Aperiodic order: from combinatorics to geometry via symbolic dynamics, number theory and algorithms

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Habilitation à diriger des recherches
4 juin 2025
http://www.slabbe.org/HDR/

Outline

1. Introduction
2. Aperiodic sets of Wang tiles
3. 2-to-1 CPS : Symbolic dynamics
4. 4-to-2 CPS : Jeandel-Rao aperiodic tilings
5. 4-to-2 CPS : Metallic mean Wang tiles
6. Open questions

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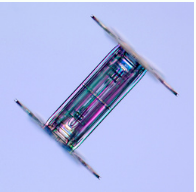
Motivation

Understand global structure emerging from local rules.

Snow crystals may have many shapes *(*`https://www.snowcrystals.com/`*)* :



Networks built by mycorrhizal fungi :

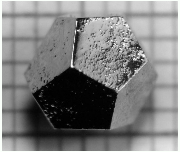
New York Times : *"Underground fungal networks are “living algorithms” that quietly help regulate Earth’s climate."*

Crystallography

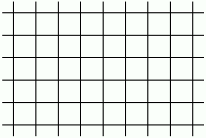
1982 (Shechtman) : observed that aluminium-manganese alloys produced a **quasicrystals structure**.

Pyrite (FeS_2) | A Ho-Mg-Zn quasicrystal

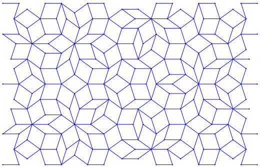
crystals :



atomic structure :



(a square grid)

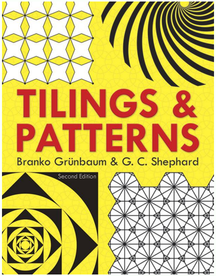


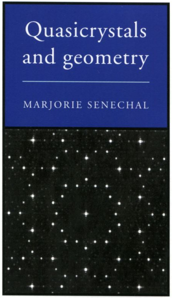
(a Penrose tiling, 1976)

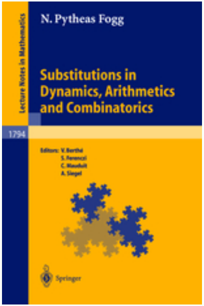
Shechtman received the 2011 **Nobel Prize** in Chemistry :

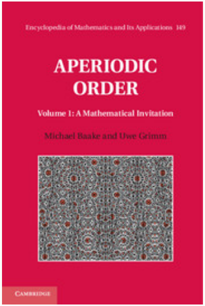
His discovery of quasicrystals revealed a new principle for packing of atoms and molecules [that] *led to a **paradigm shift** within chemistry*.

Books



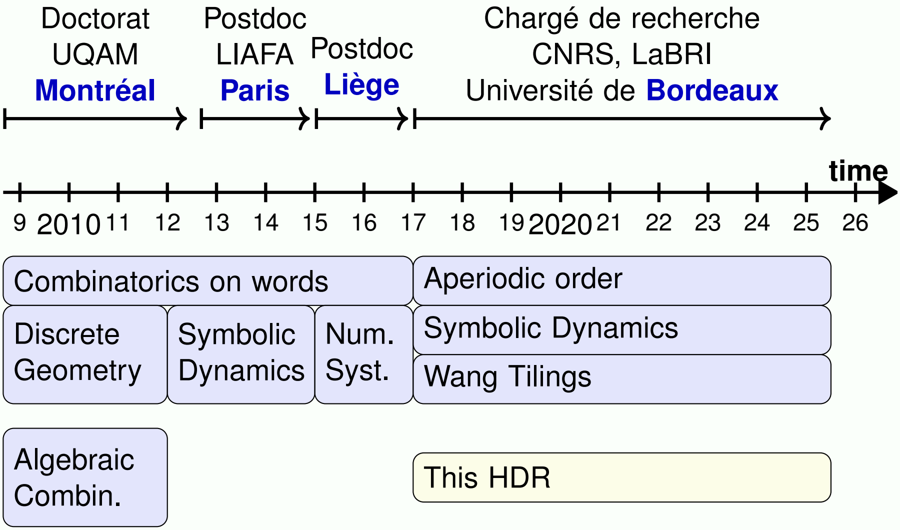






- Tilings and Patterns, by Grünbaum & Shephard, 1987
- Quasicrystals and Geometry, Senechal, 1995
- Pytheas Fogg's book, 2002
- Aperiodic Order, Baake & Grimm, 2013

Scope of the HDR thesis



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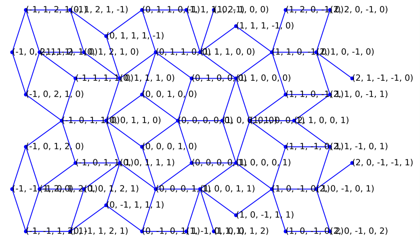
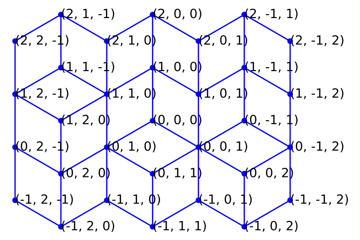
Cut and project schemes

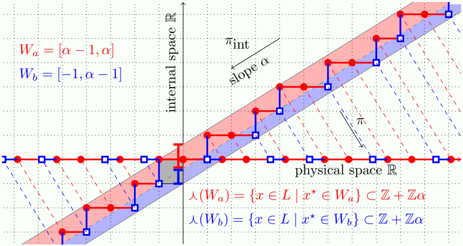
5-to-2



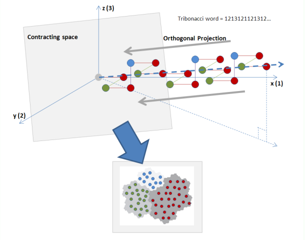
3-to-2



2-to-1



3-to-1



N. G. de Bruijn. "Algebraic theory of Penrose's nonperiodic tilings of the plane. I, II". (1981) ; Meyer (1972), Lagarias (1996), Moody (1997).

Contributions within cut and project schemes

2-to-1 A new characterization of Sturmian sequences
with Barbieri, Starosta (2021)
A q-analog of the Markoff injectivity conjecture
with Lapointe (2022), *with Lapointe, Steiner (2023)*

3-to-1 Almost everywhere balanced seq. of complexity $2n+1$
with Cassaigne, Leroy (2022)

4-to-2 Jeandel-Rao tilings,
Nonexpansive directions *with Mann, McLoud-Mann (2023)*,
Metallic-mean Wang tilings,

$(d+1)$-to-d Indistinguishable asymptotic pairs and multidimensional Sturmian configurations *with Barbieri (2025)*

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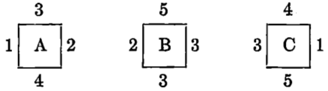
6. Open questions

Wang tiles (1961)



Then we can easily find an infinite solution by the following argument. The following configuration satisfies the constraint on the edges:

A	*B*	*C*
C	*A*	*B*
B	*C*	*A*

Now the colors on the periphery of the above block are seen to be the following:

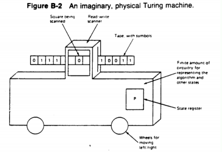
	3	5	4	
1				1
3				3
2				2
	3	5	4	

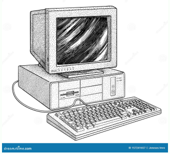
Wang's original question : is it true that a set of Wang tiles tile the plane if and only if there exists such a cyclic rectangle ?

