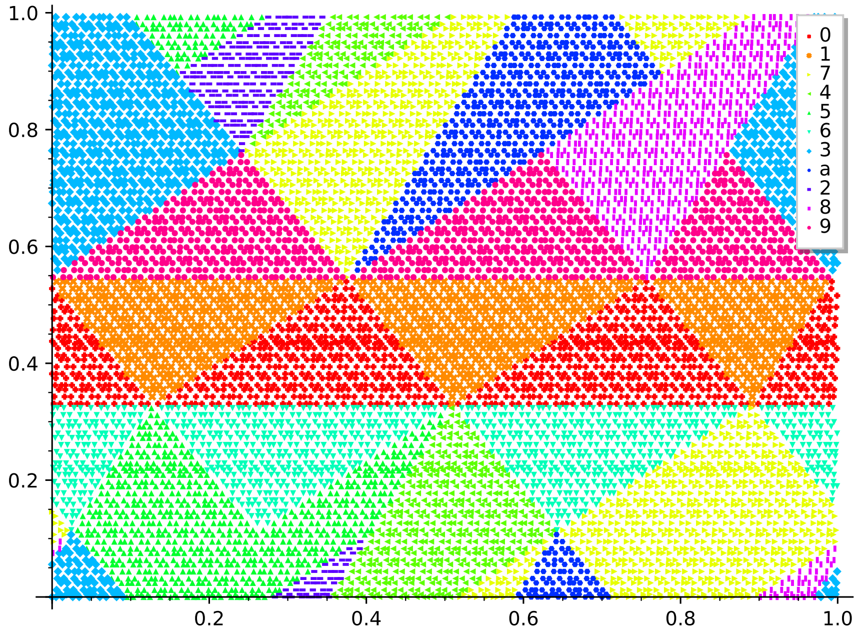
Thirdly, we use the matrix $\begin{pmatrix}\varphi & 0\\ 0 & \varphi+3\end{pmatrix}^{-1}$. This makes sense because the vertical distance (or return time) between rows involving tiles labeled #0 and #1 is 4 or 5 with an average of $\varphi+3$ as noticed already by Jeandel and Rao [163]. We observe that this gathers the rows looking similar close apart.

```
sage: M3 = matrix.column([(phi,0), (0,phi+3)])
sage: G3 = draw_pattern_on_torus(Jeandel100, M3.inverse())
```



Finally, we use the matrix $\begin{pmatrix}\varphi & 1\\ 0 & \varphi+3\end{pmatrix}^{-1}$. Replacing a zero by a one above the diagonal was the hardest step to figure out. Indeed, there is a sheer happening in Jeandel-Rao tilings. This is one of the reason that makes the description of Jeandel-Rao tilings more difficult, but certainly very interesting!

```
sage: M4 = matrix.column([(phi,0), (1,phi+3)])
sage: G4 = draw_pattern_on_torus(Jeandel100, M4.inverse())
```



Here something magical happens in the previous output. We understand that the intervals are no longer intervals: they are portions of polygons forming a partition of the torus. We can guess the coordinates of the vertices of the polygons with respect to some origin. These coordinates are simple to express after applying the transformation $x \mapsto \begin{pmatrix} \varphi & 1 \\ 0 & \varphi+3 \end{pmatrix} x$. This transformation renormalizes the polygonal partition of the unit square to a polygonal partition of a fundamental domain of the lattice $\begin{pmatrix} \varphi & 1 \\ 0 & \varphi+3 \end{pmatrix} \mathbb{Z}^2$.

11.2 Constructing valid Jeandel-Rao tilings using a polygonal partition

Let $\Gamma_0 = \begin{pmatrix} \varphi & 1 \\ 0 & \varphi+3 \end{pmatrix} \mathbb{Z}^2$ be a lattice in $\mathbb{R}^2$ involving the golden ratio $\varphi = \frac{1+\sqrt{5}}{2}$. On the 2-dimensional torus $\mathbb{R}^2/\Gamma_0$, we may define the following $\mathbb{Z}^2$-action R_0:

$$\begin{array}{rccc} R_0 : & \mathbb{Z}^2 \times \mathbb{R}^2/\Gamma_0 & \to & \mathbb{R}^2/\Gamma_0 \\ & (k, \boldsymbol{x}) & \mapsto & \boldsymbol{x} + k. \end{array}$$

The $\mathbb{Z}^2$-action R_0 defines a dynamical system $\mathbb{Z}^2 \overset{R_0}{\curvearrowright} \mathbb{R}^2/\Gamma_0$.

In [8], we proved that a minimal subshift within the Jeandel–Rao Wang shift is symbolic extension of the dynamical system $\mathbb{Z}^2 \overset{R_0}{\curvearrowright} \mathbb{R}^2/\Gamma_0$. The symbolic coding is obtained through a polygonal partition $\mathcal{P}_0$ of a rectangular fundamental domain $[0, \varphi) \times [0, \varphi + 3)$ of $\mathbb{R}^2/\Gamma_0$. The partition $\mathcal{P}_0$ is shown in Figure 11.1. It is indexed by integers from the set $\{0, 1, 2, \ldots, 10\}$.

The symbolic dynamical system $\mathcal{X}_{\mathcal{P}_0,R_0}$ corresponding to $\mathcal{P}_0, R_0$ is the topological closure of the set of all configurations $w \in \{0, 1, \ldots, 10\}^{\mathbb{Z}^2}$ obtained from the coding by the partition $\mathcal{P}_0$ of the orbit of some starting point in $\mathbb{R}^2/\Gamma_0$ by the $\mathbb{Z}^2$-action of R_0. We say that $\mathcal{X}_{\mathcal{P}_0,R_0}$ is a *subshift* as it is also closed under the shift σ by integer translations. The fact that $\mathcal{X}_{\mathcal{P}_0,R_0} \subset \Omega_0$ is illustrated in Figure 11.2 where $\Omega_0 \subset \{0, 1, \ldots, 10\}^{\mathbb{Z}^2}$ is the Jeandel-Rao Wang shift.

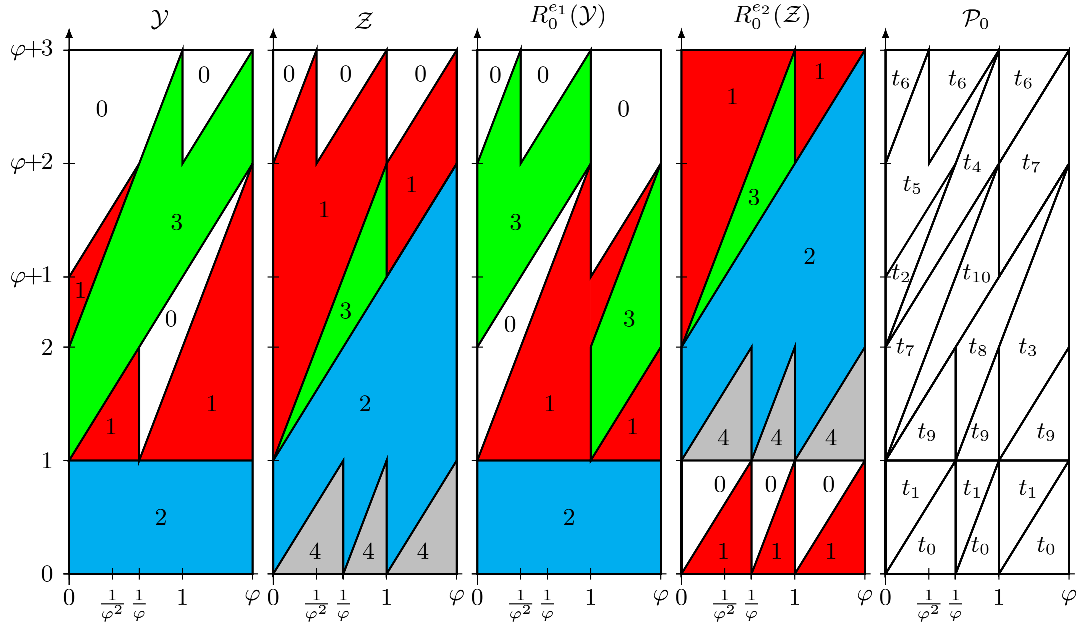


Figure 11.1: The partition for the Jeandel-Rao Wang tiles is the refinement of four polygonal partitions (one partition for each side of the Wang tiles). From left to right, the partition $\mathcal{Y}$ for the right color, $\mathcal{Z}$ for the top color, $R_0^{e_1}(\mathcal{Y})$ for the left color and $R_0^{e_2}(\mathcal{Z})$ for the bottom color. Their refinement is the partition $\mathcal{P}_0$ where each part is associated with one of Jeandel-Rao Wang tiles. Note that the typo in the coordinates of the y-axis in the published version [8] (shame on me) is fixed in this figure.

Theorem 11.2 ([8]). *The Jeandel-Rao Wang shift Ω_0 has the following properties:*

(i) *$\mathcal{X}_{\mathcal{P}_0,R_0} \subset \Omega_0$ is a minimal and aperiodic subshift of Ω_0,*

(ii) *the partition $\mathcal{P}_0$ gives a symbolic representation of $\mathbb{Z}^2 \overset{R_0}{\curvearrowright} \mathbb{R}^2/\Gamma_0$,*

(iii) *$\mathbb{Z}^2 \overset{R_0}{\curvearrowright} \mathbb{R}^2/\Gamma_0$ is the maximal equicontinuous factor of $\mathbb{Z}^2 \overset{\sigma}{\curvearrowright} \mathcal{X}_{\mathcal{P}_0,R_0}$,*

(iv) *the set of fiber cardinalities of the factor map $\mathcal{X}_{\mathcal{P}_0,R_0} \to \mathbb{R}^2/\Gamma_0$ is $\{1, 2, 8\}$,*

(v) *the dynamical system $\mathbb{Z}^2 \overset{\sigma}{\curvearrowright} \mathcal{X}_{\mathcal{P}_0,R_0}$ is uniquely ergodic,*

(vi) *the measure-preserving dynamical system $(\mathcal{X}_{\mathcal{P}_0,R_0}, \mathbb{Z}^2, \sigma, \nu)$ is isomorphic to $(\mathbb{R}^2/\Gamma_0, \mathbb{Z}^2, R_0, \lambda)$ where ν is the unique shift-invariant probability measure on $\mathcal{X}_{\mathcal{P}_0,R_0}$ and λ is the Haar measure on $\mathbb{R}^2/\Gamma_0$.*

A do-it-yourself puzzle illustrating Theorem 11.2 is available at

`http://www.slabbe.org/blogue/2024/04/`
`a-do-it-yourself-polygonal-partition-to-construct-jeandel-rao-tilings/`.

It allows hand made construction of configurations in $\mathcal{X}_{\mathcal{P}_0,R_0} \subset \Omega_0$ as the symbolic representation of starting points in $\mathbb{R}^2/\Gamma_0$; see Figure 11.3.

Note that a similar result was obtained for Penrose tilings [241, Theorem A]. In particular, it was shown that the set of fiber cardinalities for Penrose tilings (with the action of $\mathbb{R}^2$) is $\{1, 2, 10\}$. In [192], it was proved that the set of fiber cardinalities is $\{1, 2, 6, 12\}$ for a minimal hull among

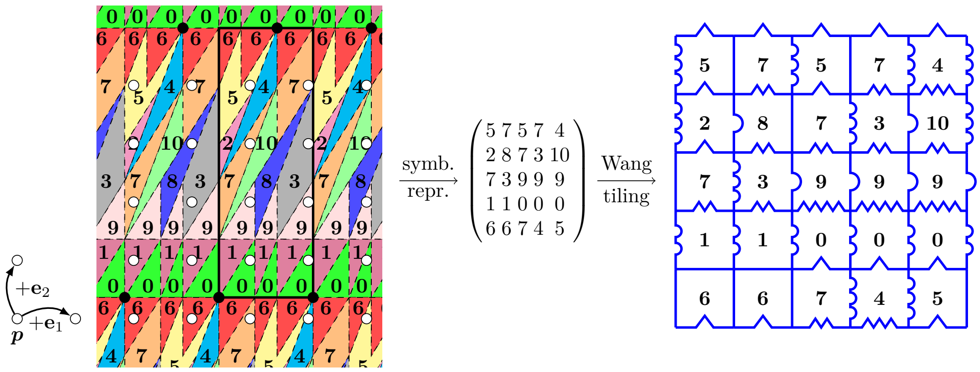


Figure 11.2: On the left, we illustrate the lattice $\Gamma_0 = \langle(\varphi, 0), (1, \varphi + 3)\rangle_{\mathbb{Z}}$, where $\varphi = \frac{1+\sqrt{5}}{2}$, with black vertices, a rectangular fundamental domain of the flat torus $\mathbb{R}^2/\Gamma_0$ with a black contour and a polygonal partition $\mathcal{P}_0$ of $\mathbb{R}^2/\Gamma_0$ with indices in the set $\{0, 1, \ldots, 10\}$. We show that for every starting point $\boldsymbol{p} \in \mathbb{R}^2$, the coding of the shifted lattice $\boldsymbol{p} + \mathbb{Z}^2$ under the polygonal partition yields a configuration $w : \mathbb{Z}^2 \to \{0, 1, \ldots, 10\}$ which is a symbolic representation of $\boldsymbol{p}$. The configuration w corresponds to a valid tiling of the plane with Jeandel-Rao's set of 11 Wang tiles.

Taylor-Socolar hexagonal tilings. We show in [8] that the set of fiber cardinalities of the maximal equicontinuous factor of a minimal dynamical system is invariant under topological conjugacy. Therefore, the Jeandel-Rao tilings, the Penrose tilings and the Taylor-Socolar tilings are inherently different.

11.3 A degenerate cut and project scheme for Jeandel-Rao tiings

A consequence of Theorem 11.2 is a description of the aperiodic Wang shift Ω_0 with cut and project schemes. Lifting a tiling by Wang tiles is more difficult because all edges of the Wang tiles are either horizontal or vertical unit vectors in the plane. The solution is to use a degenerate cut and project scheme where the projection of the lattice in the physical space is not injective.

We want to describe the positions $Q \subseteq \mathbb{Z}^2$ of patterns in configurations belonging to $\mathcal{X}_{\mathcal{P}_0,R_0} \subsetneq \Omega_0$. It is not possible to construct a classical cut and project scheme satisfying that $\pi|_{\mathcal{L}}$ is an injective map. But it is possible to construct a degenerate 4-to-2 cut and project scheme as in Section 7.2.

Let $\Gamma_0 = \langle(\varphi, 0), (1, \varphi + 3)\rangle_{\mathbb{Z}}$ be a cocompact lattice in $\mathbb{R}^2$ where $\varphi = \frac{1+\sqrt{5}}{2}$. We define the projections π and π_{int} on $\mathbb{R}^4$ as:

$$\begin{array}{rccc} \pi : & \mathbb{R}^4 & \to & \mathbb{R}^2 \\ & (x_1, x_2, x_3, x_4) & \mapsto & (x_1 + x_2, x_3 + x_4) \end{array}$$

and

$$\begin{array}{rccc} \pi_{\text{int}} : & \mathbb{R}^4 & \to & \mathbb{R}^2/\Gamma_0 \\ & (x_1, x_2, x_3, x_4) & \mapsto & \left(x_1 - \frac{1}{\varphi}x_2 + \frac{1}{\varphi}x_4, x_3 - (\varphi + 2)x_4\right). \end{array}$$

The physical space is $K = \mathbb{R}^2$. The internal space is $H = \mathbb{R}^2/\Gamma_0$. We consider the lattice $\mathcal{L} = \mathbb{Z}^4$. The projection $\pi|_{\mathcal{L}} : \mathcal{L} \to \mathbb{R}^2$ is not injective. But we have

$$\text{Ker}(\pi) \cap \mathcal{L} = \langle(1, -1, 0, 0), (0, 0, 1, -1)\rangle_{\mathbb{Z}}.$$

The image of these vectors under π_{int} are

$$\pi_{\text{int}}((1, -1, 0, 0)) = (1 + \tfrac{1}{\varphi}, 0) = (\varphi, 0) = 0 \pmod{\Gamma_0},$$
$$\pi_{\text{int}}((0, 0, 1, -1)) = (-\tfrac{1}{\varphi}, \varphi + 3) = 0 \pmod{\Gamma_0}.$$

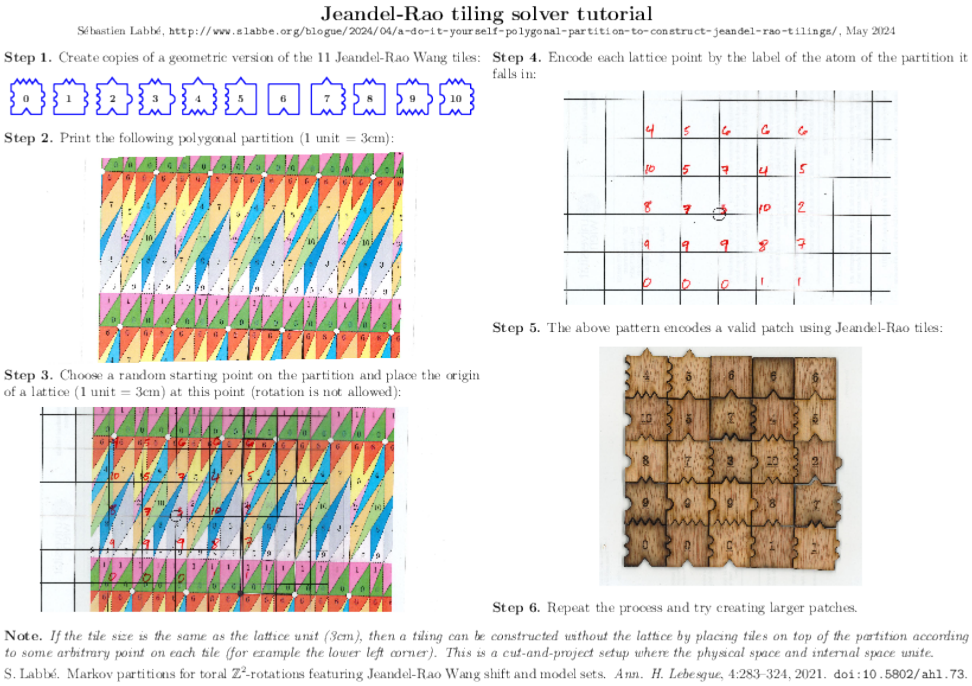


Figure 11.3: The Jeandel-Rao tiling solver tutorial.

Therefore, condition (7.1) is satisfied and we conclude that the 5-uple $(H, K, \mathcal{L}, \pi, \pi_{\text{int}})$ is a degenerate 4-to-2 cut and project scheme.

The star map defined from this degenerate cut and project scheme is the natural projection $\mathbb{Z}^2 \to \mathbb{R}^2/\Gamma_0$:

$$\star : x \mapsto x \pmod{\Gamma_0}.$$

The fact that the $\star$ map is the identity modulo the lattice Γ_0 is what allows to identify the physical space with the internal space in Figure 11.2 and Figure 11.3 as done in Example 7.5 for a degenerate 2-to-1 cut and project scheme.

Recall that we proved in Theorem 11.2 that $\mathcal{X}_{\mathcal{P}_0,R_0} \subsetneq \Omega_0$ and that there exists a factor map f_0 from $(\mathcal{X}_{\mathcal{P}_0,R_0}, \mathbb{Z}^2, \sigma)$ to $(\mathbb{R}^2/\Gamma_0, \mathbb{Z}^2, R_0)$. Therefore any Jeandel-Rao configuration $w \in \mathcal{X}_{\mathcal{P}_0,R_0} \subsetneq \Omega_0$ can be qualified as a singular or generic according to whether $f_0(w)$ is in the boundary of the partition $\Delta_{\mathcal{P}_0,R_0} \subset \mathbb{R}^2/\Gamma_0$ or not.

We show that the occurrences of patterns in the Wang shifts are regular model sets. Definitions of regular, generic and singular models sets can be found in Section 7.1.