H. Wang. Proving theorems by pattern recognition – II. Bell System Technical Journal*, 40(1) :1–41, January 1961.* doi:10.1002/j.1538-7305.1961.tb03975.x

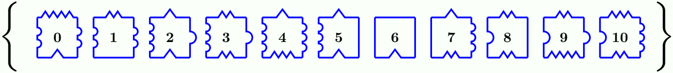
Turing machine reduction to Wang tiles

Berger (1966) : For every Turing machine





there exists a set of Wang tiles



, *e.g.*,

that tiles the plane if and only if the Turing machine does not halt.

- The **domino problem is undecidable** : there exist no algorithm that says whether a finite set of Wang tiles can tile the plane.
- There **exists an aperiodic set** of Wang tiles (*a tile set is* ***aperiodic*** *if it tiles the plane, but none of tilings is periodic*).
- Valid Wang tilings are **computing** something.

Aperiodic Wang tile sets

Aperiodic sets of Wang tiles

Positive entropy

Matching rules satisfy arithmetic Equations

- 14 tiles : **Kari** (1996)
- 13 tiles : Culik (1996)
- and their extensions [ENP07]

Substitutive

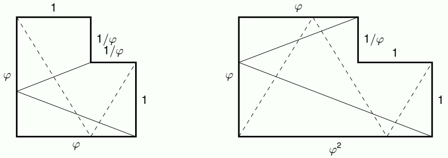
- 104 : Berger (1966)
- 92 : Knuth (1968)
- 56 : Robinson (1971)
- 16 : **Ammann** (1971)
- 11 : **Jeandel-Rao** (2015)

Theorem (Jeandel, Rao, 2015)

All sets of $\leq$ 10 Wang tiles are **periodic** or **don't tile** the plane.

Emmanuel Jeandel and Michaël Rao. An aperiodic set of 11 Wang tiles. Adv. Comb. ***37*** *(2021) Id/No 1.*

Ammann A2 encoded into 16 Wang tiles



Tilings in the Ammann A2 family can be encoded into 16 Wang tiles :

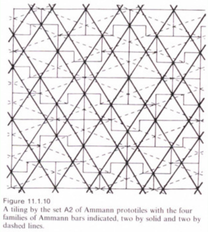


Figure 11.1.10

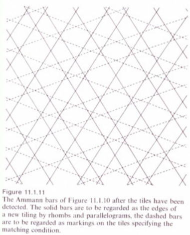


Figure 11.1.11

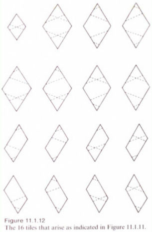


Figure 11.1.12

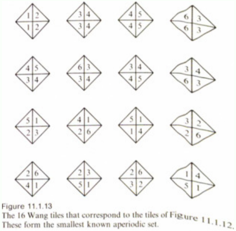
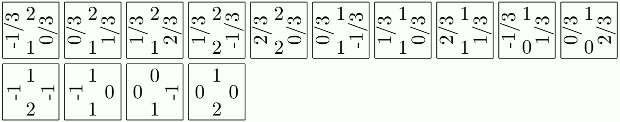
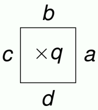


Figure 11.1.13

Branko Grünbaum and G. C. Shephard. Tilings and patterns. *W. H. Freeman and Company, New York, 1987.*

Kari's 14 Wang tiles computing $\times\frac{2}{3}$ and $\times 2$



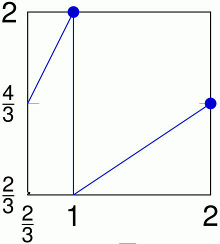


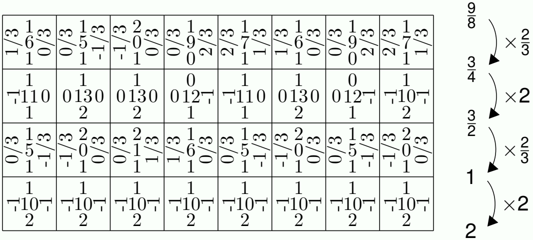
$\iff qb + c = d + a$

with $q \in \{\frac{2}{3}, 3\}$

$$g(x) = \begin{cases} 2x & \text{if } x \le 1, \\ \frac{2}{3}x & \text{if } x > 1. \end{cases}$$

Averages of horizontal labels are orbits of g :





Durand, Gamard, Grandjean (2007) Kari (2016)

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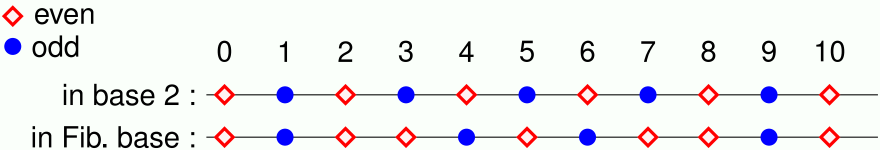
One-dimensional crystallography

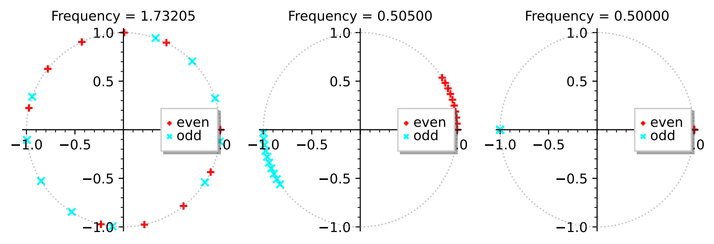


n	$\text{rep}_2(n)$	parity
0=0	0	even
1=1	1	odd
2=2	10	even
3=2+1	11	odd
4=4	100	even
5=4+1	101	odd
6=4+2	110	even
7=4+2+1	111	odd
8=8	1000	even
9=8+1	1001	odd
10=8+2	1010	even
11=8+2+1	1011	odd
12=8+4	1100	even
13=8+4+1	1101	odd

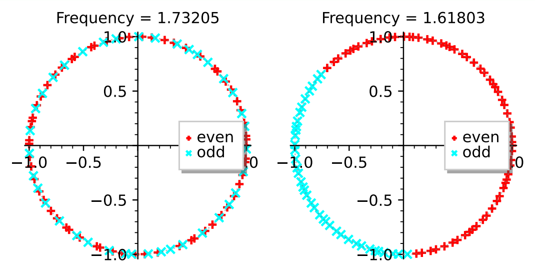
n	$\text{rep}_F(n)$	parity
0=0	0	even
1=1	1	odd
2=2	10	even
3=3	100	even
4=3+1	101	odd
5=5	1000	even
6=5+1	1001	odd
7=5+2	1010	even
8=8	10000	even
9=8+1	10001	odd
10=8+2	10010	even
11=8+3	10100	even
12=8+3+1	10101	odd
13=13	100000	even

Guessing a frequency (rotation angle)

The odd/even in base 2 has frequency $\frac{1}{2}$



The odd/even in Fibonacci base has frequency $\frac{1}{2}(1+\sqrt{5}) \approx 1.618$:



Sturmian sequences

slope $\alpha \in [0,1]$,   intercept $\rho \in \mathbb{R}$,

lower mechanical sequence :

$$s_{\alpha,\rho}(n) = \lfloor \alpha(n+1)+\rho \rfloor - \lfloor \alpha n + \rho \rfloor,$$

upper mechanical sequence :

$$s'_{\alpha,\rho}(n) = \lceil \alpha(n+1)+\rho \rceil - \lceil \alpha n + \rho \rceil.$$

Factor (pattern) complexity :

$x = \cdots 1010010100100101001010010 1\cdots$

n	$\mathcal{L}_n(x)$	$\#\mathcal{L}_n(x)$
0	ε	1
1	0, 1	2
2	00, 01, 10	3
3	001, 010, 100, 101	4
4	0010, 0100, 0101, 1001, 1010	5

Theorem (Morse, Hedlund, 1940 & Coven, Hedlund, 1970)

Let $w \in \{0,1\}^{\mathbb{Z}}$ be a non-ultimately periodic sequence.
There exists $\alpha \in [0,1]\setminus\mathbb{Q}$ and $\rho \in [0,1)$ s.t. $w = s_{\alpha,\rho}$ or $w = s'_{\alpha,\rho}$

if and only if

the sequence w **has factor complexity** $n+1$.

Proof ($\Longrightarrow$) : Easy part. ($\Longleftarrow$) : Harder. Desubstitute + Rauzy induction + Continued fractions + Ostrowki num. syst.

P. Arnoux. Sturmian sequences. In : Substitutions in dynamics, arithmetics and combinatorics. 2002, pp. 143-198. doi:10.1007/3-540-45714-3_6

Desubstituting Sturmian sequences

Theorem A (Pytheas Fogg, 2022)

Let $\alpha \in \mathbb{R}_{>0} \setminus \mathbb{Q}$ Let $w : \mathbb{Z} \to \{\mathtt{0}, \mathtt{1}\}$ be a sequence of factor complexity $n+1$ such that the ratio of frequency of $\mathtt{1}$ vs $\mathtt{0}$ exists and is equal to α.

Then the **substitutive structure** of w is

$$w = \lim_{n\to\infty} s_0^{a_0} s_1^{a_1} \dots s_{2n}^{a_{2n}} s_{2n+1}^{a_{2n+1}} (\mathtt{1} \cdot \mathtt{0})$$

where

$$s_{2n} = \tau_{\mathtt{0}} = \begin{cases} \mathtt{0} \mapsto \mathtt{0} \\ \mathtt{1} \mapsto \mathtt{01} \end{cases} \quad \text{and } s_{2n+1} = \tau_{\mathtt{1}} = \begin{cases} \mathtt{0} \mapsto \mathtt{01} \\ \mathtt{1} \mapsto \mathtt{1} \end{cases}$$

for all $n \geq 0$ and $\alpha = [a_0; a_1, a_2, \dots]$ is the **continued fraction** expansion of α.

Rauzy induction of coding of rotations

Theorem B (Pytheas Fogg, 2022)

Let $\alpha \in \mathbb{R}_{>0} \setminus \mathbb{Q}$ and partition the circle $\mathbb{R}/(1+\alpha)\mathbb{Z}$ into $I_1 = [-1, 0)$ and $I_0 = [0, \alpha)$. Let $w : \mathbb{Z} \to \{0, 1\}$ be such that

$$w_n = \begin{cases} 0 & \text{if } n \in I_0 \pmod{1+\alpha}, \\ 1 & \text{if } n \in I_1 \pmod{1+\alpha}. \end{cases}$$

Then the **substitutive structure** of w is

$$w = \lim_{n\to\infty} s_0^{a_0} s_1^{a_1} \dots s_{2n}^{a_{2n}} s_{2n+1}^{a_{2n+1}} (1 \cdot 0)$$

where

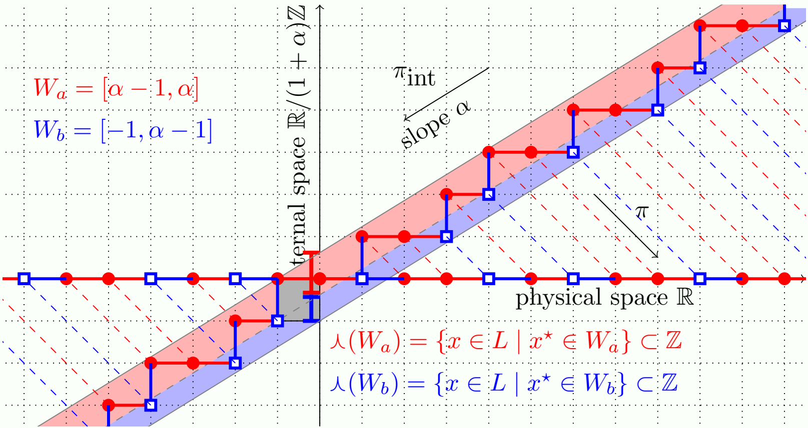
$$s_{2n} = \tau_0 = \begin{cases} 0 \mapsto 0 \\ 1 \mapsto 01 \end{cases} \quad \text{and } s_{2n+1} = \tau_1 = \begin{cases} 0 \mapsto 01 \\ 1 \mapsto 1 \end{cases}$$

for all $n \geq 0$ and $\alpha = [a_0; a_1, a_2, \dots]$ is the **continued fraction** expansion of α.

Thus, w is **linearly repetitive** $\iff$ α is **badly approximable**.