Theorem 11.3 ([8])**.** *For every Jeandel-Rao configuration $w \in \mathcal{X}_{\mathcal{P}_0,R_0} \subsetneq \Omega_0$, the set $Q \subseteq \mathbb{Z}^2$ of occurrences of a pattern in w is a regular model set within the degenerate 4-to-2 cut and project scheme defined above. If w is a generic (resp. singular) configuration, then Q is a generic (resp. singular) model set.*

It was shown that the action of $\mathbb{R}^2$ by translation on the set of Penrose tilings is an almost one-to-one extension of a minimal $\mathbb{R}^2$-action by rotations on $\mathbb{T}^4$ [241] (the fact that it is $\mathbb{T}^4$ instead of $\mathbb{T}^2$ is related to the consideration of tilings instead of shifts). This result can also be seen as a higher dimensional generalization of the Sturmian dynamical systems. Note that a shift of finite type or Wang shift can be explicitly constructed from the Penrose tiling dynamical system, as shown in [245]. This calls for a common point of view including Jeandel-Rao aperiodic tilings, Penrose tilings and others. For example, we do not know if Penrose tilings can be seen as a symbolic dynamical system associated to a Markov partition like it is the case for the Jeandel-Rao Wang shift. It is possible that such Markov partitions exist only for tilings associated to some algebraic numbers, see [57].

11.4 Does the coding by the partition generate all Jeandel–Rao configurations?

After Theorem 11.2, a natural question is whether $\mathcal{X}_{\mathcal{P}_0,R_0} \supset \Omega_0$? In other words, is every configuration in the Jeandel-Rao Wang shift obtained as the coding of the shifted lattice $\boldsymbol{p}+\mathbb{Z}^2$ for some point $\boldsymbol{p}\in\mathbb{R}^2/\Gamma_0$?

Proving $\mathcal{X}_{\mathcal{P}_0,R_0} \subset \Omega_0$ in Theorem 11.2 corresponds to the easy direction in the proof of Morse-Hedlund's theorem, namely that codings of irrational rotations have pattern complexity $n+1$. [216, 113]. Proving the converse, i.e., that almost every sequence of complexity $n+1$ is harder. It is based, one the one hand, on the desubstitution of sequences of complexity $n+1$, and on the other hand, on the Rauzy induction of irrational rotations on the circle. As it is well-known, this is related to the continued fraction expansion of the angle of rotation. Recent treatment of this relation can be found in [38, 268]

To find the starting point in the torus $\mathbb{R}^2/\Gamma_0$ associated to some Jeandel–Rao configuration, we need to generalize the proof of Morse-Hedlund theorem to the 2-dimensional setup of Wang subshifts. This has lead to split the proof of the converse into two parts (three articles). First, we computed the substitutive structure of Ω_0 [7, 10]. Then, we computed the substitutive structure of $\mathcal{X}_{\mathcal{P}_0,R_0}$ from the Rauzy induction of the $\mathbb{Z}^2$-action R_0 and polygonal partition $\mathcal{P}_0$ [9]. Both substitutive structures (given as an eventually periodic sequence of 2-dimensional substitutions) computed from totally different objects and algorithms are the same!

These results are summarized in the next sections. Note that the three articles [7], [10] and [9] totaling 130 pages. In this thesis, the goal is not to explain the details of the techniques again, but rather to give an overview of the main results and how they interact with each others. For more details about the techniques, the curious reader may want to take a look at the chapter written for a book to appear in the series of the Chair Jean Morlet (Jayadev Athreya, Fall 2023) [21]. In that chapter, we offer, in a single document, a transversal reading of our work done on Jeandel-Rao tilings by focusing on the self-similar Wang shift hidden in Jeandel-Rao tilings. The chapter contains also a lot of exercises and SageMath code allowing to reproduce the experiments: very good for students or anyone wanting to learn these techniques.

11.5 Desubstitution of Wang tilings

The description of the substitutive structure of Jeandel-Rao tilings was split into two articles. In one article, we focused on an aperiodic and self-similar set of 19 Wang tiles $\mathcal{U}$ [7]. In the other article [10], we show that there exists an aperiodic and minimal subshift X_0 of the Jeandel-Rao tilings Ω_0 such that every configuration in X_0 can be decomposed uniquely into 19 distinct patches (two of size 45, four of size 70, six of size 72 and seven of size 112) that are equivalent to the set of 19 Wang tiles $\mathcal{U}$.

Formally, configurations in $X_0 \subset \Omega_0$ can be constructed from an eventually periodic sequence of 2-dimensional substitutions. The substitutions are computed automatically using the same algorithms at each step: `FindMarkers` and `FindSubstitution`. These algorithms were implemented in the `slabbe` package [24] of SageMath [246] and their pseudo-code is available in [10]. There are a dozens of such elementary steps between Jeandel-Rao tilings $X_0 \subset \Omega_0$ and $\Omega_{\mathcal{U}}$ that are illustrated in Figure 11.4.

$$\begin{array}{ccccccccccccccccccccccccc}
\Omega_0 & \xleftarrow{\omega_0} & \Omega_1 & \xleftarrow{\omega_1} & \Omega_2 & \xleftarrow{\omega_2} & \Omega_3 & \xleftarrow{\omega_3} & \Omega_4 & & & & & & & & & & & & & & & & \circlearrowleft^{\omega_{\mathcal{U}}} \\
\cup & & \cup & & \cup & & \cup & & \cup & & & & & & & & & & & & & & & & \\
X_0 & \xleftarrow{\omega_0} & X_1 & \xleftarrow{\omega_1} & X_2 & \xleftarrow{\omega_2} & X_3 & \xleftarrow{\omega_3} & X_4 & \xleftarrow{\jmath} & \Omega_5 & \xleftarrow{\eta} & \Omega_6 & \xleftarrow{\omega_6} & \Omega_7 & \xleftarrow{\omega_7} & \Omega_8 & \xleftarrow{\omega_8} & \Omega_9 & \xleftarrow{\omega_9} & \Omega_{10} & \xleftarrow{\omega_{10}} & \Omega_{11} & \xleftarrow{\omega_{11}} \Omega_{12} \xleftarrow{\rho} & \Omega_{\mathcal{U}}
\end{array}$$

Figure 11.4: Substitutive structure of Jeandel-Rao aperiodic Wang shift Ω_0 and its minimal subshift X_0 leading to the self-similar aperiodic and minimal Wang shift $\Omega_{\mathcal{U}}$ introduced in [7].

The substitutive structure of the Jeandel-Rao Wang shift is given more precisely by the following result in which we use the following notation

$$\overline{X}^{\sigma} = \bigcup_{k\in\mathbb{Z}^2} \sigma^k X = \bigcup_{k\in\mathbb{Z}^2} \{\sigma^k(x) \mid x \in X\}$$

for the closure of a set $X \subset \mathcal{A}^{\mathbb{Z}^2}$ under the shift σ.

Theorem 11.4 ([10]). *Let Ω_0 be the Jeandel-Rao Wang shift. There exist sets of Wang tiles $\{\mathcal{T}_i\}_{1\leq i\leq 12}$ together with their associated Wang shifts $\{\Omega_i\}_{1\leq i\leq 12}$ that provide the substitutive structure of Jeandel-Rao tilings. More precisely,*

(i) There exists a sequence of recognizable 2-dimensional morphisms:

$$\Omega_0 \xleftarrow{\omega_0} \Omega_1 \xleftarrow{\omega_1} \Omega_2 \xleftarrow{\omega_2} \Omega_3 \xleftarrow{\omega_3} \Omega_4$$

that are onto up to a shift, i.e., $\overline{\omega_i(\Omega_{i+1})}^{\sigma} = \Omega_i$ for each $i \in \{0, 1, 2, 3\}$.

(ii) There exists an embedding $\jmath : \Omega_5 \to \Omega_4$ which is a topological conjugacy onto its image.

(iii) There exists a shear conjugacy $\eta : \Omega_6 \to \Omega_5$ which shears Wang tilings by the action of the matrix $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$.

(iv) There exists a sequence of recognizable 2-dimensional morphisms:

$$\Omega_6 \xleftarrow{\omega_6} \Omega_7 \xleftarrow{\omega_7} \Omega_8 \xleftarrow{\omega_8} \Omega_9 \xleftarrow{\omega_9} \Omega_{10} \xleftarrow{\omega_{10}} \Omega_{11} \xleftarrow{\omega_{11}} \Omega_{12}$$

that are onto up to a shift, i.e., $\overline{\omega_i(\Omega_{i+1})}^{\sigma} = \Omega_i$ for each $i \in \{6, 7, 8, 9, 10, 11\}$.

(v) The Wang shift Ω_{12} is equivalent to $\Omega_{\mathcal{U}}$, thus is self-similar, aperiodic and minimal.

The intermediate subshifts are given by a sequence of sets of Wang tiles $\{\mathcal{T}_i\}_{1\leq i\leq 12}$ together with their associated Wang shifts $\{\Omega_i\}_{1\leq i\leq 12}$. The cardinality of the involved sets of tiles is in the table below.

Set of tiles	$\mathcal{T}_0$	$\mathcal{T}_1$	$\mathcal{T}_2$	$\mathcal{T}_3$	$\mathcal{T}_4$	$\mathcal{T}_5$	$\mathcal{T}_6$	$\mathcal{T}_7$	$\mathcal{T}_8$	$\mathcal{T}_9$	$\mathcal{T}_{10}$	$\mathcal{T}_{11}$	$\mathcal{T}_{12}$	$\mathcal{U}$
Cardinality	11	13	20	24	28	29	29	20	20	22	18	21	19	19

Figures illustrating the substitutions between these Wang shifts are in the appendix of [10].

The theorem has the following consequences.

Corollary 11.5 ([10]). *Ω_i is aperiodic and minimal for every i with $5 \leq i \leq 12$.*

Let $X_4 = \jmath(\Omega_5)$ be the image of the embedding $\jmath$, as well as $X_3 = \overline{\omega_3(X_4)}^{\sigma}$, $X_2 = \overline{\omega_2(X_3)}^{\sigma}$, $X_1 = \overline{\omega_1(X_2)}^{\sigma}$ and $X_0 = \overline{\omega_0(X_1)}^{\sigma}$.

Corollary 11.6 ([10]). *$X_i \subseteq \Omega_i$ is an aperiodic and minimal subshift of Ω_i for every i with $0 \leq i \leq 4$.*

The fact that each tiling in X_0 can be decomposed uniquely into this procedure follows from the fact that each morphism ω_i with $0 \leq i \leq 3$ or $6 \leq i \leq 11$ is recognizable and onto up to a shift, and both $\jmath : \Omega_5 \to X_4$ and $\eta : \Omega_6 \to \Omega_5$ are one-to-one and onto. The term *recognizable morphism* essentially means that the morphism is one-to-one up to a shift.

The construction of the morphisms ω_i is inspired from a well-known method to study self-similar aperiodic tilings. One way to prove that a Wang shift is aperiodic is to use the *unique composition property* [263] also known as *composition-decomposition method* [51]. That method was used in [7] to show that $\Omega_{\mathcal{U}}$ is self-similar, minimal and aperiodic based on the notion of marker tiles and recognizability.

The same method can be used in the context of Wang shifts that are not self-similar. The idea is to prove that a Wang shift is similar to another one which is known to be aperiodic. This reminds of [86] where the authors study the recognizability for sequences of morphisms in the theory of S-adic systems on $\mathbb{Z}$ [80]. Note that applying a sequence of recognizable substitutions in the context of hierarchical tilings of $\mathbb{R}^d$ was also considered in [136, 135].

Due to the fact that we have many steps to perform to understand Jeandel-Rao tilings, we improve the method used in [7] which, based on the existence of marker tiles $M \subset \mathcal{T}$ among a set of Wang tiles $\mathcal{T}$, proved the existence of a set of Wang tiles $\mathcal{S}$ and a recognizable 2-dimensional morphism $\Omega_{\mathcal{S}} \to \Omega_{\mathcal{T}}$ that is onto up to a shift.

We provided two algorithms in [10]. Algorithm `FindMarkers` finds a set of marker tiles $M \subset \mathcal{T}$ (if it exists) from a set of Wang tiles $\mathcal{T}$ and a given surrounding radius to consider (the surrounding radius input is necessary since otherwise it is undecidable). Algorithm `FindSubstitution` computes the set of Wang tiles $\mathcal{S}$ and the recognizable 2-dimensional morphism $\Omega_{\mathcal{S}} \to \Omega_{\mathcal{T}}$ from a set of markers $M \subset \mathcal{T}$. The morphism is of the form $\square \mapsto \square, \square \mapsto \begin{smallmatrix}\square\\\square\end{smallmatrix}$ or of the form $\square \mapsto \square, \square \mapsto \square\square$ mapping each tile from $\mathcal{S}$ on a tile in $\mathcal{T}$ or on a domino of two tiles in $\mathcal{T}$.

11.6 Rauzy induction of toral $\mathbb{Z}^2$-actions

In this section, we forget about the Jeandel-Rao Wang tiles. We consider the $\mathbb{Z}^2$-action

$$\begin{array}{rccl} R_0 : & \mathbb{Z}^2 \times \mathbb{R}^2/\Gamma_0 & \to & \mathbb{R}^2/\Gamma_0 \\ & (n, \boldsymbol{x}) & \mapsto & \boldsymbol{x} + n \end{array}$$

on the torus $\mathbb{R}^2/\Gamma_0$ where

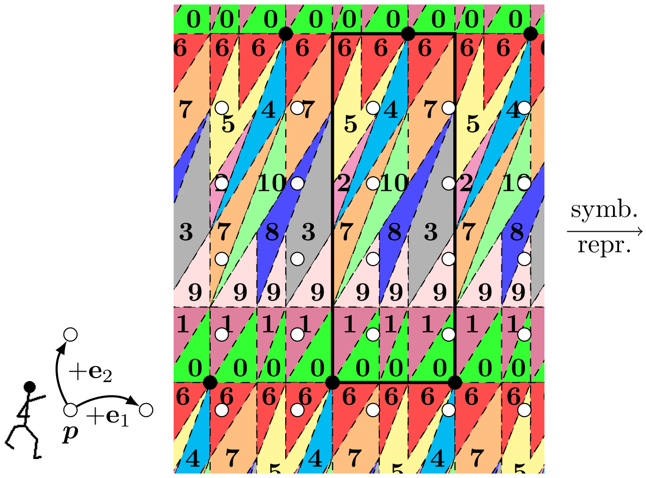
$$\Gamma_0 = \langle (\varphi, 0), (1, \varphi + 3) \rangle_{\mathbb{Z}}$$

is a lattice in $\mathbb{R}^2$ with $\varphi = \frac{1+\sqrt{5}}{2}$. We consider the symbolic dynamical system $\mathcal{X}_{\mathcal{P}_0,R_0}$ defined by coding orbits of R_0 by the polygonal partition $\mathcal{P}_0$ of the torus $\mathbb{R}^2/\Gamma_0$; see Figure 11.5.

Our goal is to describe the substitutive structure of the symbolic dynamical system $\mathcal{X}_{\mathcal{P}_0,R_0}$. The method is inspired from the proof of Morse-Hedlund's theorem and its link with continued fraction expansion and induced transformations. To achieve this, we extended in [9] the notion of Rauzy induction of IETs to the case of $\mathbb{Z}^2$-actions and we introduced the notion of induced partitions. Each induction step allows to express the original symbolic dynamical system as the image under a 2-dimensional substitution of an induced subsystem.

The substitutions are computed automatically using the same algorithms at each step: `InducedPartition` and `InducedTransformation`. These algorithms were implemented in the `slabbe` package [24] of SageMath [246] and their pseudo-code is available in [9].

The substitutive structure of $\mathcal{X}_{\mathcal{P}_0,R_0}$ is described in the next result.



$$\begin{pmatrix} 5 & 7 & 5 & 7 & 4 \\ 2 & 8 & 7 & 3 & 10 \\ 7 & 3 & 9 & 9 & 9 \\ 1 & 1 & 0 & 0 & 0 \\ 6 & 6 & 7 & 4 & 5 \end{pmatrix}$$

Figure 11.5: For every starting point $p \in \mathbb{R}^2$, the coding of the shifted lattice $p + \mathbb{Z}^2$ under the polygon partition $\mathcal{P}_0$ yields a configuration which is a symbolic representation of p.

Theorem 11.7 ([9]). *Let $\mathcal{X}_{\mathcal{P}_0,R_0}$ be the symbolic dynamical system associated to $\mathcal{P}_0$, R_0. There exist lattices $\Gamma_i \subset \mathbb{R}^2$, alphabets $\mathcal{A}_i$, $\mathbb{Z}^2$-actions $R_i : \mathbb{Z}^2 \times \mathbb{R}^2/\Gamma_i \to \mathbb{R}^2/\Gamma_i$ and topological partitions $\mathcal{P}_i$ of $\mathbb{R}^2/\Gamma_i$ indexed by letters from the alphabet $\mathcal{A}_i$ that provide the substitutive structure of $\mathcal{X}_{\mathcal{P}_0,R_0}$. More precisely,*

(i) There exists a 2-dimensional morphism $\beta_0 : \mathcal{A}_1 \to \mathcal{A}_0^{*^2}$

$$\mathcal{X}_{\mathcal{P}_0,R_0} \xleftarrow{\beta_0} \mathcal{X}_{\mathcal{P}_1,R_1}$$

that is onto up to a shift, i.e., $\mathcal{X}_{\mathcal{P}_0,R_0} = \overline{\beta_0(\mathcal{X}_{\mathcal{P}_1,R_1})}^{\sigma}$.

(ii) There exists a shear conjugacy

$$\mathcal{X}_{\mathcal{P}_1,R_1} \xleftarrow{\beta_1} \mathcal{X}_{\mathcal{P}_2,R_2}$$

shearing configurations by the action of the matrix $M = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, i.e., satisfying $\sigma^{Mk} \circ \beta_1 = \beta_1 \circ \sigma^k$ for every $k \in \mathbb{Z}^2$.

(iii) There exist 2-dimensional morphisms β_2, β_3, β_4, β_5, β_6 and β_7:

$$\mathcal{X}_{\mathcal{P}_2,R_2} \xleftarrow{\beta_2} \mathcal{X}_{\mathcal{P}_3,R_3} \xleftarrow{\beta_3} \mathcal{X}_{\mathcal{P}_4,R_4} \xleftarrow{\beta_4} \mathcal{X}_{\mathcal{P}_5,R_5} \xleftarrow{\beta_5} \mathcal{X}_{\mathcal{P}_6,R_6} \xleftarrow{\beta_6} \mathcal{X}_{\mathcal{P}_7,R_7} \xleftarrow{\beta_7} \mathcal{X}_{\mathcal{P}_8,R_8}$$

that are onto up to a shift, i.e., $\mathcal{X}_{\mathcal{P}_i,R_i} = \overline{\beta_i(\mathcal{X}_{\mathcal{P}_{i+1},R_{i+1}})}^{\sigma}$ for each $i \in \{2,3,4,5,6,7\}$.

(iv) The subshift $\mathcal{X}_{\mathcal{P}_8,R_8}$ is self-similar satisfying $\mathcal{X}_{\mathcal{P}_8,R_8} = \overline{\beta_8\beta_9\tau(\mathcal{X}_{\mathcal{P}_8,R_8})}^{\sigma}$. More precisely, there exist two 2-dimensional morphisms β_8, β_9 and a bijection $\tau : \mathcal{A}_8 \to \mathcal{A}_{10}$

$$\mathcal{X}_{\mathcal{P}_8,R_8} \xleftarrow{\beta_8} \mathcal{X}_{\mathcal{P}_9,R_9} \xleftarrow{\beta_9} \mathcal{X}_{\mathcal{P}_{10},R_{10}} \xleftarrow{\tau} \mathcal{X}_{\mathcal{P}_8,R_8}$$

that are onto up to a shift, i.e., $\mathcal{X}_{\mathcal{P}_8,R_8} = \overline{\beta_8(\mathcal{X}_{\mathcal{P}_9,R_9})}^{\sigma}$, $\mathcal{X}_{\mathcal{P}_9,R_9} = \overline{\beta_9(\mathcal{X}_{\mathcal{P}_{10},R_{10}})}^{\sigma}$ and $\mathcal{X}_{\mathcal{P}_{10},R_{10}} = \tau(\mathcal{X}_{\mathcal{P}_8,R_8})$ and the product $\beta_8\beta_9\tau$ is an expansive and primitive self-similarity.

(v) The subshift $\mathcal{X}_{\mathcal{P}_8,R_8}$ is topologically conjugate to the subshift $\mathcal{X}_{\mathcal{P}_\mathcal{U},R_\mathcal{U}}$ introduced in [8] as there exists a bijection $\zeta : \mathcal{U} \to \mathcal{A}_8$ such that $\zeta(\mathcal{X}_{\mathcal{P}_\mathcal{U},R_\mathcal{U}}) = \mathcal{X}_{\mathcal{P}_8,R_8}$.

$$
\begin{array}{ccccccccccccccccc}
\Omega_0 & \xleftarrow{\omega_0} & \Omega_1 & \xleftarrow{\omega_1} & \Omega_2 & \xleftarrow{\omega_2} & \Omega_3 & \xleftarrow{\omega_3} & \Omega_4 & & & & & & \rho\,\omega_{\mathcal{U}}\rho^{-1} \circlearrowleft & & \omega_{\mathcal{U}} \circlearrowleft \\
\cup & & \cup & & \cup & & \cup & & \cup & & & & & & & & \\
X_0 & \xleftarrow{\omega_0} & X_1 & \xleftarrow{\omega_1} & X_2 & \xleftarrow{\omega_2} & X_3 & \xleftarrow{\omega_3} & X_4 & \xleftarrow{\jmath} & \Omega_5 & \xleftarrow{\eta} & \Omega_6 & \xleftarrow{\omega_6} \ \Omega_7 \ \xleftarrow{\omega_7\omega_8\omega_9\omega_{10}\omega_{11}} & \Omega_{12} & \xleftarrow{\rho} & \Omega_{\mathcal{U}} \\
\| & & & & & & & & \| & & & & & \| & \| & & \| \\
\mathcal{X}_{\mathcal{P}_0,R_0} & & & \xleftarrow{\beta_0} & & & & & \mathcal{X}_{\mathcal{P}_1,R_1} & \xleftarrow{\beta_1} & \mathcal{X}_{\mathcal{P}_2,R_2} & \xleftarrow{\beta_2} & & \mathcal{X}_{\mathcal{P}_3,R_3} \ \xleftarrow{\beta_3\beta_4\beta_5\beta_6\beta_7} & \mathcal{X}_{\mathcal{P}_8,R_8} & \xleftarrow{\zeta} & \mathcal{X}_{\mathcal{P}_{\mathcal{U}},R_{\mathcal{U}}} \\
& & & & & & & & & & & & & & \circlearrowright\ \beta_8\beta_9\tau & & \circlearrowright\ \zeta^{-1}\beta_8\beta_9\tau\zeta
\end{array}
$$

Figure 11.6: The subshifts $X_0 \subset \Omega_0$ and $\mathcal{X}_{\mathcal{P}_0,R_0}$ are equal since they have a common substitutive structure. The substitutive structure of X_0 computed in [10] and the substitutive structure of $\mathcal{X}_{\mathcal{P}_0,R_0}$ computed in [9] satisfy $\beta_0 = \omega_0\omega_1\omega_2\omega_3$, $\beta_1\beta_2 = \jmath\,\eta\,\omega_6$, $\beta_3\beta_4\beta_5\beta_6\beta_7 = \omega_7\omega_8\omega_9\omega_{10}\omega_{11}$, $\zeta = \rho$ and $\beta_8\,\beta_9\,\tau = \rho\,\omega_{\mathcal{U}}\,\rho^{-1}$. We deduce that $\mathcal{X}_{\mathcal{P}_8,R_8} = \Omega_{12}$, $\mathcal{X}_{\mathcal{P}_3,R_3} = \Omega_7$, $\mathcal{X}_{\mathcal{P}_1,R_1} = X_4$ and finally $\mathcal{X}_{\mathcal{P}_0,R_0} = X_0$.

Theorem 11.7 must be compared with Theorem 11.4, giving the substitutive structure of a minimal subshift X_0 of the Jeandel-Rao Wang shift Ω_0. In fact, the consequence of the two theorems is that the subshifts X_0 and $\mathcal{X}_{\mathcal{P}_0,R_0}$ have the *exact same* substitutive structure given as the inverse limit of the same eventually periodic sequence of 2-dimensional morphisms; see Figure 11.6.

Theorem 11.8 ([9])**.** *The symbolic dynamical system $\mathcal{X}_{\mathcal{P}_0,R_0}$ and the minimal subshift $X_0 \subset \Omega_0$ of the Jeandel-Rao Wang shift have the same substitutive structure in the sense that the following equalities hold:*

$$
\begin{array}{lllll}
\beta_0 = \omega_0\,\omega_1\,\omega_2\,\omega_3, & \beta_1\beta_2 = \jmath\,\eta\,\omega_6, & \beta_3 = \omega_7, & \beta_4 = \omega_8, & \\
\beta_5 = \omega_9, & \beta_6 = \omega_{10}, & \beta_7 = \omega_{11}, & \zeta = \rho, & \beta_8\,\beta_9\,\tau = \rho\,\omega_{\mathcal{U}}\,\rho^{-1},
\end{array}
$$

where ω_0, ω_1, ω_2, ω_3, $\jmath$, η, ω_6, ω_7, ω_8, ω_9, ω_{10}, ω_{11}, ρ were computed in [10] and $\omega_{\mathcal{U}}$ was first defined in [7].

Remark 11.9 ([9])**.** *To obtain equalities between substitutions computed from totally different objects, we use a common convention for the definition of the β_i from the induction of toral partitions, and in [10] for the construction of the ω_i from sets of Wang tiles. When constructing the substitutions, we use the radix order to order the images of the letters, that is, short images of letter come before longer ones, and words of the same length are sorted lexicographically.*

The description of the symbolic dynamical system $\mathcal{X}_{\mathcal{P}_0,R_0}$ and the minimal subshift X_0 of Jeandel-Rao aperiodic subshift Ω_0 by their substitutive structure allows to prove their topological conjugacy.

Corollary 11.10 ([9])**.** *The subshifts $\mathcal{X}_{\mathcal{P}_0,R_0}$ and X_0 are topologically conjugate. The subshifts $\mathcal{X}_{\mathcal{P}_{\mathcal{U}},R_{\mathcal{U}}}$ and $\Omega_{\mathcal{U}}$ are topologically conjugate and are equal to the minimal aperiodic substitutive subshift $\mathcal{X}_{\omega_{\mathcal{U}}}$. The subshifts $\mathcal{X}_{\mathcal{P}_8,R_8}$ and Ω_{12} are topologically conjugate. The same holds for intermediate subshifts:*

$$\mathcal{X}_{\mathcal{P}_1,R_1} = \jmath(\Omega_5),\ \mathcal{X}_{\mathcal{P}_3,R_3} = \Omega_7,\ \mathcal{X}_{\mathcal{P}_4,R_4} = \Omega_8,\ \mathcal{X}_{\mathcal{P}_5,R_5} = \Omega_9,\ \mathcal{X}_{\mathcal{P}_6,R_6} = \Omega_{10} \text{ and } \mathcal{X}_{\mathcal{P}_7,R_7} = \Omega_{11}.$$

It also implies the following corollary which can be seen as a generalization of what happens for Sturmian sequences.

Corollary 11.11 ([9])**.** *The dynamical system $(X_0, \mathbb{Z}^2, \sigma)$ is uniquely ergodic. The measure-preserving dynamical system $(X_0, \mathbb{Z}^2, \sigma, \nu)$ is isomorphic to the toral $\mathbb{Z}^2$-rotation $(\mathbb{R}^2/\Gamma_0, \mathbb{Z}^2, R_0, \lambda)$ where ν is the unique shift-invariant probability measure on X_0 and λ is the Haar measure on $\mathbb{R}^2/\Gamma_0$.*

A consequence of Theorem 11.8 is that for every configuration $w \in X_0 \subset \Omega_0$, we can find a starting point in $\mathbb{R}^2/\Gamma_0$ such that the configuration w is a symbolic representation of that starting point. Thus, the hard work done in [7, 10, 9] allows to answer the question raised in Section 11.4 for configurations in X_0.

11.7 A Markov partition for Jeandel-Rao tilings

Markov partitions were originally defined for one-dimensional dynamical systems $\mathbb{Z} \overset{R}{\curvearrowright} \mathbb{T}^2$ and were extended to $\mathbb{Z}^d$-actions by automorphisms of compact Abelian group in [130]. We allow ourselves to use the same terminology and extend the definition proposed in [200, §6.5] for dynamical systems defined by higher-dimensional actions by rotations.

Definition 11.12 ([200, 8, 9])**.** *A topological partition $\mathcal{P}$ of $\mathbb{T}^2$ is a* ***Markov partition*** *for $\mathbb{Z}^2 \overset{R}{\curvearrowright} \mathbb{T}^2$ if*

- *$\mathcal{P}$ gives a symbolic representation of $\mathbb{Z}^2 \overset{R}{\curvearrowright} \mathbb{T}^2$ and*
- *$\mathcal{X}_{\mathcal{P},R}$ is a (2-dimensional) shift of finite type (SFT).*

Thus, we may have Markov partitions associated with aperiodic subshifts of finite type over $\mathbb{Z}^2$ coded by toral rotations (thus of zero entropy). This may seem counter-intuitive since Markov partitions are usually associated with hyperbolic systems (thus with positive entropy). Moreover, the coding of an irrational rotation on the circle leads to aperiodic Sturmian sequences which are not SFT. Our opinion is that positive entropy and all associated intuitions follows from the restriction of Definition 11.12 to the case of $\mathbb{Z}$-actions, but not from the notion of Markov partition itself. In [8] and [9], we made a choice by using the terminology of *Markov partitions* in the unusual setup of $\mathbb{Z}^2$-rotations.

Of course, SFTs over $\mathbb{Z}^2$ are much different then SFTs over $\mathbb{Z}$. The emptyness of $\mathbb{Z}^2$-SFTs is undecidable [67] and the possible entropies achievable by a $\mathbb{Z}^2$-SFT are exactly the non-negative numbers obtainable as the limit of computable decreasing sequences of rationals [159], as opposed to be given by an algebraic characterization in the case of $\mathbb{Z}$-SFT, see [200, §4]. In particular, there exist aperiodic $\mathbb{Z}^2$-SFTs of zero entropy which is not possible in the one-dimensional case, since infinite $\mathbb{Z}$-SFTs have positive entropy and contain a periodic configuration.

A consequence of the three articles [8], [9] and [10] published in 2021 is the following theorem.

Theorem 11.13 ([9])**.** *$\mathcal{P}_0$ is a Markov partition for the dynamical system $\mathbb{Z}^2 \overset{R_0}{\curvearrowright} \mathbb{R}^2/\Gamma_0$.*

Proof. From Theorem 11.2 proved in [8], the partition $\mathcal{P}_0$ gives a symbolic representation of $\mathbb{Z}^2 \overset{R_0}{\curvearrowright} \mathbb{R}^2/\Gamma_0$. It was proved in [10] that X_0 is a shift of finite type. From Corollary 11.10 proved in [9], the subshifts $\mathcal{X}_{\mathcal{P}_0,R_0}$ and X_0 are equal. Then, the 2-dimensional subshift $\mathcal{X}_{\mathcal{P}_0,R_0}$ is a shift of finite type. Thus, $\mathcal{P}_0$ is a Markov partition for the dynamical system $\mathbb{Z}^2 \overset{R_0}{\curvearrowright} \mathbb{R}^2/\Gamma_0$. □

11.8 Nonexpansive directions in Jeandel-Rao tilings

In 2019-2020, Casey Mann and Jennifer Mcloud-Mann spent a sabbatical year in Bordeaux, thanks to *Idex Bordeaux Visiting Scholars positions.* Together, we tried to prove that the Jeandel-Rao Wang shift Ω_0 is uniquely ergodic (see Conjecture 11.17 below). But that turned out to be a difficult question due to the presence of a horizontal fault line in the Jeandel-Rao Wang shift. Instead, we decided to describe the nonexpansive directions in the Jeandel-Rao Wang shift. Indeed, these can be computed from the slopes in the partition $\mathcal{P}_0$.

The notion of *Conway worms* was considered in [148, §10.5] in the context of tilings by Penrose kites and darts. It was then defined as "*a sequence of bow ties placed end to end*" and it was proved that every tiling by Penrose kites and darts contains arbitrarily long finite Conway worms, see [148, p. 10.5.8]. Also it was noted that there are 5 different possible slopes for these Conway worms and the difference between any two of them is a multiple of $\frac{\pi}{5}$.

The understanding of Penrose tilings was greatly improved by N. G. de Bruijn who for the first time expressed them in terms of cut and project schemes where the aperiodic tilings are described as the projection of a lattice living in the product of the physical space of dimension two and some internal space of dimension three [98]. Based on this work, Robinson further developed the dynamical properties of Penrose tilings [241]. In particular, he expressed Conway worms appearing in the singular Penrose tilings in terms of coincidences happening in the internal space and noted that Conway worms come in pairs [241, §6] that he called positive and negative resolutions of a Conway worm (see also the same idea appearing in [98, Figures 12 and 13]); see also [51, Figure 7.22]. A reproduction of Figure 8 from [241] illustrating the two ways to resolve Conway worms in the context of Penrose tilings is shown in Figure 11.7. Notice that the existence of an infinite Conway worm of a given slope $\alpha \in \mathbb{R} \cup \{\infty\}$ implies the existence of a tiling of some half-plane delimited by a line of slope α which has more than one completion to a tiling of the whole plane. The notion of Conway worms may give more insights on a family of tilings. For instance, it allows one to prove that tiles occur in only finitely many orientations in parallelogram tilings using a finite number of shapes [137].

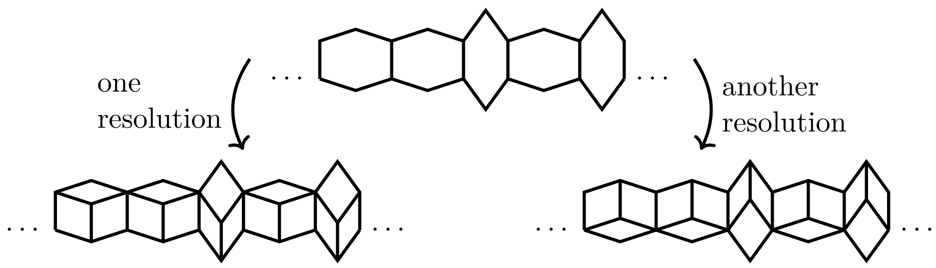


Figure 11.7: An illustration of an unresolved Conway worm made of two kinds of hexagons together with its two resolutions within a Penrose tiling.

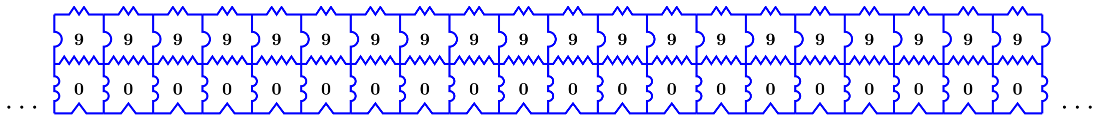
In the context of subshifts, the concept of Conway worms is formalized in terms of nonexpansiveness. Let F be a subspace of $\mathbb{R}^d$. Given $t > 0$, the *t-neighborhood* of F is defined by $F^t := \{g \in \mathbb{Z}^d \colon \operatorname{dist}(g, F) \leq t\}$. Let $X \subset \mathcal{A}^{\mathbb{Z}^d}$ be a subshift, and for any subset $S \subset \mathbb{Z}^d$ and configuration $x \in X$, let $x|_S$ denote the restriction of x to S. Following Boyle and Lind [94], a subspace $F \subset \mathbb{R}^d$ is *expansive* on X if there exists $t > 0$ such that for any $x, y \in X$, $x|_{F^t} = y|_{F^t}$ implies that $x = y$. Moreover, a subspace F is *nonexpansive* if for all $t > 0$, there exist $x, y \in X$ such that $x|_{F^t} = y|_{F^t}$ but $x \neq y$. If F is expansive, then every translate of F is expansive. Thus, in the 2-dimensional case, which is our focus, we refer to *nonexpansive directions*.

Boyle and Lind [94, Theorem 3.7] showed that if $X \subset \mathcal{A}^{\mathbb{Z}^d}$ is an infinite subshift, then, for each $0 \leq n < d$, there exists a n-dimensional subspace of $\mathbb{R}^d$ that is nonexpansive on X. Answering a question of Boyle and Lind, Hochman proved that any one-dimensional subspace in the plane $\mathbb{R}^2$ occurs as the unique nonexpansive one-dimensional subspace of a $\mathbb{Z}^2$-action [157]. As a consequence, Hochman proved that a set of one-dimensional subspaces occurs as the set of nonexpansive directions for a subshift $X \subset \mathcal{A}^{\mathbb{Z}^2}$ if and only if it is closed and non-empty. The notions of expansive and nonexpansive directions was used to obtain partial results toward solving Nivat's conjecture, an important problem in symbolic dynamics, see for instance [117, 111].

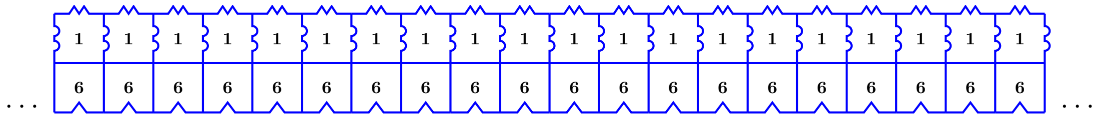
The notion of nonexpansive direction can also be stated equivalently in terms of nonexpansive half-spaces. Let $X \subset \mathcal{A}^{\mathbb{Z}^d}$ be a subshift and σ be a $\mathbb{Z}^d$-action on X. We say that a half-space $H \subset \mathbb{R}^d$ is *nonexpansive* for σ if there exist $x, y \in X$ such that $x|_{\mathbb{Z}^d \cap H} = y|_{\mathbb{Z}^d \cap H}$ but $x \neq y$. It was proved in the preliminary section of [129] that a codimension 1 subspace V of $\mathbb{R}^d$ is nonexpansive for σ if and only if there is a half-space H whose boundary is V and which is nonexpansive for σ, see [13, Lemma 2.2]. The set of nonexpansive directions is difficult to compute in general and brings a deeper understanding of a subshift since it is a topological invariant, see [13, Lemma 2.3].

Conway worms can be defined in the context of subshifts from nonexpansiveness. Let $x, y \in X$ be two configurations. The support of positions where x and y are distinct is the set $D(x, y) = \{n \in \mathbb{Z}^d \mid x_n \neq y_n\}$. We say that the set $D(x, y)$ is a *Conway worm* associated to a subspace F if there exists $t > 0$ such that $\varnothing \neq D(x, y) \subset F^t$. Observe that if $S \subset \mathbb{Z}^d$ is a Conway worm associated to a subspace F, then F is nonexpansive. Also, reusing the vocabulary proposed in [241], we say that the restriction of the configurations x and y to the support $D(x, y)$ are two *resolutions of the Conway worm*.

In [13], we described the Conway worms and their resolutions in the Jeandel-Rao Wang shift. As noticed in [163], there exist tilings of the plane containing a bi-infinite horizontal strip of tiles numbered 0. Since only a tile numbered 9 can be on top of a tile numbered 0, we have the following bi-infinite strip of height 2:



It turns out that the above strip is a Conway worm. Indeed its other resolution can be obtained by replacing the tiles numbered 9 with tiles numbered 1 and replacing the tiles numbered 0 with tiles numbered 6:



We observe that both strips have the same constraints on top and at the bottom making them replaceable by one another. The tiling above and below of the two strips is shown in Figure 11.8. This means that 0 is the slope of a nonexpansive direction within Jeandel-Rao Wang shift.

With Mann and Mcloud-Mann, we computed the nonexpansive directions for the minimal subshift X_0 of the Jeandel-Rao Wang shift Ω_0. As opposed to the nonexpansive directions in Penrose tilings which are the directions perpendicular to the fifth roots of unity, see [162, Theorem 5.1.1], we obtain a more surprising and far less symmetric result for the minimal subshift X_0.

Theorem 11.14 ([13])**.** *The minimal subshift X_0 of the Jeandel-Rao Wang shift contains exactly 4 nonexpansive directions whose slopes are* $\{0, \varphi + 3, -3\varphi + 2, -\varphi + \frac{5}{2}\}$.