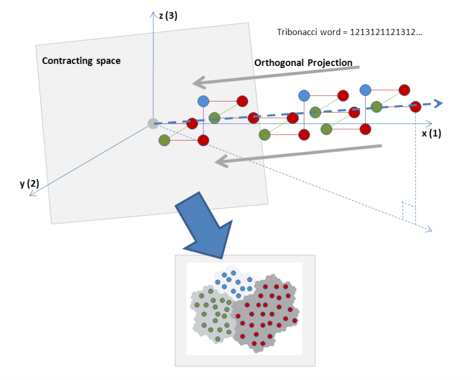
Haynes, Koivusalo, Walton, A char. of lin. repetitive cut and project sets. (2018).

Degenerate 2-to-1 cut and project scheme (Fibonacci word)



Tribonacci word and Rauzy fractal : 3-to-1 CPS

3-to-1



G. Rauzy. *"Nombres algébriques et substitutions". Bull. Soc. Math. France (1982)*

Pisot Conjecture : For every Pisot substitution $s : \mathcal{A} \to \mathcal{A}$, the substitutive subshift $X_s \subset \mathcal{A}^{\mathbb{Z}}$ is isomorphic to a translation on a torus.

Berthé, Steiner, Thuswaldner. (2023) Fogg, Noûs. (2024)

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6 Open questions

Jeandel-Rao's set of 11 Wang tiles

Theorem (Jeandel, Rao, 2015)

The following set of 11 Wang tiles is **aperiodic** :

A geometrical encoding of the Jeandel-Rao tiles and its Wang shift :

$\mathcal{T}_0 = \Bigg\{$ 

$\Bigg\}$

$$\Omega_0 := \Omega_{\mathcal{T}_0} := \Big\{ w : \mathbb{Z}^2 \to \{0, 1, \ldots, 10\} \Big| w \text{ is a valid tiling with } \mathcal{T}_0 \Big\}$$

on which the shift $\mathbb{Z}^2 \overset{\sigma}{\curvearrowright} \Omega_0$ acts naturally as

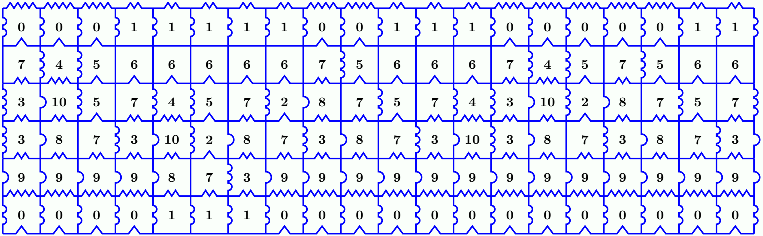
$$\begin{array}{rccl} \sigma : & \mathbb{Z}^2 \times \Omega_0 & \to & \Omega_0 \\ & (\mathbf{k}, w) & \mapsto & \sigma^{\mathbf{k}}(w) := (\boldsymbol{n} \mapsto w_{\boldsymbol{n}+\mathbf{k}}). \end{array}$$

Question : what is $\mathbb{Z}^2 \overset{\sigma}{\curvearrowright} \Omega_0$ computing ?

Downloading a 100×100 patch

```
sage: import urllib
sage: url = "https://members.loria.fr/EJeandel/research/100.txt"
sage: content = urllib.request.urlopen(url).read()
sage: J = [row.decode() for row in content.splitlines()]
sage: len(J), len(J[0])
(100, 100)
sage: J[0][:70]
0001111100111000001110001100000111000111110001100000111000111110011100
sage: J[1][:70]
7456666756667457566674566745756667456666674566745756667456666667566674
sage: J[2][:70]
3a574572875743a2875743a5743a2875743a5745743a5743a2875743a574572875743a
```

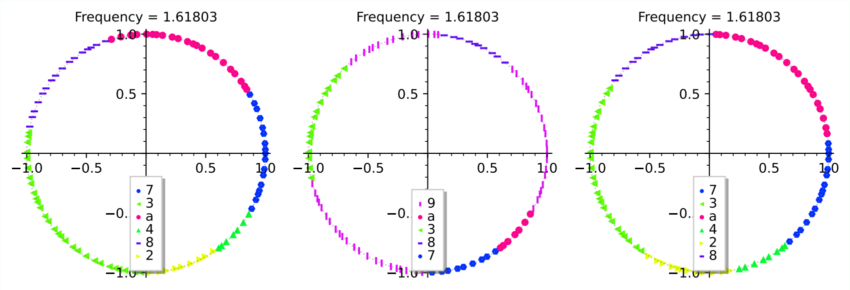
The first 6 rows (limited to 20 columns) are :



Wrapping the rows on a circle

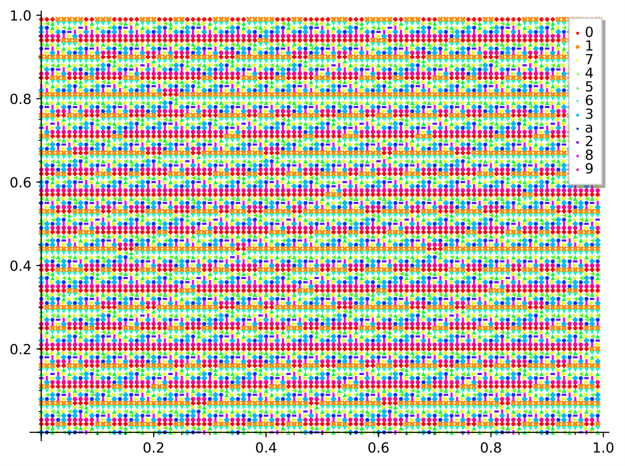
```
sage: J[35][:70]
73a43a3873a2873a43a3873a2873873a2873a43a3873a2873a43a2873a2873873a2873
sage: J[36][:70]
999a399999873999a39999987399999873999a399999873999a3873998739999987399
sage: J[58][:70]
73a43a2873a43a3873a2873a43a2873a2873873a2873a43a2873a2873a43a2873a43a3
```

Wrapping these rows on a circle using the golden ratio frequency gives :



Experiment (step 1)

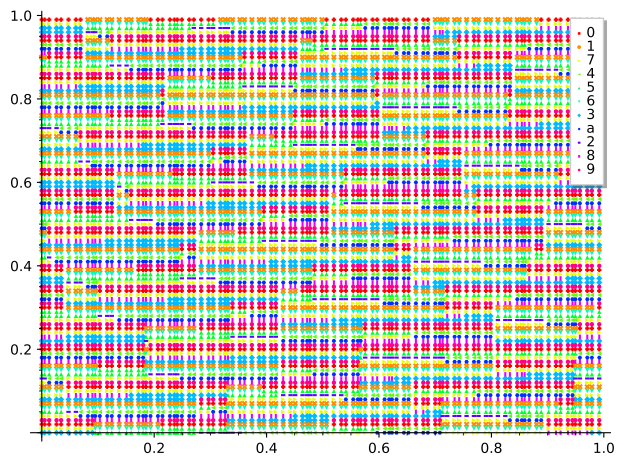
Wrapping on the 2-torus with frequency $\begin{pmatrix} 100 & 0 \\ 0 & 100 \end{pmatrix}^{-1}$:



Using frequency $\frac{1}{100}$ horizontally and vertically is a trick to make the points represent the tiling itself.