While slope 0 is not a surprise, the other slopes are irrational and their values are unexpected. In particular, we show that there is a link between the slopes that appear in the Markov partition provided in [8] and studied more deeply in [9] and the slopes of nonexpansive directions, but the relation is not equality. This contrasts with well-known cases like Penrose tilings where the symmetry of the tilings hides a more complex relation. More precisely, we show that slopes of nonexpansive directions within Jeandel-Rao Wang shift are related to slopes that appear in the associated Markov partition according to the following table (see [13, Proposition 5.4]):

slope in the Markov Partition	slope of associated nonexpansive direction
0	0
∞	$\varphi + 3$
φ	$-3\varphi + 2$
φ^2	$-\varphi + \frac{5}{2}$

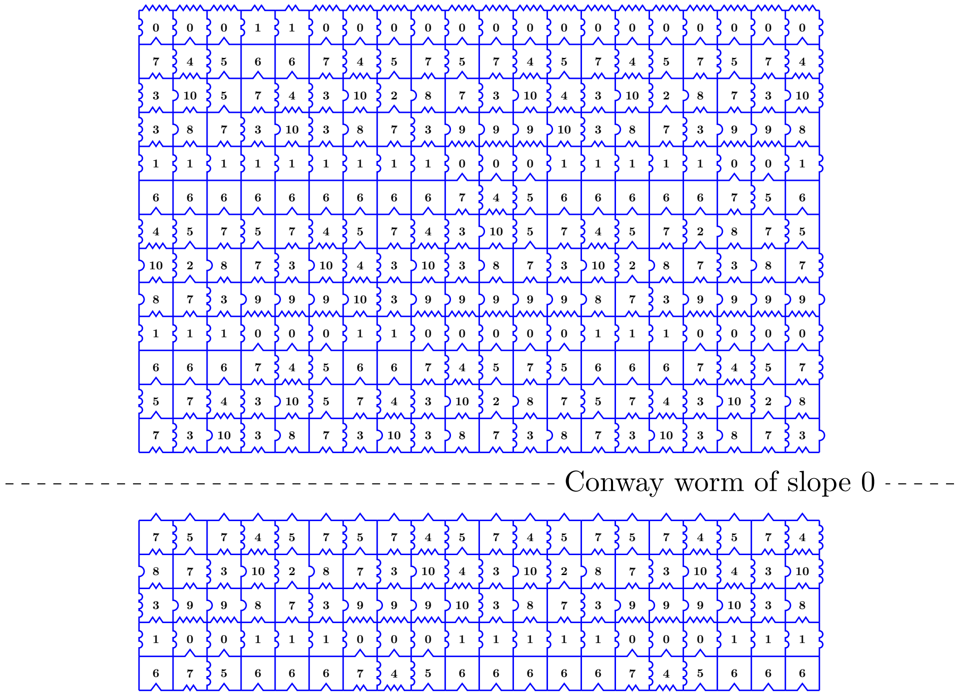


Figure 11.8: A partial tiling of the plane with an unresolved Conway worm of slope 0.

The three other nonexpansive directions are illustrated in Figure 11.9 and Figure 11.10.

The computation of the nonexpansive directions made in [13] was done specifically for the partition $\mathcal{P}_0$ associated to Jeandel-Rao tilings.

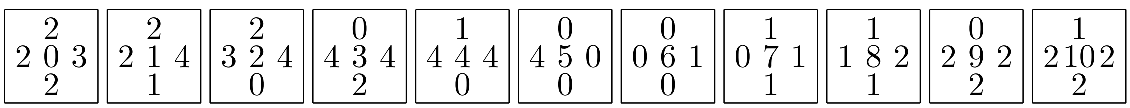
Question 11.15. *Let R be a $\mathbb{Z}^2$-action on $\mathbb{T}^2$ and $\mathcal{P}$ be a polygonal partition of $\mathbb{T}^2$ whose vertices belong to some quadratic field. Generalize [13, Proposition 5.1] to compute the nonexpansive directions of the symbolic dynamical system $\mathcal{X}_{\mathcal{P},R}$ for arbitrary polygonal partitions $\mathcal{P}$.*

11.9 Thompson's memoir

In their article, Jeandel and Rao provided a list of 33 sets of 11 Wang tiles which are candidates for being aperiodic [3]. This means their algorithm was not able to decide if these sets tile the plane or not. They took one in the list and they proved it to be aperiodic [163].

One of the set of tiles appeared in the slide #50 of the presentation of Michael Rao in April 2017 (Montreal) calling it a "*strange (interesting) candidate*". Here it is:

```
sage: from slabbe import WangTileSet
sage: tiles = "3222␣4221␣4230␣4042␣4140␣0040␣1000␣1101␣2111␣2022␣2122"
sage: tiles = [tuple(tile) for tile in tiles.split()]
sage: slide50 = WangTileSet(tiles)
```



[3] https://framagit.org/mrao/small-wang-tile-sets

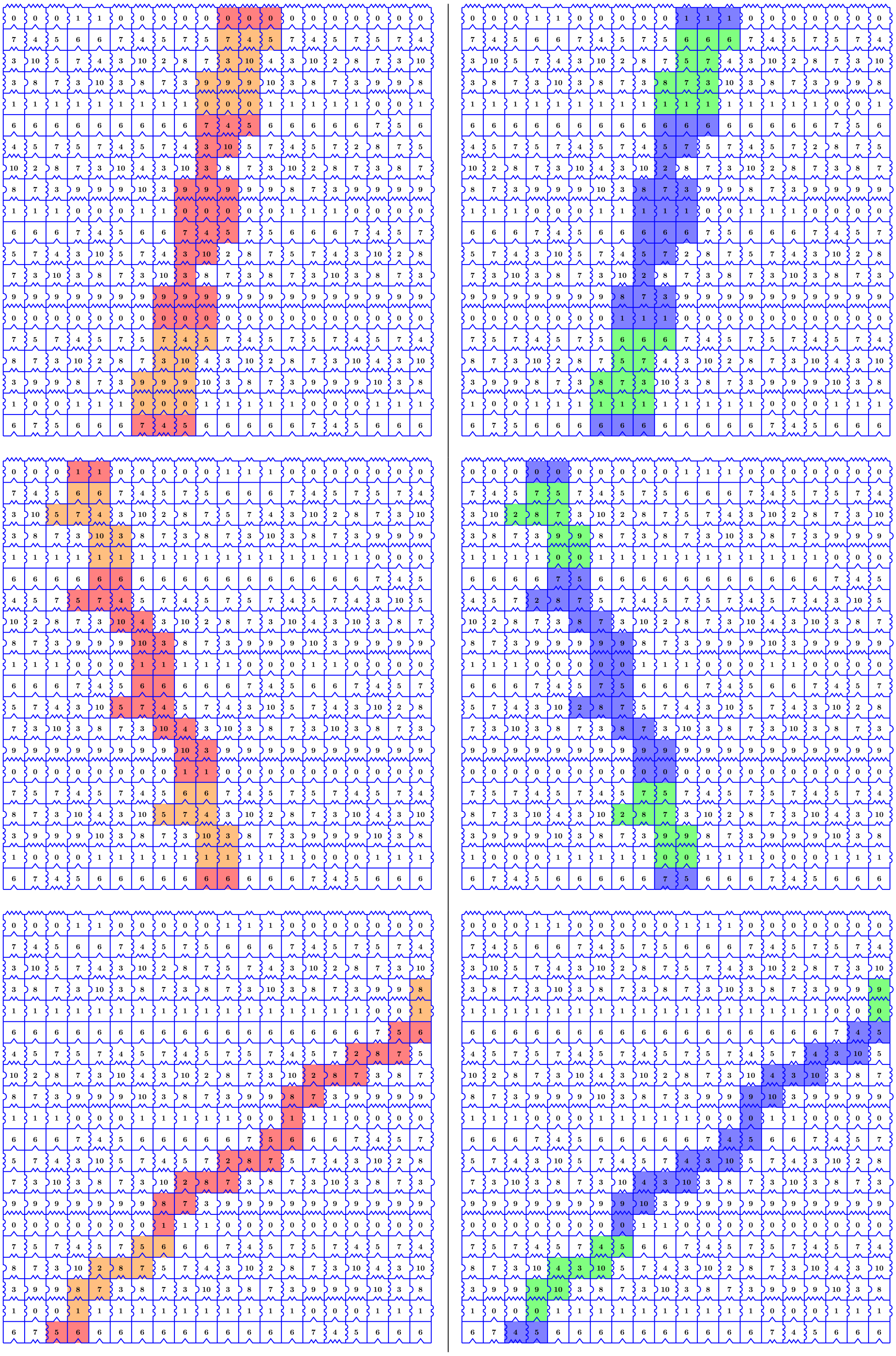


Figure 11.9: Tilings of a 20×20 square illustrating the Conway worms of slope $\varphi + 3$, $-3\varphi + 2$ and $-\varphi + \frac{5}{2}$. The difference between the left and the right images is shown with a colored background.

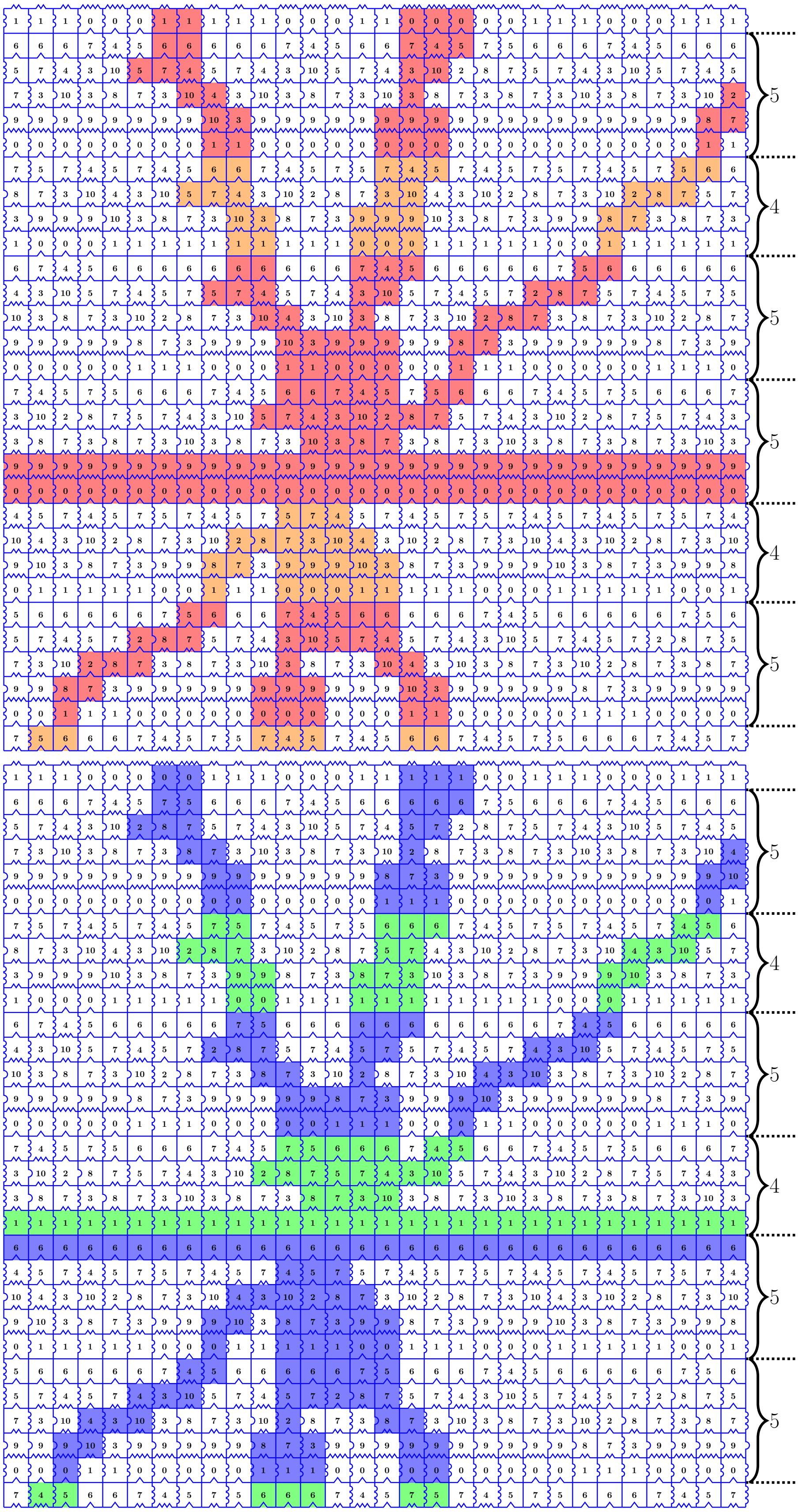
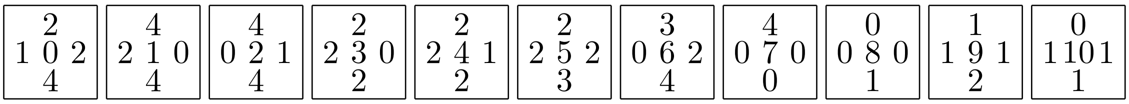


Figure 11.10: Tilings of a 30 × 30 square illustrating the four Conway worms. The difference between both images is shown with a colored background. This reminds of the cartwheel tiling in the context of Penrose tilings [148, Figure 10.5.1 (c)].

A natural question is to study these remaining 32 candidates. A master student of Uwe Grimm wrote a magnificent memoir in 2022 where he studied one of the 32 candidates in more details. Thompson proved it to be related to the Jeandel-Rao one through a substitution. Also Thompson found a polygonal partition of a $\varphi \times (\varphi + 4)$ rectangle describing the tilings allowed by these tiles [267]. Thompson was master student at Open University in UK and followed courses by Uwe Grimm at, just before Uwe passed away.

The tile set $T_{\mathcal{Y}}$ in the memoir of Thompson is the following:

```
sage: tiles = "2214␣0424␣1404␣0222␣1222␣2223␣2304␣0400␣0001␣1112␣1011"
sage: tiles = [tuple(tile) for tile in tiles.split()]
sage: TY = WangTileSet(tiles)
```



Note that the set $T_{\mathcal{Y}}$ studied by Roger Thompson corresponds (up to an isometry) to the strange interesting candidate mentioned in the slide #50 of the presentation of Michael Rao:

```
sage: TY.is_equivalent_up_to_isometry(slide50, certificate=True)
(True, ((1,4,3,2), {'2': '2', '4': '4', '3': '3', '0': '0', '1': '1'}, {'1': '1', '2': '2', '0': '
    0'}, Substitution 2d: {0: [[1]], 1: [[3]], 2: [[4]], 3: [[9]], 4: [[10]], 5: [[0]], 6: [[2]],
    7: [[5]], 8: [[6]], 9: [[8]], 10: [[7]]}))
```

11.10 Open questions

It was also shown that $\mathcal{X}_{\mathcal{P}_0,R_0}$ is a *strict subset* of the Jeandel-Rao Wang shift Ω_0 [163],

In [10], we also show that $\Omega_0 \setminus X_0 \neq \varnothing$ due to the presence of some horizontal fault lines in Ω_0, but we believe that X_0 gives an almost complete description of the Jeandel-Rao tilings. More precisely and as opposed to the minimal subshift of the Kari-Culik tilings [256], we think the following holds.

Conjecture 11.16 ([10]). *$\Omega_0 \setminus X_0$ is of measure zero for any shift-invariant probability measure on Ω_0.*

Although Jeandel-Rao Wang shift Ω_0 is not minimal as it contains the proper minimal subshift $\mathcal{X}_{\mathcal{P}_0,R_0} = X_0$, we believe that it is uniquely ergodic.

Conjecture 11.17 ([9]). *The Jeandel-Rao subshift Ω_0 is uniquely ergodic.*

That conjecture is equivalent to prove that $\Omega_0 \setminus X_0$ has measure 0 for any shift-invariant probability measure on Ω_0 which was stated as a conjecture in [10] and where some progress was done. That would also imply the existence of an isomorphism of measure-preserving dynamical systems between Jeandel-Rao Wang shift $(\Omega_0, \mathbb{Z}^2, \sigma, \nu)$ and $(\mathbb{R}^2/\Gamma_0, \mathbb{Z}^2, R_0, \lambda)$ where ν would be the unique shift-invariant probability measure on Ω_0 and λ is the Haar measure on $\mathbb{R}^2/\Gamma_0$.

This calls for a general theory of d-dimensional subshifts of finite type coded by Markov partitions of the d-dimensional torus and admitting induced subsystems.

Among the first examples to study are the candidates discovered by Jeandel and Rao (other than the one chosen by Jeandel and Rao and the one already studied by Thompson as mentioned in Section 11.9).

Question 11.18. *Which one of the 33 candidate sets of 11 Wang tiles found by Jeandel and Rao are aperiodic?*

It is known in certain cases [63, 64, 65] that cut and project scheme leading to aperiodic tilings given by a finite set of forbidden rules implies algebraic restrictions on the possible projections. On this subject, during Sage Days 128[4] with Carole Porrier, we implemented in the `slabbe` package the question whether a cut and project scheme is determined by its subperiods [231]. Here is what we obtain.

The Penrose hull is determined by its subperiods:

```
sage: from slabbe import cut_and_project_schemes
sage: c = cut_and_project_schemes.Penrose()
sage: c.is_determined_by_subperiods()
True
```

Ammann-Beenker is not determined by its subperiods:

```
sage: c = cut_and_project_schemes.AmmannBeenker()
sage: c.is_determined_by_subperiods()
False
```

Jeandel-Rao is not determined by its subperiods:

```
sage: c = cut_and_project_schemes.JeandelRao()
sage: c.is_determined_by_subperiods()
False
```

Question 11.19. *Why are Jeandel-Rao tilings not determined by their subperiods, while Penrose are?*

[4] `https://wiki.sagemath.org/days128`

Chapter 12

Metallic mean Wang tiles

"The ultimate test of whether I understand something is if I can explain it to a computer. I can say something to you and you'll nod your head, but I'm not sure that I explained it well. But the computer doesn't nod its head. It repeats back exactly what I tell it. In most of life, you can bluff, but not with computers."

— *Donald Knuth*

In this chapter, we present a new family of aperiodic Wang tiles related to the metallic mean that were introduced in [22, 23].

12.1 Definition

Recall that the metallic mean β is the positive root of the polynomial $x^2 - nx - 1$ where $n \geq 1$ is an integer [265], that is,

$$\beta = [n; n, n, \cdots] = n + \frac{1}{n + \frac{1}{n + \frac{1}{n + \cdots}}} = n + \frac{1}{\beta}.$$

Metallic means were also called *silver means* in [249] and *noble means* in [51].

For every integer $n \geq 1$, the n^{th} metallic mean Wang shift Ω_n is defined from a set $\mathcal{T}_n$ of $(n+3)^2$ Wang tiles. An illustration of the set $\mathcal{T}_n$ for $n \in \{1, 2, 3\}$ is shown in Figure 12.1. The labels of the Wang tiles are vectors in $\mathbb{N}^3$. In Figure 12.1, we represent vectors as words for economy of space reasons. For instance, the vector $(1, 1, 4)$ is represented as 114. Note that integers vectors were already used as labels of Wang tiles in [169, 170], see also [171]. A finite rectangular valid tiling is shown in Figure 12.2 for the set $\mathcal{T}_3$. More images of valid tilings with metallic mean Wang tiles are available in [22].

12.2 Results from the first article

In the first article, we showed that the metallic mean Wang shift Ω_n is self-similar, aperiodic and minimal.

Theorem 12.1 ([22]). *For every integer $n \geq 1$,*

(i) the metallic mean Wang shift Ω_n is self-similar, aperiodic and minimal,

(ii) the inflation factor of the self-similarity of Ω_n is the n-th metallic mean, that is, the positive root of $x^2 - nx - 1$.

Also, when $n = 1$, Ω_1 is equivalent to the Wang shift defined from the 16 Ammann Wang tiles; see Figure 6.2.

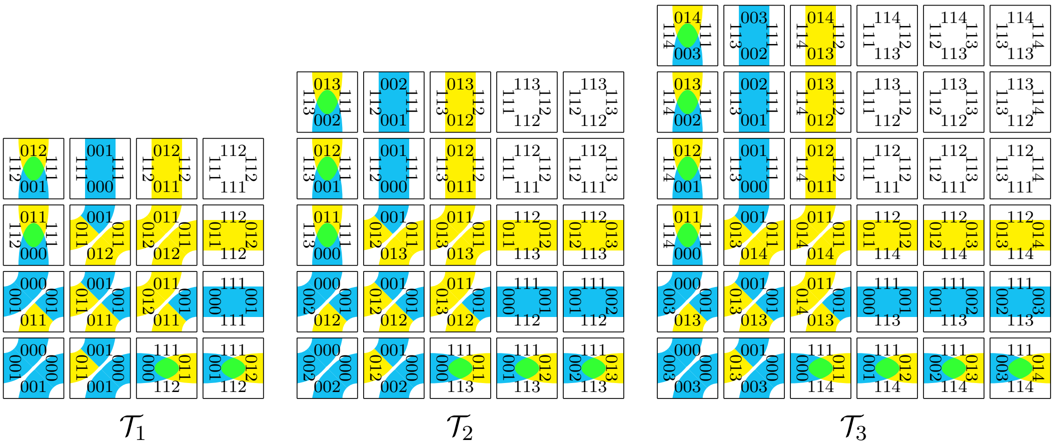


Figure 12.1: Metallic mean Wang tile sets $\mathcal{T}_n$ for $n = 1, 2, 3$.

Therefore, the family $\{\Omega_n\}_{n\geq 1}$ is a generalization of the Ammann aperiodic set of 16 Wang tiles [148].

In order to describe the substitutive structure of the Wang shift Ω_n generated from the set $\mathcal{T}_n$, it was needed in [22] to introduce a larger set $\mathcal{T}'_n$ satisfying $\mathcal{T}_n \subseteq \mathcal{T}'_n$. It was shown that the set $\mathcal{T}'_n$ is in bijection with the set of possible return blocks allowing to decompose uniquely the configurations of Ω_n. The return blocks are rectangular blocks of tiles with a unique junction tile (a tile where horizontal and vertical color stripes intersect) at the lower left corner. Also, it was proved in [22] that in a valid configuration of Ω'_n, only the tiles from $\mathcal{T}_n$ appear. From this observation follows the self-similarity of Ω_n.

12.3 Results from the second article

In the second article, we demonstrated that the tiles from $\mathcal{T}_n$ satisfy equations and can even be defined by them. Therefore, it provides an example of substitutive aperiodic sets of Wang tiles defined by equations like Kari-Culik sets, see Figure 12.3. The equations satisfied by the tiles are derived from a function that expresses a relation between the labels of the Wang tiles. The function provides an independent definition of the family of metallic mean Wang tiles as the instances of an aperiodic computer chip. See [23] for the details.

Among the main results, we proved that Ω_n is aperiodic for another reason. Namely, the $\mathbb{Z}^2$ shift action on Ω_n is an almost 1-to-1 extension of a minimal $\mathbb{Z}^2$-action by rotations on $\mathbb{T}^2$. This reminds of a result proved for Penrose tilings [241] and the two reasons for them to be aperiodic. Aperiodicity of Penrose tilings follows from its self-similarity [228] and from the fact of being a cut-and-project scheme [98, 51].

In Kari–Culik tilings [167, 114], there is a well-defined notion of average [125] of the top tile labels along a bi-infinite horizontal row. The value from one row to the next row is described by a piecewise rationally multiplicative map. In this context, metallic mean Wang shifts also behave like Kari–Culik tilings. It involves the consideration of the average of specific inner products and irrational rotations instead of multiplications, see Figure 12.4.

We show that the average of the dot products of the vector $\frac{1}{n}d = \frac{1}{n}(0, -1, 1)$ with the top labels of a given row in a valid configuration $\mathbb{Z}^2 \to \mathcal{T}_n$ in Ω_n is well-defined and takes a value in the interval $[0, 1]$. By symmetry of the set $\mathcal{T}_n$, the same holds for the right labels of a given column. By considering the row and column going through the origin of a configuration, the two averages define a map $\Phi_n : \Omega_n \to \mathbb{T}^2$. We prove that this map is a factor map from the Wang shift to the

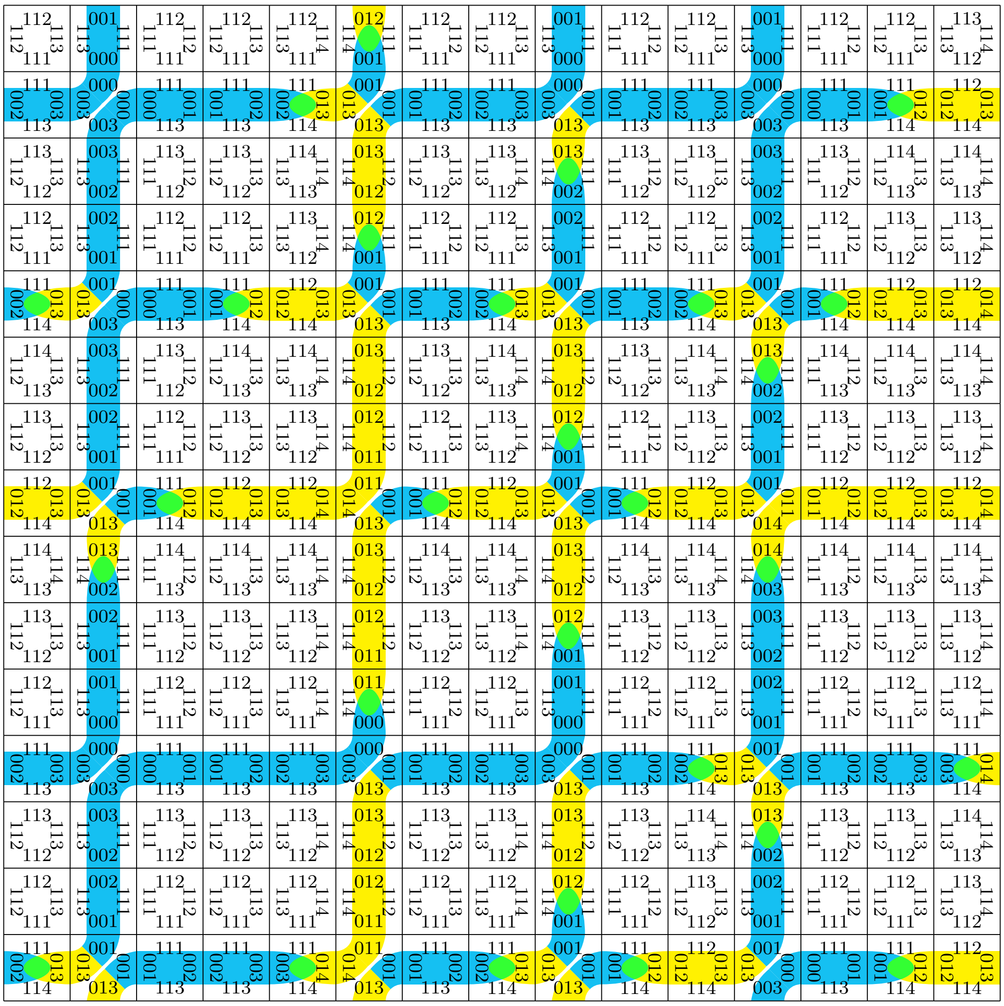


Figure 12.2: A valid 15×15 pattern with Wang tile set $\mathcal{T}_3$.

2-torus.

Theorem 12.2 ([23])**.** *Let $d = (0, -1, 1)$, $n \geq 1$ be an integer and Ω_n be the n^{th} metallic mean Wang shift. The map*

$$
\begin{array}{rccl}
\Phi_n : & \Omega_n & \to & \mathbb{T}^2 \\
& w & \mapsto & \displaystyle\lim_{k\to\infty} \frac{1}{2k+1} \sum_{i=-k}^{k} \begin{pmatrix} \langle \frac{1}{n} d, \textsc{Right}(w_{0,i}) \rangle \\ \langle \frac{1}{n} d, \textsc{Top}(w_{i,0}) \rangle \end{pmatrix}
\end{array}
\tag{12.1}
$$

is a factor map, that is, it is continuous, onto and commutes the shift $\mathbb{Z}^2 \overset{\sigma}{\curvearrowright} \Omega_n$ with the toral $\mathbb{Z}^2$-rotation $\mathbb{Z}^2 \overset{R_n}{\curvearrowright} \mathbb{T}^2$ by the equation $\Phi_n \circ \sigma^k = R_n^k \circ \Phi_n$ for every $k \in \mathbb{Z}^2$ where

$$
\begin{array}{rccl}
R_n & : \mathbb{Z}^2 \times \mathbb{T}^2 & \to & \mathbb{T}^2 \\
& (k, x) & \mapsto & R_n^k(x) := x + \beta k
\end{array}
$$

and $\beta = \frac{n+\sqrt{n^2+4}}{2}$ is the n^{th} metallic mean, that is, the positive root of the polynomial $x^2 - nx - 1$.

As a consequence of Theorem 12.2, we deduce that Ω_n is aperiodic because β is irrational and R_n is a free $\mathbb{Z}^2$-action. Note that since $\beta - \beta^{-1} = n$, we have $\beta = \beta^{-1} \pmod 1$.

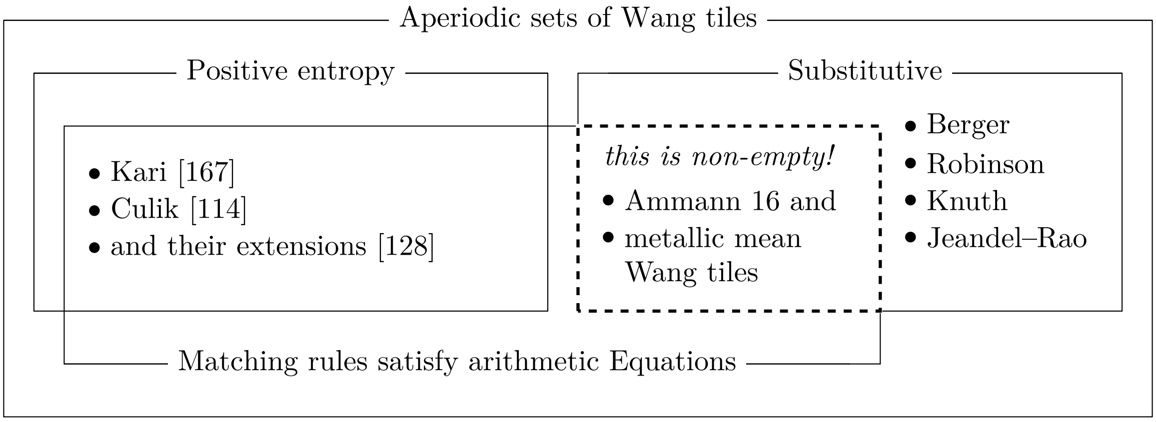


Figure 12.3: A Venn diagram of aperiodic sets of Wang tiles. Aperiodicity of Kari and Culik sets of tiles and their extensions follows from the arithmetic Equations satisfied by their matching rules. We show that the dashed region in the Venn diagram is non-empty, that is, there exists a family of substitutive (self-similar) aperiodic sets of Wang tiles whose matching rules satisfy arithmetic Equations.

Theorem 12.2 is an analogue of a result known for Kari and Culik aperiodic Wang tilings which satisfy equations involving balanced representations of real numbers and orbits of piecewise rationally multiplicative maps, see also Theorem 16 in [128] and Proposition 3 in [256]. Here the result applies to all of the configurations in the Wang shift Ω_n.

As proved for Jeandel–Rao Wang shift [8], we have the following additional topological and measurable properties for the factor map Φ_n. A similar result holds for Penrose tilings [241].

Theorem 12.3 ([23])**.** *The Wang shift Ω_n and the $\mathbb{Z}^2$-action R_n have the following properties:*

(i) $\mathbb{Z}^2 \overset{R_n}{\curvearrowright} \mathbb{T}^2$ is the maximal equicontinuous factor of $\mathbb{Z}^2 \overset{\sigma}{\curvearrowright} \Omega_n$,

(ii) the factor map $\Phi_n : \Omega_n \to \mathbb{T}^2$ is almost one-to-one and its set of fiber cardinalities is $\{1, 2, 8\}$,

(iii) the shift-action $\mathbb{Z}^2 \overset{\sigma}{\curvearrowright} \Omega_n$ on the metallic mean Wang shift is uniquely ergodic,

(iv) the measure-preserving dynamical system $(\Omega_n, \mathbb{Z}^2, \sigma, \nu)$ is isomorphic to $(\mathbb{T}^2, \mathbb{Z}^2, R_n, \lambda)$ where ν is the unique shift-invariant probability measure on Ω_n and λ is the Haar measure on $\mathbb{T}^2$.

12.4 Open questions

Note that the n^{th} metallic mean is a quadratic Pisot unit, that is, it is an algebraic unit of degree two and all its algebraic conjugates have modulus strictly less than one. The other quadratic Pisot units are the positive roots of $x^2 - nx + 1$ for $n \geq 3$. The family of quadratic Pisot units has nice properties [90, 182, 209]; see also [32]. The continued fraction expansion of the positive root of $x^2 - nx + 1$ is $[n-1; (1, n-2)^\infty]$. In particular, it is not purely periodic.

Question 12.4 ([22])**.** *Let β be a positive quadratic Pisot unit other than the metallic means. Can we construct a self-similar set of Wang tiles whose inflation factor is β?*

An alternative question is about those quadratic integers whose continued fraction expansion is purely periodic.

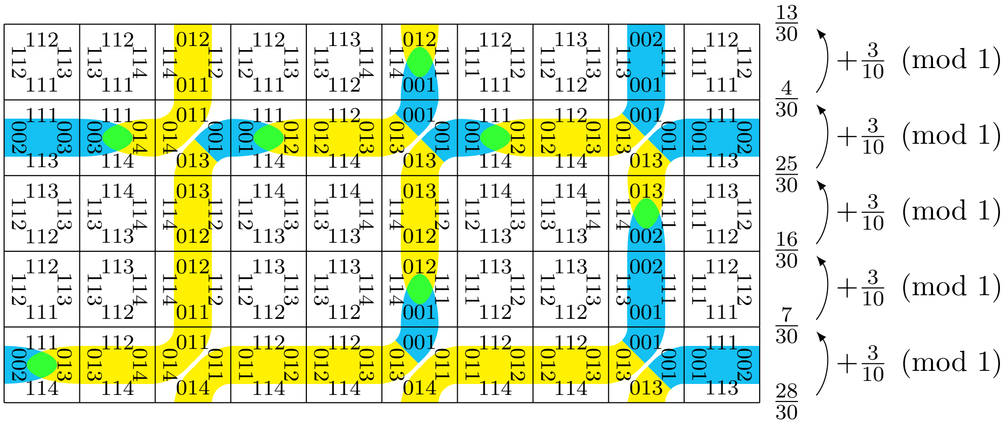


Figure 12.4: A 10×5 valid rectangular tiling with the set $\mathcal{T}_n$ with $n = 3$. The numbers indicated in the right margin are the average of the inner products $\langle \frac{1}{n} d, v\rangle$ over the vectors v appearing as top (or bottom) labels of a horizontal row of tiles and where $d = (0, -1, 1)$. We observe that these numbers increase by $\frac{3}{10}$ (mod 1) from row to row. The number $\frac{3}{10}$ is equal to the frequency of columns containing junction tiles (a junction tile is a tile whose labels all start with 0). Observe that this is a cylindrical tiling (left and right outer labels of the rectangle match) which simplifies the equations involved because the left and right carries cancel.

Question 12.5 ([22])**.** *Let β be a positive quadratic integer whose continued fraction expansion is purely periodic. Does there exist a set of Wang tiles such that its Wang shift is self-similar with inflation factor equal to β?*

The procedure explained in [148, p.594–598] starts from the Ammann A2 L shapes and constructs a set of 16 Wang tiles which we show is equivalent to the set $\mathcal{T}_1$. A question we can ask is whether this construction can be inverted. More precisely, starting from the Ammann set of 16 Wang tiles, can we recover the two Ammann L-shapes with their Ammann bars on them? In general, we ask the following question.

Question 12.6 ([22])**.** *For every integer $n \geq 1$, can we find geometrical shapes with Ammann bars on them such that encoding their tilings by rhombi along a pair of Ammann bars is equivalent to the tiles $\mathcal{T}_n$?*

Jeandel–Rao Wang tiles considered in Chapter 11 correspond to computing the orbit of points in the plane $\mathbb{R}^2$ under the translations by $+1$ horizontally and $+1$ vertically modulo the lattice Γ_0. How come this is possible is still a mystery. The link between the 11 Jeandel–Rao Wang tiles themselves and the golden ratio or toral rotation R_0 remains unclear. Unlike the Kari example, the values 0, 1, 2, 3, 4 of the labels of the Jeandel–Rao Wang tiles are five distinct symbols rather than arithmetic values. They do not satisfy a known equation.

After the discovery of the family of metallic mean Wang tiles and the expression of the Ammann set as $\mathcal{T}_1$, the following questions can be raised.

Question 12.7 ([23])**.** *Let $\mathcal{T}$ be a set of Wang tiles such that the Wang shift $\Omega_{\mathcal{T}}$ is aperiodic.*

- *Is it multiplicative (Kari-Culik-like)? More precisely, can we replace the labels of the tiles in $\mathcal{T}$ by arithmetic values in such a way that an equation similar to* (6.1) *is satisfied?*
- *Is it additive (metallic mean-like)? More precisely, can we replace the labels of the tiles in $\mathcal{T}$ by integer vectors computed from floors of linear forms as in [23, Proposition 7.4] and satisfying additive equations as in [23, Theorem B]?*

Does there exists an aperiodic set of Wang tiles which is neither multiplicative nor additive?

Solving Question 12.7 for Jeandel–Rao Wang tiles would improve our understanding of the Jeandel–Rao Wang shift. Hopefully it would allow to generate more examples maybe not related to the golden ratio and that are not self-similar. Remember that the computations made by Jeandel and Rao took one year using 100 cpus to explore exhaustively the sets of 11 Wang tiles [163]. Finding new examples by exploring all sets of 12, 13 or 14 Wang tiles becomes soon out of reach. We need to understand what is happening in order to find other examples and characterize them.

Question 12.8 ([23])**.** *If an aperiodic set of Wang tiles is additive (metallic mean-like) with labels given by integer vectors satisfying equations, can we use the equations to directly prove that the Wang shift $\Omega_{\mathcal{T}}$ is aperiodic following the short arithmetical argument for the nonperiodicity of Kari's tile set?*

Finding an answer to Question 12.8 for the Ammann set of 16 Wang tiles was the original motivation of the author which led to the discovery of the family of metallic mean Wang tiles. Nevertheless, Question 12.8 remains open even for the Ammann 16 Wang tiles and the family of metallic mean Wang tiles.

In general, we may ask the following question.

Question 12.9 ([23])**.** *For which invertible matrix $M \in \mathrm{GL}_2(\mathbb{R})$ does there exist a set of Wang tiles $\mathcal{T}$ such that the Wang shift $\Omega_{\mathcal{T}}$ is isomorphic, as a measure-preserving dynamical system, to the toral $\mathbb{Z}^2$-rotation $R : \mathbb{Z}^2 \times \mathbb{T}^2 \to \mathbb{T}^2$ defined by $R^{\boldsymbol{k}}(\boldsymbol{x}) = \boldsymbol{x} + M\boldsymbol{k}$ on the 2-dimensional torus $\mathbb{T}^2 = (\mathbb{R}/\mathbb{Z})^2$?*

The Markov partition associated to Jeandel–Rao tiles and action R_0 on $\mathbb{R}^2/\Gamma_0$ is related to the golden ratio [8]. We now have a family of $\mathbb{Z}^2$-actions related to the metallic mean quadratic integers associated to the polygonal partition $\mathcal{P}_n = \{\Phi_n([t])\}_{t\in\mathcal{T}_n}$ of $\mathbb{T}^2$. Can we find examples related to other numbers?

Question 12.10 ([23])**.** *For which $\mathbb{Z}^2$-actions defined by rotations on a 2-dimensional torus does there exist a Markov Partition? When is this partition smooth/polygonal?*

As for toral hyperbolic automorphisms, we can expect that smooth Markov partitions are associated to algebraic integers of degree 2 and that the partition is piecewise linear in this case [106]. Markov partitions for typical toral hyperbolic automorphisms have fractal boundaries [93].

The relation with toral hyperbolic automorphisms does not come out of nowhere. Indeed, the self-similarity of Ω_n proved in [22] has an incidence matrix of size $(n+3)^2 \times (n+3)^2$. Its eigenvalues are all quadratic integers, 0 or ± 1. This incidence matrix acts hyperbolically as a toral automorphism on a subspace of $\mathbb{R}^{(n+3)^2}$ thus admits a Markov partition with piecewise linear boundaries. A link between this Markov partition and the partition $\mathcal{P}_n$ can be expected, because this is what happens for 1-dimensional sequences. Indeed, the Markov partition associated to the toral automorphism $\left(\begin{smallmatrix}1&1&1\\1&0&0\\0&1&0\end{smallmatrix}\right)$ is a suspension of the Rauzy fractal [237] as nicely illustrated in a talk by Timo Jolivet [164].

Question 12.11 ([23])**.** *What is the relation between the Markov partition for the hyperbolic toral automorphism defined from the incidence matrix of the self-similarity of Ω_n and the Markov partition $\mathcal{P}_n$ associated to $\mathbb{Z}^2 \overset{\sigma}{\curvearrowright} \Omega_n$?*

The symmetric properties of Ω_n and of the partition $\mathcal{P}_n$ make them a good object of study to tackle these questions in more generality.