Experiment (step 2)

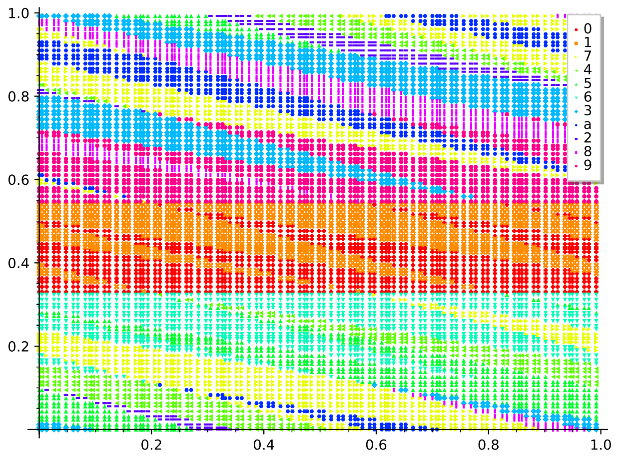
Wrapping on the 2-torus with frequency $\begin{pmatrix} \varphi & 0 \\ 0 & 100 \end{pmatrix}^{-1}$:



This makes each row in the patch to wrap around a circle (shown horizontally on the image below) with golden mean frequency.

Experiment (step 3)

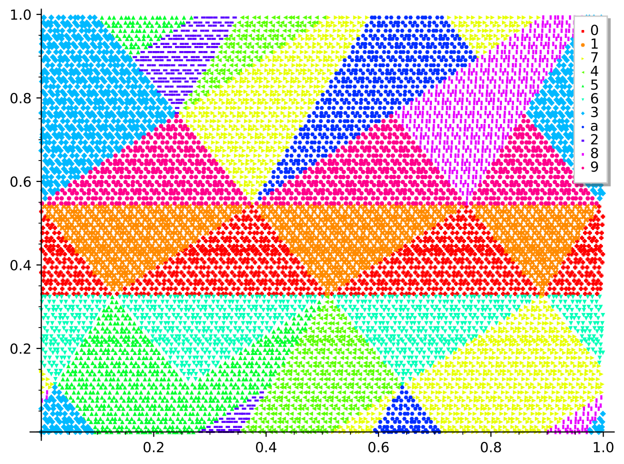
Wrapping on the 2-torus with frequency $\begin{pmatrix} \varphi & 0 \\ 0 & \varphi+3 \end{pmatrix}^{-1}$.



This makes sense because the vertical distance (or return time) between rows involving tiles labeled #0 and #1 is 4 or 5 with an average of $\varphi + 3$ as noticed already by Jeandel and Rao.

Experiment (step 4)

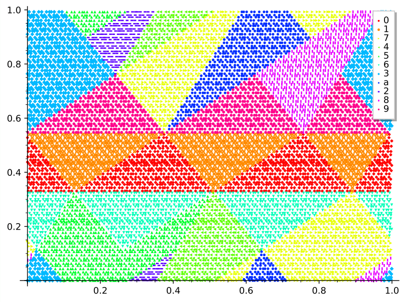
Wrapping on the 2-torus with frequency $\begin{pmatrix} \varphi & 1 \\ 0 & \varphi+3 \end{pmatrix}^{-1}$.



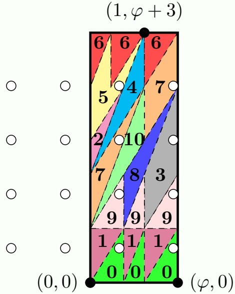
A shear is happening in Jeandel-Rao tilings.
This is one of the reasons that makes the description of Jeandel-Rao tilings more difficult, but certainly very interesting !

Rescaling to get $\mathbb{Z}^2$-action R_0 and partition $\mathcal{P}_0$

Step 4 of the experiment	Rescaling
$\mathbb{Z}^2 \curvearrowright \mathbb{R}^2/\mathbb{Z}^2$	$\mathbb{Z}^2 \overset{R_0}{\curvearrowright} \mathbb{R}^2/\begin{pmatrix}\varphi & 1\\ 0 & \varphi+3\end{pmatrix}\mathbb{Z}^2$
$(\mathbf{k},\mathbf{x}) \mapsto \mathbf{x} + \begin{pmatrix}\varphi & 1\\ 0 & \varphi+3\end{pmatrix}^{-1}\mathbf{k}.$	$R_0 : (\mathbf{k},\mathbf{x}) \mapsto \mathbf{x} + \mathbf{k}.$



$\mathcal{P}_0 =$

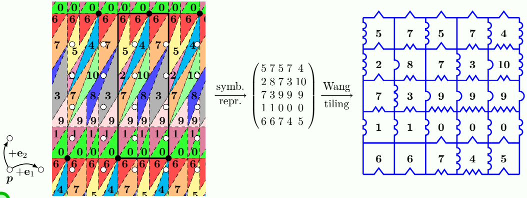


$$\begin{pmatrix}5 & 7\\ 7 & 3\\ 9 & 9\\ 0 & 0\end{pmatrix} \in \mathcal{L}_{\mathcal{P}_0,R_0} = \left\{ w : S \to \mathcal{A} \,\middle|\, S \subset \mathbb{Z}^2 \text{ and } w \text{ is } \textbf{allowed} \right\}$$

Jeandel–Rao aperiodic set of 11 Wang tiles

Coding $\mathbb{Z}^2 \overset{R_0}{\curvearrowright} \mathbb{R}^2/\Gamma_0$ by partition $\mathcal{P}_0$ defines a **symb. dyn. system** :

$$\mathcal{X}_{\mathcal{P}_0,R_0} = \left\{ w : \mathbb{Z}^2 \to \{0,1,\ldots,10\} \,\middle|\, \mathcal{L}(w) \subset \mathcal{L}_{\mathcal{P}_0,R_0} \right\}.$$



Theorem

- $\mathcal{X}_{\mathcal{P}_0,R_0}$ is a **minimal, aperiodic and uniquely ergodic** subshift of the Jeandel-Rao Wang shift, i.e., $\mathcal{X}_{\mathcal{P}_0,R_0} \subset \Omega_0$.
- Occurences of patterns in $\mathcal{X}_{\mathcal{P}_0,R_0}$ is a **4-to-2 C&P set**.

A Wang shift $\Omega_{\mathcal{T}}$ is **minimal** if every orbit by the shift is dense in Ω.