PART V

CONTRIBUTIONS WITHIN $(D+1)$-TO-D CUT AND PROJECT SCHEMES

CONTENTS

Chapter 13

Multidimensional Sturmian configurations

"The essential point of the positivist theory is that there is no other source of knowledge except the straight and short way of perception through the senses."

— *Max Plank*

In Chapter 8, we described our work on indistinguishable asymptotic pairs in the one-dimensional case which led to a new characterization of Sturmian sequences [3]. With Barbieri, we explored this notion further and we soon discovered that it leads to a characterization of multidimensional Sturmian configurations over $\mathbb{Z}^d$ [2]. The results provide a generalization to $\mathbb{Z}^d$ of the characterization of Sturmian sequences by their factor complexity $n+1$. This work was performed during the pandemic with all the introspection necessary to prove our intuitions on this subject. We present the main results of this work in this chapter.

13.1 Indistinguishable asymptotic pairs

Asymptotic pairs, also known as homoclinic pairs, are pairs of points in a dynamical system whose orbits coalesce. These were first studied by Poincaré [35] in the context of the three body problem and used to model chaotic behavior. Namely, two orbits which remain arbitrarily close outside a finite window of time may be used to represent pairs of trajectories that despite having similar behavior for an arbitrarily long time, present abrupt local differences.

In this chapter, we consider asymptotic pairs of zero-dimensional expansive actions of $\mathbb{Z}^d$. Concretely, given a finite set Σ, we consider the space of configurations $\Sigma^{\mathbb{Z}^d} = \{x\colon \mathbb{Z}^d \to \Sigma\}$ endowed with the prodiscrete topology and the shift action $\mathbb{Z}^d \overset{\sigma}{\curvearrowright} \Sigma^{\mathbb{Z}^d}$. In this setting, two configurations $x, y \in \Sigma^{\mathbb{Z}^d}$ are **asymptotic** if x and y differ in finitely many sites of $\mathbb{Z}^d$. The finite set $F = \{v \in \mathbb{Z}^d : x_v \neq y_v\}$ is called the **difference set** of (x, y). An example of an asymptotic pair when $d = 2$ is shown in Figure 13.1.

Given two asymptotic configurations $x, y \in \Sigma^{\mathbb{Z}^d}$, we want to compare the number of occurrences of patterns. A pattern is a function $p\colon S \to \Sigma$ where S, called the **support** of p, is a finite subset of $\mathbb{Z}^d$. The occurrences of a pattern $p \in \Sigma^S$ in a configuration $x \in \Sigma^{\mathbb{Z}^d}$ is the set $\mathsf{occ}_p(x) \coloneqq \{n \in \mathbb{Z}^d\colon \sigma^n(x)|_S = p\}$. The **language** of a configuration $x \in \Sigma^{\mathbb{Z}^d}$ over a finite support $S \subset \mathbb{Z}^d$ is $\mathcal{L}_S(x) = \{p \in \Sigma^S\colon \mathsf{occ}_p(x) \neq \varnothing\}$. When $x, y \in \Sigma^{\mathbb{Z}^d}$ are asymptotic configurations, the difference $\mathsf{occ}_p(x) \setminus \mathsf{occ}_p(y)$ is finite because the occurrences of p are the same far from the difference set of x and y. We say that (x, y) is an **indistinguishable asymptotic pair** if (x, y) is asymptotic and the following equality holds

$$\#\left(\mathsf{occ}_p(x) \setminus \mathsf{occ}_p(y)\right) = \#\left(\mathsf{occ}_p(y) \setminus \mathsf{occ}_p(x)\right) \tag{13.1}$$

for every pattern p of finite support.

x

1	0	2	2	1	0	2	1	0	2	1	1	0	2	1
0	2	1	1	0	2	1	0	2	1	0	2	2	1	0
2	1	0	2	2	1	0	2	1	0	2	1	1	0	2
1	0	2	1	1	0	2	1	0	2	1	0	2	2	1
0	2	1	0	2	2	1	0	2	1	0	2	1	1	0
2	1	0	2	1	1	0	2	1	0	2	1	0	2	2
1	0	2	1	0	2	2	1	0	2	1	0	2	1	1
0	2	1	0	2	1	1	0	2	1	0	2	1	0	2
2	1	0	2	1	0	2	2	1	0	2	1	0	2	1
1	0	2	1	0	2	1	0	2	2	1	0	2	1	0
2	2	1	0	2	1	0	2	1	1	0	2	1	0	2
1	1	0	2	1	0	2	1	0	2	2	1	0	2	1
0	2	2	1	0	2	1	0	2	1	1	0	2	1	0
2	1	1	0	2	1	0	2	1	0	2	2	1	0	2
1	0	2	2	1	0	2	1	0	2	1	1	0	2	1

y

1	0	2	2	1	0	2	1	0	2	1	1	0	2	1
0	2	1	1	0	2	1	0	2	1	0	2	2	1	0
2	1	0	2	2	1	0	2	1	0	2	1	1	0	2
1	0	2	1	1	0	2	1	0	2	1	0	2	2	1
0	2	1	0	2	2	1	0	2	1	0	2	1	1	0
2	1	0	2	1	1	0	2	1	0	2	1	0	2	2
1	0	2	1	0	2	2	1	0	2	1	0	2	1	1
0	2	1	0	2	1	0	2	2	1	0	2	1	0	2
2	1	0	2	1	0	2	1	1	0	2	1	0	2	1
1	0	2	1	0	2	1	0	2	2	1	0	2	1	0
2	2	1	0	2	1	0	2	1	1	0	2	1	0	2
1	1	0	2	1	0	2	1	0	2	2	1	0	2	1
0	2	2	1	0	2	1	0	2	1	1	0	2	1	0
2	1	1	0	2	1	0	2	1	0	2	2	1	0	2
1	0	2	2	1	0	2	1	0	2	1	1	0	2	1

Figure 13.1: The indistinguishable asymptotic configurations $x, y \in \{0,1,2\}^{\mathbb{Z}^2}$ are shown on the support $[\![-7,7]\!] \times [\![-7,7]\!]$. The two configurations are equal except on their difference set $F = \{0, -e_1, -e_2\}$ shown in red.

In other words, an asymptotic pair (x, y) is indistinguishable if every pattern appears the same number of times in x and in y while overlapping the difference set. The pair of configurations x and y shown in Figure 13.1 is an example of an indistinguishable asymptotic pair: we may check by hand that Equation (13.1) holds for patterns with small supports such as symbols (patterns of shape $\{0\}$), dominoes (patterns of shape $\{0, e_1\}$ and $\{0, e_2\}$), etc. For instance, the configurations x and y in Figure 13.1 contain eight different patterns with support $\{0, e_1, 2e_1, e_2\}$, each occurring exactly once while overlapping the difference set, see Figure 13.2.

The notion of indistinguishable asymptotic pairs appears naturally in Gibbs theory. This theory studies measures on symbolic dynamical systems which are at equilibrium in the sense that the conditional pressure for every finite region of the lattice is maximized, so that every finite region is in equilibrium with its surrounding. See [141, 189, 244, 61] for further background. An important component of Gibbs measures, the **specification**, can be formalized by means of a shift-invariant cocycle in the equivalence relation of asymptotic pairs, see [108, 60]. With an appropriate norm, the space of continuous shift invariant cocycles on the asymptotic relation becomes a Banach space, and every asymptotic pair induces a continuous linear functional through the canonical evaluation map.

The set of indistinguishable asymptotic pairs are precisely those which induce the trivial linear functional and thus a natural question is if there is an underlying dynamical structure behind this property. We shall not speak any further of Gibbs theory in this work and study indistinguishable asymptotic pairs without further reference to their origin in Gibbs theory. An interested reader can find out more about the role of indistinguishable asymptotic pairs in the aforementioned setting by reading sections 2 and 3 of [60].

In the case of dimension $d = 1$, it was shown that for the difference set $F = \{-1, 0\} \subset \mathbb{Z}$, indistinguishable asymptotic pairs are precisely the étale limits of characteristic bi-infinite Sturmian sequences ([3, Theorem B]). In the case where one of the configurations in the indistinguishable pair is recurrent, the asymptotic pair can only be a pair of characteristic bi-infinite Sturmian sequences associated to a fixed irrational value ([3, Theorem A]). Furthermore, it was shown that

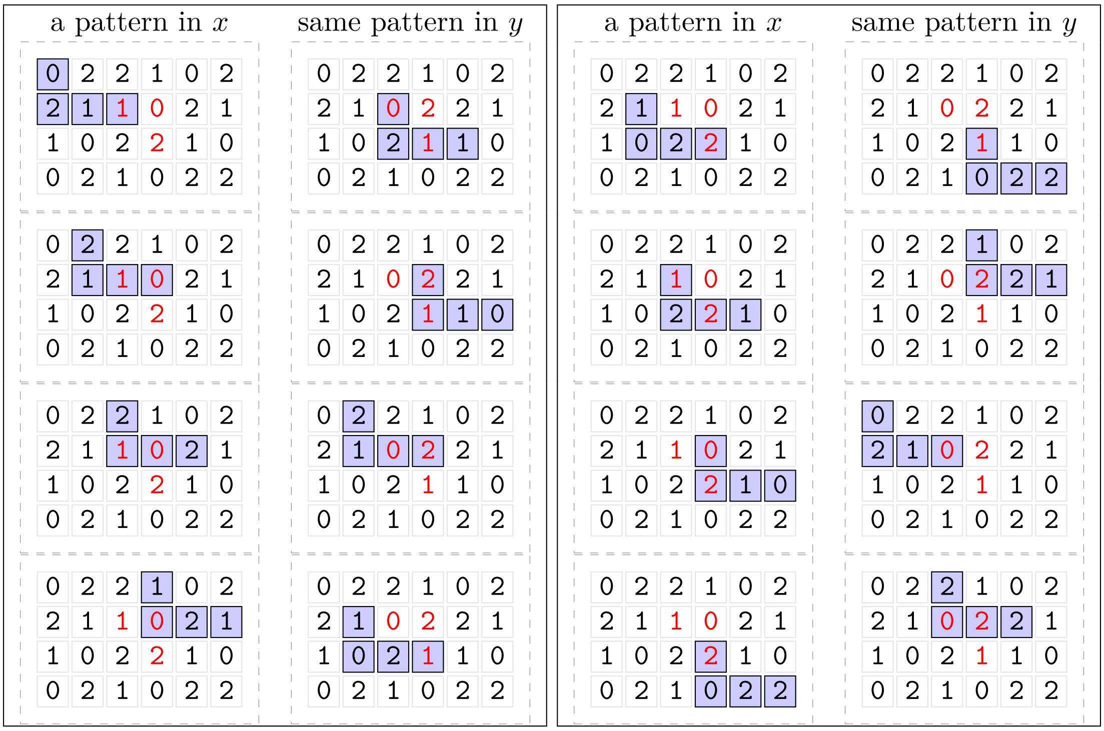


Figure 13.2: The 8 patterns of shape $\{0, e_1, 2e_1, e_2\}$ appearing in the configurations x and y. All of them appear intersecting the difference set in x and y.

any indistinguishable asymptotic pair in $\Sigma^{\mathbb{Z}}$ can be obtained from these base cases through the use of a substitution and the shift map ([3, Theorem C]), thus providing a full characterization of indistinguishable asymptotic pairs in $\mathbb{Z}$.

13.2 Main results

In this work, we extend [3, Theorem A] to the multidimensional setting. It is based on the following additional condition made on the difference set. Let $\{e_1, \ldots, e_d\}$ denote the canonical basis of $\mathbb{Z}^d$. We say that two indistinguishable asymptotic configurations $x, y \in \{0, 1, \ldots, d\}^{\mathbb{Z}^d}$ satisfy the **flip condition** if

- their difference set is $F = \{0, -e_1, \ldots, -e_d\}$,
- every symbol in $\{0, 1, \ldots, d\}$ occurs in x and y at the support F, and
- the map defined by $x_n \mapsto y_n$ for every $n \in F$ is a cyclic permutation on the alphabet $\{0, 1, \ldots, d\}$.

Without lost of generality, we assume that $x_0 = 0$ and $y_n = x_n - 1 \bmod (d+1)$ for every $n \in F$. For example, the indistinguishable asymptotic pair (x, y) illustrated in Figure 13.1 satisfies the flip condition with $(x_0, x_{-e_1}, x_{-e_2}) = (0, 1, 2)$ and $(y_0, y_{-e_1}, y_{-e_2}) = (2, 0, 1)$.

It is a well known fact that Sturmian configurations in dimension one can be characterized by their complexity [216, 113], that is, they are exactly the bi-infinite recurrent words in which exactly $n+1$ subwords of length n occur for every $n \in \mathbb{N}$. The first result provides a similar characterization of indistinguishable asymptotic pairs satisfying the flip condition by their pattern complexity which does not require uniform recurrence, or even recurrence, as an hypothesis.

Theorem 13.1 ([2]). *Let $d \geq 1$ and $x, y \in \{0, 1, \ldots, d\}^{\mathbb{Z}^d}$ be an asymptotic pair satisfying the flip condition with difference set $F = \{0, -e_1, \ldots, -e_d\}$. The following are equivalent:*

(i) For every nonempty finite connected subset $S \subset \mathbb{Z}^d$ and $p \in \mathcal{L}_S(x) \cup \mathcal{L}_S(y)$, we have

$$\# \left(\mathsf{occ}_p(x) \setminus \mathsf{occ}_p(y)\right) = 1 = \# \left(\mathsf{occ}_p(y) \setminus \mathsf{occ}_p(x)\right).$$

(ii) The asymptotic pair (x, y) is indistinguishable.

(iii) For every nonempty finite connected subset $S \subset \mathbb{Z}^d$, the pattern complexity of x and y is

$$\#\mathcal{L}_S(x) = \#\mathcal{L}_S(y) = \#(F - S).$$

The proof of Theorem 13.1 relies on an extension of the notion of bispecial factor to the setting of multidimensional configurations. Given a language, a bispecial factor is a word that can be extended in more than one way to the left and to the right. The bilateral multiplicity of bispecial factors in a one-dimensional language is closely related to the complexity of that language, see [104]. Here, for a connected support $S \subset \mathbb{Z}^d$ and two distinct positions $a, b \in \mathbb{Z}^d \setminus S$ such that $S \cup \{a\}$, $S \cup \{b\}$ and $S \cup \{a, b\}$ are connected, we say that a pattern $w \colon S \to \mathcal{A}$ is bispecial if it can be extended in more than one way at position a and at position b. The description of the bispecial patterns of indistinguishable asymptotic pairs and their multiplicities, provides us a tool for bounding their pattern complexity. Reciprocally, the rigid pattern complexity given in Theorem 13.1 forces the extension graphs associated to the bispecial patterns to have no cycle, which in turn provides us a way to show that the configurations are indistinguishable. In one dimension, sequences such as the extension graphs of bispecial factors are trees are known as dendric words [73] and thus we may think of our construction as multidimensional analogues of those.

When S is a d-dimensional rectangular block, the number $\#(F - S)$ from Theorem 13.1 admits a nice form. When $d = 1$, we compute $\#(F - S) = \#(\{0, -1\} - \{0, 1, \ldots, n-1\}) = n + 1$ which is the factor complexity function for 1-dimensional Sturmian words. When $d = 2$, $\#(F - S) = \#(\{(0,0), (-1,0), (0,-1)\} - \{(i,j) \colon 0 \leq i < n, 0 \leq j < m\}) = mn + m + n$ is the rectangular pattern complexity of a discrete plane with totally irrational (irrational and rationally independent) slope, see [87] for further references. With our result above, we can provide an explicit formula for the rectangular pattern complexity in every dimension.

Corollary 13.2 ([2]). *Let $d \geq 1$ and $(m_1, \ldots, m_d) \in \mathbb{N}^d$. The rectangular pattern complexity of an indistinguishable asymptotic pair $x, y \in \{0, 1, \ldots, d\}^{\mathbb{Z}^d}$ satisfying the flip condition is*

$$\#\mathcal{L}_{(m_1, \ldots, m_d)}(x) = \#\mathcal{L}_{(m_1, \ldots, m_d)}(y) = m_1 \cdots m_d \left(1 + \frac{1}{m_1} + \cdots + \frac{1}{m_d}\right).$$

The next result provides a beautiful connection between indistinguishable asymptotic pairs satisfying the flip condition and codimension-one (dimension of the internal space) cut and project schemes, see [154] for further background, and more precisely with multidimensional Sturmian configurations. The definition of multidimensional Sturmian configurations from codimension-one cut and project schemes is fully described in Section 13.3. A quick and easy definition of multidimensional Sturmian configurations can be given with the following formulas. Given a totally irrational vector $\alpha = (\alpha_1, \ldots, \alpha_d) \in [0, 1)^d$, the **lower** and **upper characteristic d-dimensional Sturmian configurations** with slope α are given by

$$\begin{array}{rccl} c_\alpha \colon & \mathbb{Z}^d & \to & \{0, 1, \ldots, d\} \\ & n & \mapsto & \sum_{i=1}^{d} \left(\lfloor \alpha_i + n \cdot \alpha \rfloor - \lfloor n \cdot \alpha \rfloor\right) \end{array} \quad \text{and} \quad \begin{array}{rccl} c'_\alpha \colon & \mathbb{Z}^d & \to & \{0, 1, \ldots, d\} \\ & n & \mapsto & \sum_{i=1}^{d} \left(\lceil \alpha_i + n \cdot \alpha \rceil - \lceil n \cdot \alpha \rceil\right). \end{array} \tag{13.2}$$

It turns out that these configurations are examples of indistinguishable asymptotic pairs which satisfy the flip condition. In fact, we show that a pair of uniformly recurrent asymptotic configurations is indistinguishable and satisfies the flip condition if and only if it is a pair of characteristic d-dimensional Sturmian configurations for some totally irrational slope.

Theorem 13.3 ([2])**.** *Let $d \geq 1$ and $x, y \in \{\mathtt{0}, \mathtt{1}, \dots, d\}^{\mathbb{Z}^d}$ such that x is uniformly recurrent. The pair (x, y) is an indistinguishable asymptotic pair satisfying the flip condition if and only if there exists a totally irrational vector $\alpha \in [0, 1)^d$ such that $x = c_\alpha$ and $y = c'_\alpha$ are the lower and upper characteristic d-dimensional Sturmian configurations with slope α.*

The indistinguishable asymptotic pair shown in Figure 13.1 is an example as such, where $x = c_\alpha$ and $y = c'_\alpha$ with $\alpha = (\alpha_1, \alpha_2) = (\sqrt{2}/2, \sqrt{19} - 4)$. Notice that c_α and c'_α are uniformly recurrent when α contains at least an irrational coordinate, so that hypothesis is really only used in one direction of the theorem. Note that a version of Theorem 13.3 for rational vector $\alpha \in \mathbb{Q}^d$ was considered in [14] with an infinite difference set of the form $F + K$ where $K \subset \mathbb{Z}^d$ is some lattice.

The link with codimension-one cut and project schemes can be illustrated as follows. The configurations $x = c_\alpha$ and $y = c'_\alpha$ encode the rhombi obtained as the projection of the cube faces in a discrete plane of normal vector $(1 - \alpha_1, \alpha_1 - \alpha_2, \alpha_2)$, see Figure 13.3. This three symbol coding of a discrete plane was proposed in [161], see also [75].