Markov partitions for toral $\mathbb{Z}^2$-rotations featuring Jeandel-Rao Wang shift and model sets. Ann. H. Lebesgue *4 (2021) 283–324.* doi:10.5802/ahl.73

Substitutive structure of Ω_0 and $\mathcal{X}_{\mathcal{P}_0,R_0}$

Using **algorithms** `FindMarkers` and `FindSubstitution` :

$$
\begin{array}{ccccccccccccccccc}
\Omega_0 & \xleftarrow{\omega_0} & \Omega_1 & \xleftarrow{\omega_1} & \Omega_2 & \xleftarrow{\omega_2} & \Omega_3 & \xleftarrow{\omega_3} & \Omega_4 & & & & & & & & \omega_{\mathcal{U}} \circlearrowleft \\
\cup & & \cup & & \cup & & \cup & & \cup & & & & & & & & \\
X_0 & \xleftarrow{\omega_0} & X_1 & \xleftarrow{\omega_1} & X_2 & \xleftarrow{\omega_2} & X_3 & \xleftarrow{\omega_3} & X_4 & \xleftarrow{\pi} & \Omega_5 & \xleftarrow{\eta} & \Omega_6 & \xleftarrow{\omega_6} & \Omega_7 & \xleftarrow{\omega_7\omega_8\omega_9\omega_{10}\omega_{11}} & \Omega_{12} \xleftarrow{\rho} \Omega_{\mathcal{U}}
\end{array}
$$

Using **algorithms** `induced_partition` and `induced_transformation` :

$$
\mathcal{X}_{\mathcal{P}_0,R_0} \xleftarrow{\beta_0} \mathcal{X}_{\mathcal{P}_1,R_1} \xleftarrow{\beta_1} \mathcal{X}_{\mathcal{P}_2,R_2} \xleftarrow{\beta_2} \mathcal{X}_{\mathcal{P}_3,R_3} \xleftarrow{\beta_3} \mathcal{X}_{\mathcal{P}_4,R_4} \xleftarrow{\beta_4} \mathcal{X}_{\mathcal{P}_5,R_5} \xleftarrow{\beta_5} \mathcal{X}_{\mathcal{P}_6,R_6} \xleftarrow{\beta_6} \mathcal{X}_{\mathcal{P}_7,R_7} \xleftarrow{\beta_7} \mathcal{X}_{\mathcal{P}_8,R_8} \xleftarrow{\rho} \mathcal{X}_{\mathcal{P}_{\mathcal{U}},R_{\mathcal{U}}} \circlearrowleft \omega_{\mathcal{U}}
$$

Theorem

$X_0 \subsetneq \Omega_0$ and $\mathcal{X}_{\mathcal{P}_0,R_0}$ have **the same** substitutive structure :

$$\omega_0\omega_1\omega_2\omega_3 = \beta_0, \quad \pi\eta\omega_6 = \beta_1\beta_2, \quad \omega_7\omega_8\omega_9\omega_{10}\omega_{11} = \beta_3\beta_4\beta_5\beta_6\beta_7$$

thus are **equal**.

Substitutive structure of Jeandel-Rao aperiodic tilings.
Discrete Comput. Geom., *65 (2021) 800–855.* doi:10.1007/s00454-019-00153-3

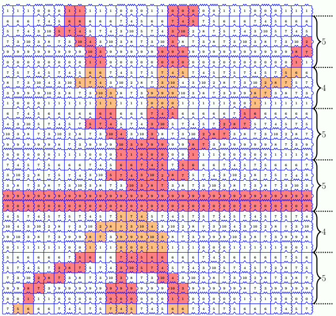
Rauzy induction of polygon partitions and toral $\mathbb{Z}^2$-rotations
Journal of Modern Dynamics 17 (2021) 481–528. doi:10.3934/jmd.2021017

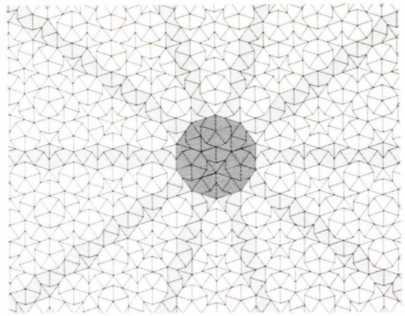
4 slopes of Conway worms in Jeandel-Rao WS

Theorem (L., Mann, McLoud-Mann, 2023)

The minimal subshift X_0 of the Jeandel-Rao Wang shift contains exactly **4 nonexpansive directions** whose slopes are

$$\left\{0, \quad \varphi + 3, \quad -3\varphi + 2, \quad -\varphi + \frac{5}{2}\right\}.$$





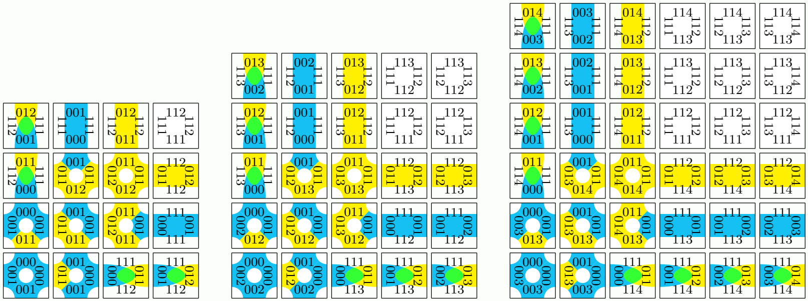
(c)

It reminds of the cartwheel tiling in the context of Penrose tilings.

Outline

1. Introduction
2. Aperiodic sets of Wang tiles
3. 2-to-1 CPS : Symbolic dynamics
4. 4-to-2 CPS : Jeandel-Rao aperiodic tilings
5. 4-to-2 CPS : Metallic mean Wang tiles
6. Open questions

A family $\{\mathcal{T}_n\}_{n\geq 1}$ of metallic mean Wang tiles



$\mathcal{T}_1 \equiv$ **Ammann** $\qquad \mathcal{T}_2 \qquad \mathcal{T}_3$

- For every $n \geq 1$, $\Omega_{\mathcal{T}_n}$ is **self-similar**, **aperiodic** and **minimal**.
- Inflation factor is the ***n*-th metallic mean** (pos. root of $x^2 - nx - 1$),
- structure associated with **substitution** $a \mapsto ab^n, b \mapsto ab^{n-1}$,

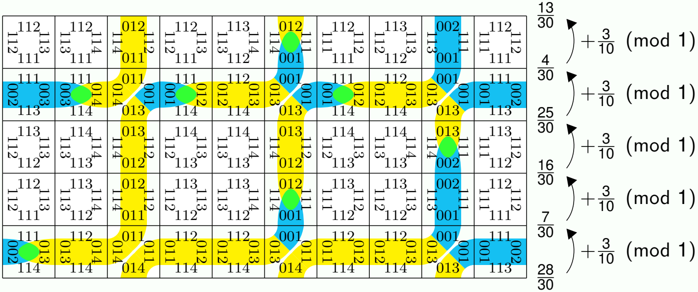
Metallic mean Wang tiles I : self-similarity, aperiodicity and minimality.
arXiv:2312.03652 , *to appear in Forum of Mathematics, Sigma*

Metallic mean Wang tiles II : the dynamics of an aperiodic computer chip.
arXiv:2403.03197

An explicit factor map (example)

A 10×5 valid rectangular tiling with the set $\mathcal{T}_n$ with $n = 3$.