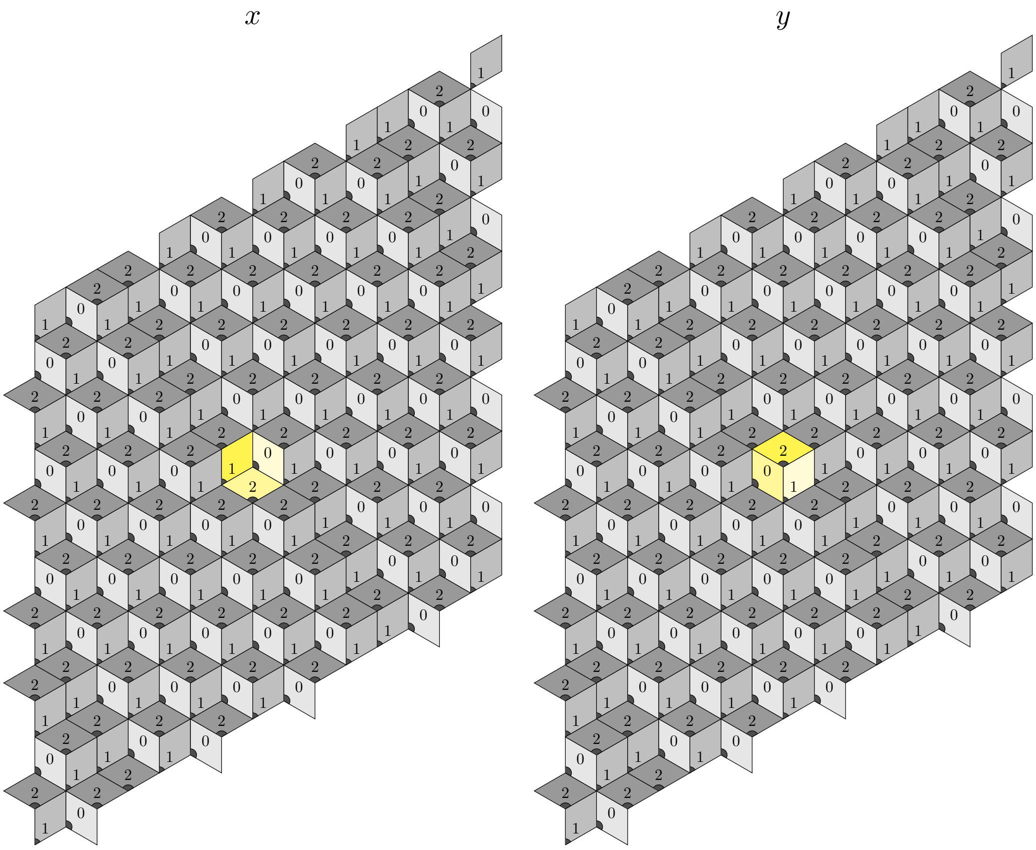


Figure 13.3: The configurations x and y from Figure 13.1 are encoding a tiling of the plane [41] by three types of pointed rhombus drawn using Jolivet's notation [165, p. 112]. The tilings shown above correspond to the projection of the surface of a discrete plane of normal vector $(1 - \alpha_1, \alpha_1 - \alpha_2, \alpha_2) \approx (0.293, 0.348, 0.359)$, with $\alpha = (\alpha_1, \alpha_2) = (\sqrt{2}/2, \sqrt{19} - 4)$, in 3 dimensional space, and their difference can be interpreted as the flip of a unit cube shown in yellow.

We also prove a slightly more general version of Theorem 13.3. We say that two indistinguishable asymptotic configurations $x, y \in \Sigma^{\mathbb{Z}^d}$ satisfy the **affine flip condition** if their difference set F has cardinality $\#F = d+1$, there is $m \in F$ such that $(F-m) \setminus \{0\}$ is a base of $\mathbb{Z}^d$, the restriction $x|_F$ is a bijection $F \to \Sigma$ and the map defined by $x_n \mapsto y_n$ for every $n \in F$ is a cyclic permutation on Σ.

Corollary 13.4 ([2])**.** *Let $d \geq 1$ and $x, y \in \Sigma^{\mathbb{Z}^d}$ such that x is uniformly recurrent. The pair (x, y) is an indistinguishable asymptotic pair satisfying the affine flip condition if and only if there exist a bijection $\tau\colon \{\mathtt{0}, \mathtt{1}, \dots, d\} \to \Sigma$, an invertible affine transformation $A \in \mathrm{Aff}(\mathbb{Z}^d)$ and a totally irrational vector $\alpha \in [0,1)^d$ such that $x = \tau \circ c_\alpha \circ A$ and $y = \tau \circ c'_\alpha \circ A$.*

If we further assume that the configurations in the asymptotic pair are uniformly recurrent, we can put together Theorem 13.1 and Theorem 13.3 and obtain the following characterization of uniformly recurrent multidimensional Sturmian configurations in terms of their pattern complexity. This generalizes the well-known theorem of Morse-Hedlund-Coven to higher dimensions [216, 113].

Corollary 13.5 ([2])**.** *Let $d \geq 1$ and $x, y \in \{\mathtt{0}, \mathtt{1}, \dots, d\}^{\mathbb{Z}^d}$ be an asymptotic pair such that x is uniformly recurrent and which satisfies the flip condition with difference set $F = \{0, -e_1, \dots, -e_d\}$. The following are equivalent:*

(i) For every nonempty finite connected subset $S \subset \mathbb{Z}^d$ and $p \in \mathcal{L}_S(x, y)$, we have

$$\#\left(\mathsf{occ}_p(x) \setminus \mathsf{occ}_p(y)\right) = 1 = \#\left(\mathsf{occ}_p(y) \setminus \mathsf{occ}_p(x)\right).$$

(ii) The asymptotic pair (x, y) is indistinguishable.

(iii) For every nonempty finite connected subset $S \subset \mathbb{Z}^d$, we have

$$\#\mathcal{L}_S(x) = \#\mathcal{L}_S(y) = \#(F - S).$$

(iv) There exists a totally irrational vector $\alpha \in [0,1)^d$ such that $x = c_\alpha$ and $y = c'_\alpha$.

13.3 A codimension-one cut and project scheme

Cut and project schemes of codimension-one (dimension of the internal space) can be defined in several ways (for a different definition see [154]). In what follows we follow the formalism of [51, §7], but note that we need to adapt it in order to describe symbolic configurations over a lattice $\mathbb{Z}^d$. Let $d \geq 1$ be an integer and

$$\begin{array}{cccc} \pi: & \mathbb{R}^{d+1} & \to & \mathbb{R}^d \\ & (x_0, x_1, \dots, x_d) & \mapsto & (x_1, \dots, x_d) \end{array}$$

be the projection of $\mathbb{R}^{d+1}$ in the **physical space** $\mathbb{R}^d$. Let $\alpha_0 = 1$, $\alpha_{d+1} = 0$ and $\alpha = (\alpha_1, \dots, \alpha_d) \in [0,1)^d$ be a **totally irrational** vector, that is such that $\{1, \alpha_1, \dots, \alpha_d\}$ is linearly independent over $\mathbb{Q}$. Let

$$\begin{array}{cccc} \pi_{\mathrm{int}}: & \mathbb{R}^{d+1} & \to & \mathbb{R}/\mathbb{Z} \\ & (x_0, x_1, \dots, x_d) & \mapsto & \sum_{i=0}^d x_i \alpha_i \end{array}$$

be the projection of $\mathbb{R}^{d+1}$ in the **internal space** $\mathbb{R}/\mathbb{Z}$. Consider the lattice $\mathcal{L} = \mathbb{Z}^{d+1} \subset \mathbb{R}^{d+1}$ whose image is $\pi(\mathcal{L}) = \mathbb{Z}^d$. The projection on the physical space is not injective when restricted to the lattice $\mathcal{L} = \mathbb{Z}^{d+1}$. But we have that condition (7.1) holds since

$$\operatorname{Ker} \pi \cap \mathcal{L} = \mathbb{Z} \times \{0\}^d \subseteq \operatorname{Ker} \pi_{\mathrm{int}}.$$

Thus, this is the setting of a codimension-one $(d+1)$-to-d degenerate cut and project scheme summarized in the following diagram:

$$\begin{array}{ccccc} W \subset \mathbb{R}/\mathbb{Z} & \xleftarrow{\pi_{\text{int}}} & \mathbb{R}^{d+1} & \xrightarrow{\pi} & \mathbb{R}^d \\ \cup \text{ dense} & & \cup & & \cup \\ \pi_{\text{int}}(\mathcal{L}) & \longleftarrow & \mathcal{L} & \longrightarrow & \pi(\mathcal{L}) \supset \curlywedge(W) \\ & \nwarrow & \star & \swarrow & \end{array}$$

We deduce that the star map of this cut and project scheme is defined as

$$n^\star = \alpha \cdot n \bmod 1 \tag{13.3}$$

for every $n \in \pi(\mathcal{L}) = \mathbb{Z}^d$. For a given **window** $W \subset \mathbb{R}/\mathbb{Z}$ in the internal space,

$$\curlywedge(W) := \{x \in L \mid x^\star \in W\}$$

is the projection set within the cut and project scheme, where $L = \pi(\mathcal{L})$. If $W \subset \mathbb{R}/\mathbb{Z}$ is a relatively compact set with non-empty interior, any translate $t + \curlywedge(W)$ of the projection set, $t \in \mathbb{R}^d$, is called a **model set**.

If $W = [0,1)$, then $\curlywedge(W) = \mathbb{Z}^d$. Thus, if $W \subset [0,1)$, then $\curlywedge(W) \subset \mathbb{Z}^d$. Moreover, if $\{W_i\}_{i\in\{0,\dots,d\}}$ is a partition of $[0,1)$, then $\{\curlywedge(W_i)\}_{i\in\{0,\dots,d\}}$ is a partition of $\mathbb{Z}^d$. Using this idea, we now build configurations $\mathbb{Z}^d \to \{\mathtt{0},\mathtt{1},\dots,d\}$ according to a partition of $\mathbb{R}/\mathbb{Z}$, or equivalently of the interval $[0,1)$, into consecutive intervals.

Definition 13.6 ([2])**.** *Let $\alpha = (\alpha_1,\dots,\alpha_d) \in [0,1)^d$ be a totally irrational vector and τ be the permutation of $\{\mathtt{1},\dots,d\} \cup \{\mathtt{0}, d+1\}$ which fixes $\{\mathtt{0}, d+1\}$ and such that $0 = \alpha_{\tau(d+1)} < \alpha_{\tau(d)} < \cdots < \alpha_{\tau(\mathtt{1})} < \alpha_{\tau(\mathtt{0})} = 1$. For every $i \in \{0,1,\dots,d\}$, let*

$$W_i = [1-\alpha_{\tau(i)}, 1-\alpha_{\tau(i+1)}), \qquad W_i' = (1-\alpha_{\tau(i)}, 1-\alpha_{\tau(i+1)}]$$

be such that $\{W_i\}_{i\in\{0,\dots,d\}}$ and $\{W_i'\}_{i\in\{0,\dots,d\}}$ are two partitions of the interval $[0,1)$. The configurations

$$\begin{array}{rccl} c_\alpha : & \mathbb{Z}^d & \to & \{\mathtt{0},\mathtt{1},\dots,d\} \\ & n & \mapsto & i \text{ if } n^\star \in W_i \end{array} \quad \text{and} \quad \begin{array}{rccl} c'_\alpha : & \mathbb{Z}^d & \to & \{\mathtt{0},\mathtt{1},\dots,d\} \\ & n & \mapsto & i \text{ if } n^\star \in W_i' \end{array}$$

*are respectively the **lower** and **upper characteristic** d-**dimensional Sturmian configurations** with slope $\alpha \in [0,1)^d$. Moreover, if $\rho \in \mathbb{R}/\mathbb{Z}$, the configurations*

$$\begin{array}{rccl} s_{\alpha,\rho} : & \mathbb{Z}^d & \to & \{\mathtt{0},\mathtt{1},\dots,d\} \\ & n & \mapsto & i \text{ if } n^\star + \rho \in W_i \end{array} \quad \text{and} \quad \begin{array}{rccl} s'_{\alpha,\rho} : & \mathbb{Z}^d & \to & \{\mathtt{0},\mathtt{1},\dots,d\} \\ & n & \mapsto & i \text{ if } n^\star + \rho \in W_i' \end{array}$$

*are respectively the **lower** and **upper** d-**dimensional Sturmian configurations** with **slope** $\alpha \in [0,1)^d$ and **intercept** $\rho \in \mathbb{R}/\mathbb{Z}$.*

It turns out that configurations $s_{\alpha,\rho}$ and $s'_{\alpha,\rho}$ can be expressed by a formula involving a sum of differences of floor functions thus extending the definition of Sturmian sequences by mechanical sequences [216]. It also reminds of recent progresses on Nivat's conjecture where configurations with low pattern complexity are proved to be sums of periodic configurations [173, 266], although here it involves a sum of non-periodic configurations.

Lemma 13.7 ([2])**.** *Let $\alpha = (\alpha_1,\dots,\alpha_d) \in [0,1)^d$ be a totally irrational vector and $\rho \in \mathbb{R}/\mathbb{Z}$. The lower and upper d-dimensional Sturmian configurations with slope α and intercept ρ are given by the following rules:*

$$\begin{array}{rccl} s_{\alpha,\rho} : & \mathbb{Z}^d & \to & \{\mathtt{0},\mathtt{1},\dots,d\} \\ & n & \mapsto & \sum_{i=1}^{d} \left(\lfloor \alpha_i + n\cdot\alpha + \rho \rfloor - \lfloor n\cdot\alpha+\rho \rfloor\right), \end{array}$$

and

$$\begin{array}{rcl} s'_{\alpha,\rho}: & \mathbb{Z}^d & \to & \{0,1,\dots,d\} \\ & n & \mapsto & \sum_{i=1}^{d} \left(\lceil \alpha_i + n\cdot\alpha + \rho \rceil - \lceil n\cdot\alpha+\rho\rceil\right). \end{array}$$

When $d = 1$, $s_{\alpha,\rho}$ and $s'_{\alpha,\rho}$ correspond to lower and upper mechanical words defined in [216], see also [202, 38, 33]. When $d = 2$, they are in direct correspondence to discrete planes as defined in [87, 39, 41]. See also Jolivet's Ph.D. thesis [165]. In general, we say that a configuration in $\{0,1,\dots,d\}^{\mathbb{Z}^d}$ is **Sturmian**, if it coincides either with $s_{\alpha,\rho}$ or $s'_{\alpha,\rho}$ for some $\rho \in \mathbb{R}$ and totally irrational $\alpha \in [0,1)^d$.

When $\rho = 0$, we have $s_{\alpha,0} = c_\alpha$ and $s'_{\alpha,0} = c'_\alpha$. Thus, Equation (13.2) follows from Lemma 13.7.

The fact that the configurations c_α and c'_α are encodings of codimension-one cut and project schemes is illustrated with $\alpha = (\alpha_1, \alpha_2) = (\sqrt{2}/2, \sqrt{19}-4)$ in Figure 13.3 in which we see a discrete plane in dimension 3 of normal vector $(1-\alpha_1, \alpha_1-\alpha_2, \alpha_2) \approx (0.293, 0.348, 0.359)$.

13.4 A factor map

Let $\alpha \in [0,1)^d$ and consider the dynamical system $\mathbb{Z}^d \overset{R}{\curvearrowright} \mathbb{R}/\mathbb{Z}$ where $R\colon \mathbb{Z}^d \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ is the continuous $\mathbb{Z}^d$-action on $\mathbb{R}/\mathbb{Z}$ defined by

$$R^n(x) := R(n,x) = x + n\cdot\alpha$$

for every $n \in \mathbb{Z}^d$.

Recall that an action is minimal if every orbit is dense. Following Section 2.4, we have a factor map from the subshift of a multidimensional Sturmian configuration to the torus.

Lemma 13.8 ([2])**.** *Let $\alpha \in [0,1)^d$ be totally irrational and consider the topological partition of the circle*

$$\mathcal{P} = \{\mathrm{Int}(W_{\mathtt{i}})\}_{\mathtt{i}\in\{0,1,\dots,d\}}.$$

1. *The partition $\mathcal{P}$ gives a symbolic representation of the dynamical system $\mathbb{Z}^d \overset{R}{\curvearrowright} \mathbb{R}/\mathbb{Z}$.*
2. *The symbolic dynamical system $\mathcal{X}_{\mathcal{P},R}$ is minimal and satisfies $\mathcal{X}_{\mathcal{P},R} = \overline{\{\sigma^k c_\alpha \colon k \in \mathbb{Z}^d\}}$.*
3. *$f\colon \mathcal{X}_{\mathcal{P},R} \to \mathbb{R}/\mathbb{Z}$ where $f(x) \in \bigcap_{n=0}^{\infty} \overline{D}_n(w)$ is a factor map.*

13.5 Open questions

To fully generalize the theorem of Morse-Hedlund-Coven, we would hope the equivalence holds for single configurations and not only for pairs of asymptotic configurations satisfying the flip condition. More precisely, in the case of uniformly recurrent configurations, we believe that the pattern complexity characterizes multidimensional Sturmian configurations. Let $\alpha = (\alpha_1, \dots, \alpha_d) \in [0,1)^d$ be a totally irrational vector and $\rho \in \mathbb{R}/\mathbb{Z}$. The lower and upper d-dimensional Sturmian configurations with slope α and intercept ρ are given by the following rules [2, Lemma 4.3]:

$$\begin{array}{rcl} s_{\alpha,\rho}: & \mathbb{Z}^d & \to & \{0,1,\dots,d\} \\ & n & \mapsto & \sum_{i=1}^{d} \left(\lfloor \alpha_i + n\cdot\alpha + \rho \rfloor - \lfloor n\cdot\alpha+\rho\rfloor\right), \end{array}$$

and

$$\begin{array}{rcl} s'_{\alpha,\rho}: & \mathbb{Z}^d & \to & \{0,1,\dots,d\} \\ & n & \mapsto & \sum_{i=1}^{d} \left(\lceil \alpha_i + n\cdot\alpha + \rho \rceil - \lceil n\cdot\alpha+\rho\rceil\right). \end{array}$$

Question 13.9 ([2])**.** *Let $d \geq 1$ and $x \in \{0, 1, \dots, d\}^{\mathbb{Z}^d}$ be uniformly recurrent configuration. Let $F = \{0, -e_1, \dots, -e_d\}$. Consider the following two statements:*

(i) for every nonempty finite connected subset $S \subset \mathbb{Z}^d$, we have $\#\mathcal{L}_S(x) = \#(F - S)$.

(ii) there exists a totally irrational vector $\alpha \in [0,1)^d$ and a intercept $\rho \in [0,1)$ such that $x = s_{\alpha,\rho}$ or $x = s'_{\alpha,\rho}$.

Since $s_{\alpha,\rho}$, $s'_{\alpha,\rho}$ and c_α have the same language when α is totally irrational, we can deduce from Corollary 13.5 that (ii) implies (i). Is it true that (i) and (ii) are equivalent?

Consider a sequence of totally irrational slopes $(\alpha_n)_{n\in\mathbb{N}}$ for which both c_{α_n} and c'_{α_n} converge in the prodiscrete topology. Then $(c_{\alpha_n}, c'_{\alpha_n})_{n\in\mathbb{N}}$ converges in the asymptotic relation to an étale limit (c, c'), see [2, Definition 2.8]. It turns out that étale limits preserve both the flip condition and indistinguishability, and will thus satisfy all of the equivalences stated in Theorem 13.1. An example of such a limit is illustrated in Figure 13.4.

c

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	2	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	2	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	1	2	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	2	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	1	2	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	1	2	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	1	2	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	1	2
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

c'

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	2	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	2	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	1	2	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	1	2	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	2	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	1	2	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	1	2	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	1	2
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Figure 13.4: An indistinguishable asymptotic pair (c, c') which satisfies the flip condition obtained by taking the limit of the Sturmian configurations given by $\alpha_n = (\frac{1}{n}(\sqrt{2}-1), \frac{1}{n}(\sqrt{3}-1))$.

We believe that in fact every indistinguishable asymptotic pair on $\mathbb{Z}^d$ which satisfies the flip condition can be obtained through an étale limit as above.

Conjecture 13.10 ([2])**.** *Let $d \geq 1$ and $x, y \in \{0, 1, \dots, d\}^{\mathbb{Z}^d}$ be an indistinguishable asymptotic pair which satisfies the flip condition. Then there exists a sequence of totally irrational vectors $(\alpha_n)_{n\in\mathbb{N}}$ such that (x, y) is the étale limit of the sequence of asymptotic pairs $(c_{\alpha_n}, c'_{\alpha_n})_{n\in\mathbb{N}}$.*

It was proved that Conjecture 13.10 holds when $d = 1$, see [3, Theorem B]. Proving it for $d > 1$ is harder due to the various ways a sequence $(\alpha_n)_{n\in\mathbb{N}}$ can converge to some vector $\alpha \in [0,1)^d$ leading to infinitely many étale limits associated to a single vector. When $d = 1$, there are only two such ways: from above or from below. Describing combinatorially what happens in these two cases was sufficient in [3] to prove the result. An analogue combinatorial description of all different behaviors when $d > 1$ is still open.

In [3, Theorem C], indistinguishable asymptotic pairs were totally described when $d = 1$ by the image under substitutions of characteristic Sturmian sequences. Describing indistinguishable asymptotic pairs in general when $d > 1$ (other than those satisfying the flip condition or some affine version of it) remains an open question.