The numbers indicated in the right margin are the average of the inner products $\langle \frac{1}{n} d, v \rangle$ over the vectors v appearing as top (or bottom) labels of a horizontal row of tiles and where $d = (0, -1, 1)$.



We observe that these numbers increase by $\frac{3}{10}$ (mod 1) from row to row. The number $\frac{3}{10}$ is equal to the frequency of columns containing junction tiles (a junction tile is a tile whose labels all start with 0).

An explicit factor map

Theorem

Let $d = (0, -1, 1)$, $n \geq 1$ be an integer and Ω_n be the n^{th} metallic mean Wang shift. The map

$$\begin{array}{rccl} \Phi_n : & \Omega_n & \to & \mathbb{T}^2 \\ & w & \mapsto & \displaystyle\lim_{k\to\infty} \frac{1}{2k+1} \sum_{i=-k}^{k} \begin{pmatrix} \langle \frac{1}{n} d, \textsc{Right}(w_{0,i}) \rangle \\ \langle \frac{1}{n} d, \textsc{Top}(w_{i,0}) \rangle \end{pmatrix} \end{array}$$

is a factor map commuting the shift $\mathbb{Z}^2 \overset{\sigma}{\curvearrowright} \Omega_n$ with $\mathbb{Z}^2 \overset{R_n}{\curvearrowright} \mathbb{T}^2$ by the equation $\Phi_n \circ \sigma^k = R_n^k \circ \Phi_n$ for every $k \in \mathbb{Z}^2$ where

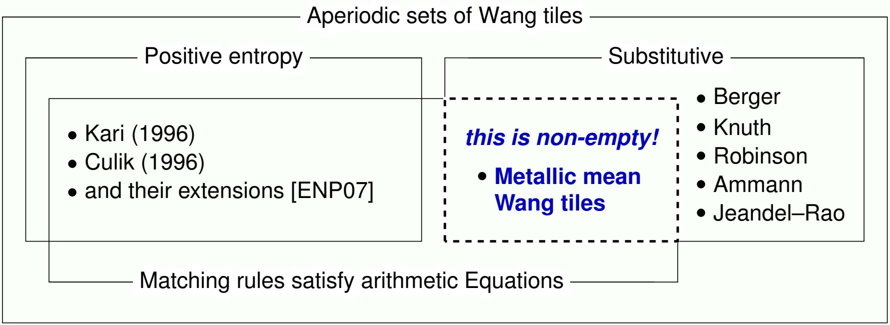
$$\begin{array}{rccl} R_n & : \mathbb{Z}^2 \times \mathbb{T}^2 & \to & \mathbb{T}^2 \\ & (k, x) & \mapsto & R_n^k(x) := x + \beta k \end{array}$$

and $\beta = \frac{n+\sqrt{n^2+4}}{2}$ is the n^{th} metallic mean, that is, the positive root of the polynomial $x^2 - nx - 1$.

Proof uses **Weyl equidistribution thm** (Kuipers, Niederreiter, 1974).

Remark : Φ_n satisfies $\Phi_n(c_{(x,y)}) = (x, y)$.

Venn Diagram again



Question

Which other aperiodic sets of Wang tiles have matching rules satisfying arithmetic equations ?

Outline

1 Introduction

2 Aperiodic sets of Wang tiles

3 2-to-1 CPS : Symbolic dynamics

4 4-to-2 CPS : Jeandel-Rao aperiodic tilings

5 4-to-2 CPS : Metallic mean Wang tiles

6 **Open questions**

Open questions on Jeandel-Rao tilings

Conjecture

$\Omega_0 \setminus X_0$ is of **measure 0** for any shift-invariant probability measure on Ω_0 (it consists of sliding half-plane along a fault line).

Open question

Jeandel–Rao must not be alone : **characterize** its family.

Sub-questions :

- Can we interpret the labels of the Jeandel-Rao Wang tiles by real numbers doing arithmetic computations ?
- Can we prove aperiodicity of Jeandel-Rao tiles in 10 lines using a Kari-like arithmetic argument ?
- What other algebraic numbers can we get ?
- Describe a **Tribonacci** set of Wang tiles and its 4-dimensional Rauzy fractal

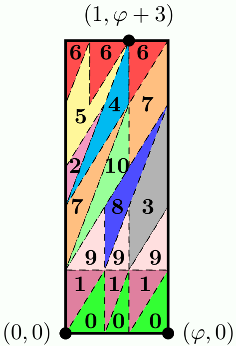
(a Tribonacci set of Wang tiles must exist following Mozes 1989)

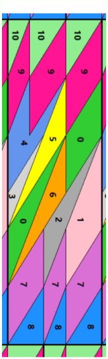
Open questions on Jeandel-Rao tilings

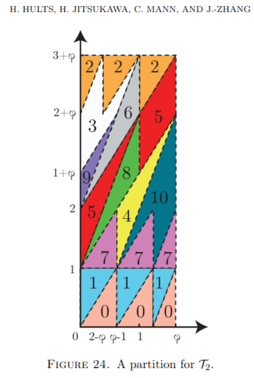
- Describe all of the 33 candidates of 11 Wang tiles listed by JR

  progresses was made by Thompson (2022) and Mann (2024)

partition for JR | Thompson (2022) | Mann et al. (2024)







R. D. Thompson. “The Jeandel-Rao Aperiodic Wang Tilings of the Plane”. MSc in Mathematics. The Open University, Milton Keynes, UK, May 2022

Hults, Jitsukawa, Mann, Zhang, Experimental Results on Potential Markov Partitions for Wang Shifts `arXiv:2302.13516`

Open questions on Metallic-mean Wang tiles

- Find **geometric shapes** with Ammann bars on them associated with metallic-mean Wang tiles for $n > 2$.

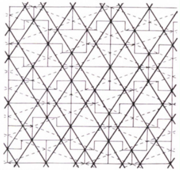


Figure 11.1.10
A tiling by the set A2 of Ammann prototiles with the four families of Ammann bars indicated, two by solid and two by dashed lines.

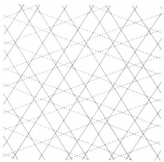


Figure 11.1.11
The Ammann bars of Figure 11.1.10 after the tiles have been detected. The solid bars are to be regarded as the edges of a new tiling by rhombs and parallelograms, the dashed bars are to be regarded as markings on the tiles specifying the matching condition.

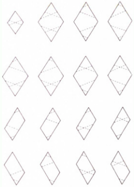


Figure 11.1.12
The 16 tiles that arise as indicated in Figure 11.1.11.

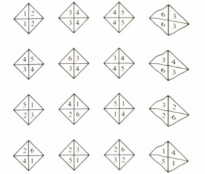


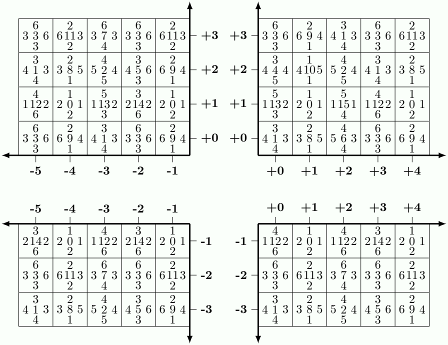
Figure 11.1.13
The 16 Wang tiles that correspond to the tiles of Figure 11.1.12. These form the smallest known aperiodic set.

Figure 11.1.10 Figure 11.1.11 Figure 11.1.12 Figure 11.1.13

Branko Grünbaum and G. C. Shephard. 1987

Jana Lepšová's Ph. D. thesis

Let $\mathrm{rep}_{\mathcal{F}c}$ be the (padded) **Fibonacci comp. num. syst.** for $\mathbb{Z}^2$.
There exists a **deterministic finite automaton with output** (DFAO) $\mathcal{A}$ such that $\binom{n_1}{n_2} \mapsto \mathcal{A}\left(\mathrm{rep}_{\mathcal{F}c}\binom{n_1}{n_2}\right)$ is a **valid tiling** with Ammann tiles



L., Lepšová. *A Fibonacci analogue of the two's complement numeration system. RAIRO - Theoretical Informatics and Applications 57 (2023) 12.*

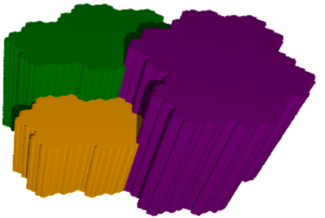
L., Lepšová. *Dumont-Thomas complement numeration systems for $\mathbb{Z}$. Integers 24 (2024) Paper No. A112, 27 pages.* doi:10.5281/zenodo.14340125

vs Markov Partitions for automorphisms

Bowen (1978) : The boundaries of the sets in a Markov partition for linear Anosov diffeomorphisms of $\mathbb{T}^3$ **cannot be smooth**.

For example, a Markov partition for the automorphism $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ of $\mathbb{T}^3$ is shown on the right.

Image credit : Timo Jolivet's talk, Japan, 2012.



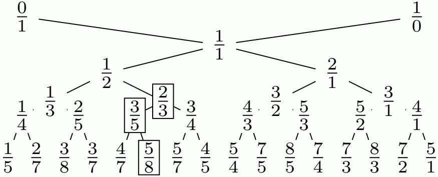
Cawley (1991) : The only hyperbolic toral automorphisms f for which there exist Markov partitions with piecewise **smooth boundary** are those for which a power f^k is linearly covered by a **direct product of automorphisms of the 2-torus**.

Question

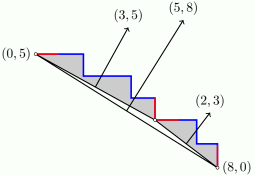
Relate the Markov partition of the hyperbolic automorphism extracted from the self-similarity of Ammann (or metallic) Wang tiles to the Markov partition describing the Wang shifts.

Factorization of Christoffel graphs

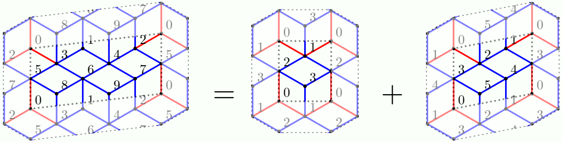
The Stern-Brocot tree



Standard factorization of Christoffel words



Likewise, can we factorize every (3-to-2) Christoffel graph ?

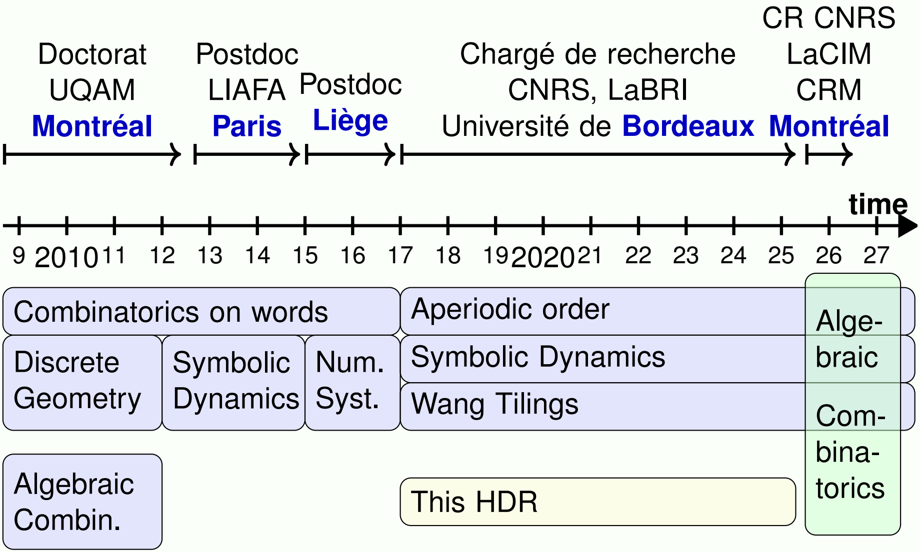


$\mathcal{H}_{(2,3,5)} = \mathcal{H}_{(1,1,2)} + \mathcal{H}_{(1,2,3)}$

Labbé, Reutenauer. A d-dimensional extension of Christoffel words. (2015).

Roussillon and Labbé. Decomposition of Rational Discrete Planes (2024).

What's next


Doctorat
UQAM
Montréal
Postdoc
LIAFA
Paris
Postdoc
Liège
Chargé de recherche
CNRS, LaBRI
Université de Bordeaux
CR CNRS
LaCIM
CRM
Montréal
time
9 2010 11 12 13 14 15 16 17 18 19 2020 21 22 23 24 25 26 27
Combinatorics on words
Discrete Geometry
Symbolic Dynamics
Num. Syst.
Algebraic Combin.
Aperiodic order
Symbolic Dynamics
Wang Tilings
This HDR
Alge-braic Com-bina-torics

The end