Question 13.11. *Let $d \geq 1$ and $x, y \in \{0, 1, \ldots, d\}^{\mathbb{Z}^d}$ be an indistinguishable asymptotic pair. Does there exists a sequence of totally irrational vectors $(\alpha_n)_{n\in\mathbb{N}}$ such that (x, y) can be derived from the étale limit of the sequence of asymptotic pairs $(c_{\alpha_n}, c'_{\alpha_n})_{n\in\mathbb{N}}$?*

Our current work also leads to another interesting question. In dimension 1, it is known at least since [215] that a sequence of factor complexity less than or equal to n is eventually periodic. In two dimensions, it is still an open problem [117, 173] known as Nivat's conjecture [219] whether a configuration x for which there are $n, m \in \mathbb{N}$ with $\#\mathcal{L}_{(n,m)}(x) \leq nm$ is periodic or not. Another question which seems to have been overlooked due to the difficulty of settling Nivat's conjecture is to describe the minimal complexity of an aperiodic configuration (trivial stabilizer under the shift map, that is $\sigma^n(x) = x$ only holds for $n = 0$) which admits a totally irrational vector of symbols frequencies. When $d = 1$, we know that such sequences have complexity at least $n + 1$ and are realized by Sturmian configurations. However, when $d = 2$, configurations with rectangular pattern complexity $mn+1$ are not uniformly recurrent and do not have a totally irrational vector of symbol frequencies [103]. As the symbol frequencies of the multidimensional Sturmian configurations c_α and c'_α is α, it follows by Theorem 13.3 and Theorem 13.1 that they provide an upper bound for this problem, namely, that these sequences can be realized with complexity $\#(F - S)$ for every pattern of connected support S. According to Cassaigne and Moutot (personal communication, January 2023), there exist 2-dimensional configurations with totally irrational vector of symbol frequencies with pattern complexity strictly less than $\#(F - S)$ for infinitely many connected supports S. Therefore, we ask the following question.

Question 13.12 ([2])**.** *Let $d \geq 1$. Let $x \in \{0, 1, \ldots, d\}^{\mathbb{Z}^d}$ be a configuration with trivial stabilizer and assume that the frequencies of symbols in x exist and form a totally irrational vector. Let $S \subset \mathbb{Z}^d$ be a nonempty connected finite support. What is the greatest lower bound for the pattern complexity $\#\mathcal{L}_S(x)$?*

It is known that bispecial factors within the language of a Sturmian sequence of slope $\alpha \in [0, 1)$ are related to the convergents of the continued fraction expansion of α [203]. Since our work extends the notion of bispecial factors to the setup of multidimensional Sturmian configurations (see Figure 13.5), it is natural to ask the following question about simultaneous Diophantine approximation [248].

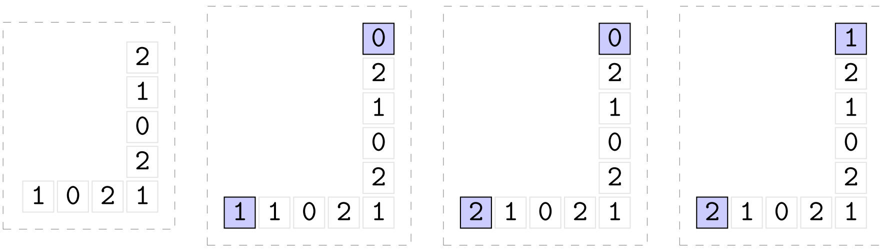


Figure 13.5: On the left, an L-shape pattern of support $\{(1,0), (2,0), (3,0), (4,0), (4,1), (4,2), (4,3), (4,4)\}$ is shown. It is bispecial at positions $a = (0, 0)$ and $b = (4, 5)$ because it can be extended in more than one way at these positions within the language of the configurations x and y shown in Figure 13.1. Thus $b - a = (4, 5) \in V_\alpha$ when $\alpha = (\sqrt{2}/2, \sqrt{19} - 4)$.

Question 13.13 ([2]). *Let $d \geq 1$ and $\alpha \in [0,1)^d$ be a totally irrational vector. What is the relation between the set*

$$V_\alpha = \{b - a\colon \text{ there exists } w \in \mathcal{L}_S(c_\alpha) \text{ which is bispecial at positions } a, b \in \mathbb{Z}^d\}$$

and simultaneous Diophantine approximations of the vector α?

PART VI

CONCLUSION

CONTENTS

Chapter 14

Perspectives

"How and to what extent can a dynamical system be represented by a symbolic one?"

— Adler, [25]

The subject of this thesis lies at the interface of mathematics and theoretical computer science; more precisely, in the fields of dynamical systems and number theory with close connections to discrete mathematics and tiling theory. The main objects of study are discrete dynamical systems given by a (continuous or measurable) map of a space into itself. Hadamard [149] first proposed the idea of representing the orbit of a point in a dynamical system as a sequence of symbols according to a partition of the space. Later the pioneering work of Morse and Hedlund [215] gave birth to the field of **symbolic dynamical systems** with strong links to number theory (continued fractions [216], Diophantine approximation [105]) and geometry (geodesic flow on the modular surface [254]). That was generalized to interval exchange transformations [174] with its relation to Teichmüller theory [272, 278, 47].

Another famous application of Symbolic Dynamics is the study of hyperbolic automorphisms and diffeomorphisms of the torus [258, 26, 91] where orbits are coded using a partition recalling the Markov process property. Symbolic Dynamics in higher dimensions is very different in terms of decidability [67] and entropy [159] due to the possibility of embedding the computation of any Turing machine [270] in a two-dimensional tiling. Thus, $\mathbb{Z}^d$-actions are intrinsically more complex than $\mathbb{Z}$-actions. Many important challenging open problems are still open including Pisot conjecture [28], Nivat's conjecture [172] and the Markoff injectivity conjecture [27, 240].

Many open questions are raised at the end of many chapters in the current thesis (Section 8.3, Section 9.3, Section 10.8, Section 11.10, Section 12.4, Section 13.5). In this chapter, we present additional open questions. These subjects will be proposed to future Ph. D. students.

14.1 Digital geometry and combinatorics

In the last decades, computer imaging has undergone tremendous development in a number of fields: image analysis and processing, pattern recognition, image synthesis and computer graphics, computer vision and algorithmic geometry. These fields have contributed to the growth of discrete geometry, whose varied development is reminiscent of the richness of geometry and topology in mathematics [269]. Digital geometry is the set of theories developed in computer science for topological and geometric questions on a finite or countable set of $\mathbb{Z}^d$ points [178, 109, 120]. These theories seek analogies with Euclidean geometry in $\mathbb{R}^d$, without the analytical tools.

We know that topology and Euclidean geometry cannot be transported to discrete spaces without major distortions and a phenomenon of theory multiplication. For example, the existence of a Jordan theorem in the discrete plane for discrete curves [273] and the recognition of a discrete line lead to several concepts. Digital geometry must therefore be carefully distinguished from Euclidean geometry. It addresses fundamental imaging problems based on appropriate mathematical theories, leading to robust algorithms and efficient software [96].

Discrete segments in $\mathbb{Z}^2$

Introduced in Section 4.2, Christoffel words are the basic building blocks of discrete geometry in dimension two, as they represent the discrete segments of $\mathbb{Z}^2$. Christoffel's words are related to

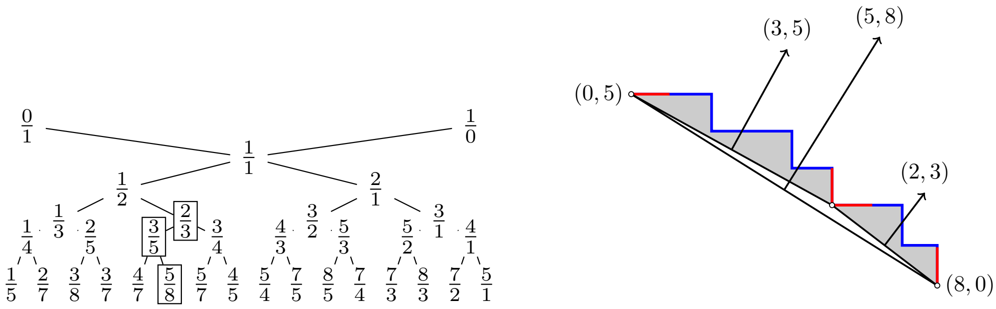


Figure 14.1: Left, the Stern-Brocot tree. On the right, the standard factorization $w_{\vec{a}} = w_{\vec{b}} \cdot w_{\vec{c}}$ of the Christoffel word of normal vector $\vec{a} = (5, 8)$ with $w_{\vec{b}} =$ `00100101`, $w_{\vec{c}} =$ `00101`, $\vec{b} = (3, 5)$ and $\vec{c} = (2, 3)$.

.

Farey sequences, the Stern-Brocot tree (see Fig. 14.1) and continued fractions via their standard factorization. Indeed, a Christoffel word $w_{\vec{a}}$ of normal vector $\vec{a}$ factorizes into a unique product $w_{\vec{b}} \cdot w_{\vec{c}}$ such that $w_{\vec{b}}$ and $w_{\vec{c}}$ are two Christoffel words of normal vector $\vec{b}$ and $\vec{c}$ respectively. This is the *standard factorization* of Christoffel words introduced by Borel and Laubie [89] (see Fig. 14.1). Christoffel words are also used to segment discrete curves. For example, an optimal algorithm [97] for deciding the convexity of a discrete curve is based on the recognition of Lyndon words and Christoffel words.

A Lyndon word is a word that is lexicographically smaller than all its circular permutations. For example, the word `atoire` is a Lyndon word, because its circular permutations are:

$$\{\texttt{atoire}, \texttt{toirea}, \texttt{oireat}, \texttt{ireato}, \texttt{reatoi}, \texttt{eatoir}\}.$$

It is known [201, Theorem 5.1.1] that any non-empty word w admits a unique factorization as a lexicographically decreasing Lyndon word sequence:

$$w = l_1^{n_1} l_2^{n_2} \cdots l_k^{n_k}, \qquad l_1 > l_2 > \cdots > l_k,$$

where $n_i \geq 1$ and l_i is a Lyndon word, for all i such that $1 \leq i \leq k$. For example, we check that the unique Lyndon factorization of the word `combinatory` is: (`com`)(`bin`)(`atoire`). The following result linking Lyndon factorization and the convexity property of a path has provided an important link between word combinatorics and digital geometry (see Fig. 14.2).

Theorem 14.1 ([97])**.** *A word $w \in \{0, 1\}^*$ describes a convex path if and only if its unique Lyndon factorization $l_1^{n_1} l_2^{n_2} \cdots l_k^{n_k}$ is such that all l_i are Christoffel words.*

An open question [234] is:

Question 14.2. *Find a higher-dimensional equivalent to Theorem 14.1 that would allow a discrete convex surface to be segmented.*

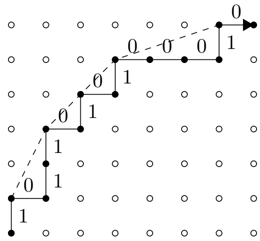


Figure 14.2: The Lyndon factorization of the discrete convex curve $v = 1011010100010$ is $v = (1)^1 \cdot (011)^1 \cdot (01)^2 \cdot (0001)^1 \cdot (0)^1$ where 0, 011, 01, 0001 and 0 are words from Christoffel [97].

To answer this question, it's natural to look for an equivalent of Christoffel's words that would allow us to generalize the concepts we know in $\mathbb{Z}^2$ to a higher dimension. With Christophe Reutenauer, we generalized Christoffel's words to dimension $d \geq 3$ [14]. The object we have defined is not a word, but rather a Christoffel graph associated with a normal vector.

The concatenation of Christoffel graphs, whose definition is not as simple as for words, still needs to be better described. Experiments suggest that the standard factorization generalizes to higher dimensions (see Fig.14.3, as well as recent joint results with Tristan Roussillon [15]). However, this factorization is not unique and requires further study.

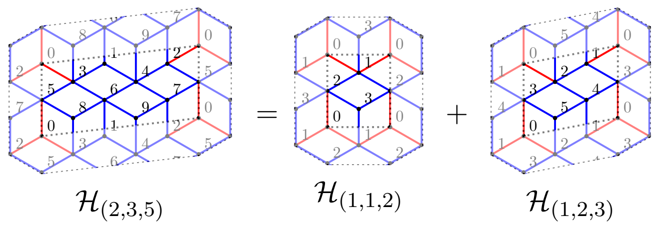


Figure 14.3: A factorization of Christoffel's graph $\mathcal{H}_{(2,3,5)}$.

Question 14.3. *Describe the factorization of Christoffel graphs.*

As Christoffel's words are directly related to continued fractions via their standard factorization, this project have applications in number theory and discrete dynamical systems. Indeed, the multidimensional continued fraction algorithm behind Christoffel's graphs is not known. Higher-dimensional generalizations of Farey sequences have already been considered, demonstrating the difficulties that come with them. Christoffel graphs offer a promising new approach to this question.

Question 14.4. *Generalize the Stern-Brocot tree to higher dimensions, based on Christoffel graphs.*

14.2 Cut and project schemes and subshifts of finite type

Jeandel-Rao tilings are very interesting and were rich enough to be the subject of many articles [7, 10, 8, 9, 13]. However, it remains a single example. A step forward was the discovery of the family of metallic mean Wang tiles which extend the sets of aperiodic Wang tiles beyond the omnipresent golden ratio [22, 23]. However, this list of subshifts of finite type described by cut and project schemes remains incomplete. Can we characterize them all?

Question 14.5. *Characterize the family of subshifts of finite type in $\mathcal{A}^{\mathbb{Z}^d}$ that are described by (Euclidean) cut and project schemes.*

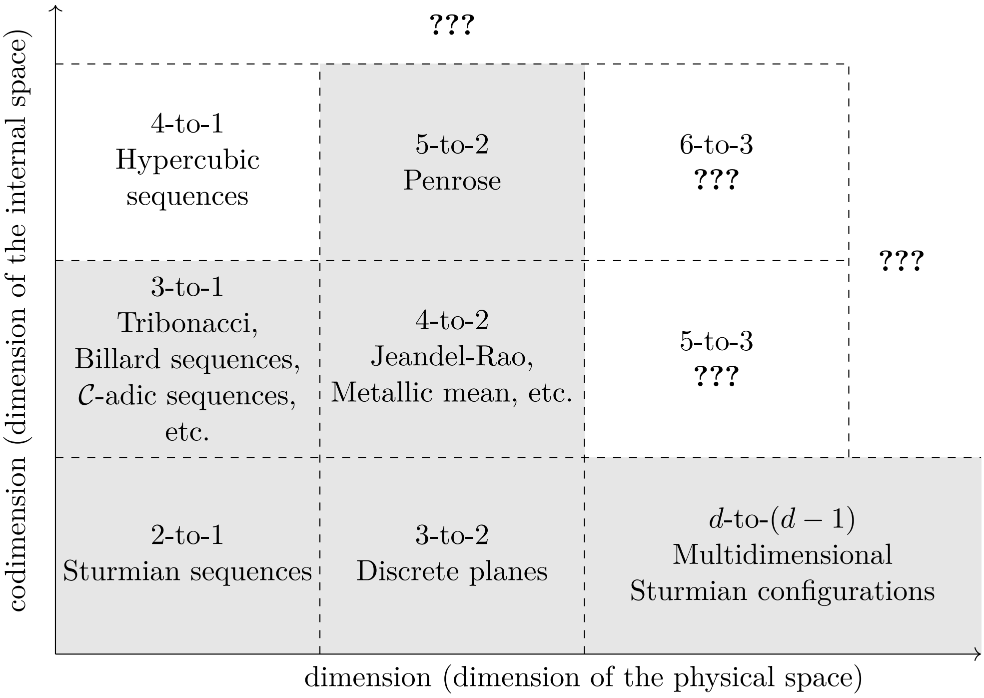


Figure 14.4: Combinatorial properties of cut and project schemes are studied mostly when the physical dimension is 2 and when the codimension is less than 2 or 3. It remains a large territory to explore.

A simpler question might be to restrict it 2 dimensions and 4-to-2 cut and project schemes.

Question 14.6. *Characterize the family of subshifts of finite type in $\mathcal{A}^{\mathbb{Z}^2}$ that are described by (Euclidean) 4-to-2 cut and project schemes. Are they always self-similar or substitutive?*

On this subject, many other more precise open questions are listed in Part IV in Section 11.10 and Section 12.4. Following the work on Jeandel-Rao tilings and metallic mean Wang tiles, there are several entry points to study these questions, depending on the interest of a future Ph. D. Student:

- existing sets of aperiodic Wang tiles (among which are the other candidates listed by Jeandel and Rao [163], the encoding of Penrose tilings into Wang tiles [148], the golden octagonal [56], etc.)
- $\mathbb{Z}^d$-actions on $\mathbb{R}^d/\mathbb{Z}^d$ coded by other polyhedral partitions (this is how metallic mean Wang tiles were discovered),
- automata theory and numeration systems for $\mathbb{Z}^d$ [17, 18, 16],
- d-dimensional substitutions and d-dimensional self-similar subshifts.
- cut and project schemes and their combinatorial (linear repetitivity [155]) and numerical properties [154],
- geometrical coincidences in the cut and project scheme, Grassmann coordinates [57] and matroid theory.

This project has already started in the community. Some preliminary results were obtained by other authors recently about potential polygonal partition coding other Wang shifts [160]. Also, a polygonal partition of a rectangular fundamental domain of height $\varphi + 4$ was discovered for one of the candidates listed by Jeandel and Rao in a memoir of Thompson [267], see Section 11.9.

14.3 Other extensions of Sturmian systems

In this document, we have listed open questions about Sturmian sequences within 2-to-1 cut and project schemes (Section 8.3), about extensions of 3-to-1 cut and project schemes with good combinatorial properties (Section 10.8) and about multidimensional Sturmian configurations and codimension 1 cut and project schemes (Section 13.5).

Nevertheless, in terms of the dimension and codimensions, much of the combinatorial and numerical properties cut and project schemes remain to be explored, see Figure 14.4. When the dimension and codimension is large (≥ 3), can subshifts defined by cut and project schemes be characterized by their pattern complexity? as subshifts of finite type? as Wang shifts?

14.4 A question of Terence Tao

Let $d \geq 1$ be an integer. We say that a set $S \subset \mathbb{Z}^d$ tiles the space, if there exists a set of translations translations $T \subset \mathbb{Z}^d$ such that $\mathbb{Z}^d = S \oplus T$ (here the notation $S \oplus T$ means that $\{S + t | t \in T\}$ forms a partition of $\mathbb{Z}^d$). A set of translations $T \subset \mathbb{Z}^d$ is said to be *periodic* if there exists a vector $v \in \mathbb{Z}^d \setminus \{0\}$ such that $T = T + v$. A set of translations $T \subset \mathbb{Z}^d$ is said to be *strongly periodic* if it is the finite union of cosets of a finite-index subgroup of $\mathbb{Z}^d$.

In an article published in the summer of 2024 in *Annals of Mathematics* [147], Rachel Greenfeld and Terence Tao have shown that in a (very large) dimension $d \geq 1$, there exists a finite subset $S \subset \mathbb{Z}^d$ that is (weakly) aperiodic, i.e.:

- S tiles the space: there is a set of translations $T \subset \mathbb{Z}^d$ such that $\mathbb{Z}^d = S \oplus T$;
- S never tiles the space in a strongly periodic way: for any set of translations $T \subset \mathbb{Z}^d$ such that $\mathbb{Z}^d = S \oplus T$, then T is not strongly periodic.

This answered an open question by Lagarias and Wang [188]. The existence of a strongly aperiodic subset remains open.

A question asked at the end of Tao and Greenfeld's article seeks to make a connection with the substitutive structure of aperiodic Jeandel-Rao tilings:

> "*Following [10], it would be of interest to study the tilings in* $Tile(F; \mathbb{Z}^2 \times G_0)$ *that have a substitution structure. Question 10.9. Can any of the tilings by our aperiodic tile be interpreted as a substitution tiling?*
> *The 2-adic nature of the Sudoku solutions suggests a positive answer.*"

Indeed, the tools used by Tao and Greenfeld in their construction use what they themselves call "tetris move" and sheers [147]. In the study of Jeandel-Rao tilings, it was also needed to describe two very similar operations, which we formalized with substitutions and symbolic dynamics.

It is possible that we can reinterpret their approach in the symbolic dynamic systems terminology. The aim here would be to simplify the proof and find a smaller-dimensional aperiodic set. We know that none exists in dimension 2 [88], but there could exist some in dimensions as low as 3.

Question 14.7. *Find an aperiodic subset* $S \subseteq \mathbb{Z}^d$ *in dimension* $d \geq 3$ *smaller than the one in very large dimension one proposed in [147], and if possible in dimension* $d = 3$.

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