

An aperiodic set of Wang tiles for every quadratic irrational

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- Kari 1994: ~~14 Wang tiles~~, an aperiodic set of 14 Wang tiles

$\rightarrow$ existence of tilings
 $\rightarrow$ ~~no tiling is periodic~~ (no tiling is invariant under a non-trivial translation)

10 lines proof of non-periodicity:

using $k, l \in \mathbb{N}, 2^k 3^l = 1 \Rightarrow k = l = 0$

- Ammann 16 tiles 1970's

Jean-Louis Lagarias, 11 tiles, 2015

} proof of non-periodicity is tedious:
 based on subst. structure + self-similarity
 + recognizability + computation of allowed/forbidden patterns

Question Can we find short arithmetic proof of non-periodicity?

Ammann 16 tiles:

I = set of vertical labels
 J = set of horizontal labels

$T \subseteq I \times J \times I \times J$
 set of Wang tiles

$(a, b, c, d) \in T \Rightarrow \begin{array}{|c|} \hline a & b \\ \hline c & d \\ \hline \end{array}$

$T_I = \left\{ \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array} \right\}$

$T_H = \left\{ \begin{array}{|c|c|} \hline 6 & 4 \\ \hline 3 & 5 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 2 & 3 \\ \hline 6 & 2 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 4 & 5 \\ \hline 1 & 6 \\ \hline \end{array} \right\}$

$T_{II} = \left\{ \begin{array}{|c|c|} \hline 4 & 2 \\ \hline 1 & 6 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 5 & 2 \\ \hline 3 & 6 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 2 & 6 \\ \hline 3 & 4 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 5 & 4 \\ \hline 1 & 3 \\ \hline \end{array} \right\}$

$T_{III} = \left\{ \begin{array}{|c|c|c|} \hline 4 & 3 & 4 \\ \hline 3 & 4 & 3 \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 5 & 4 & 5 \\ \hline 4 & 5 & 4 \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 3 & 6 & 3 \\ \hline 6 & 3 & 6 \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 4 & 3 & 4 \\ \hline 3 & 4 & 3 \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 5 & 4 & 5 \\ \hline 4 & 5 & 4 \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 3 & 6 & 3 \\ \hline 6 & 3 & 6 \\ \hline \end{array} \right\}$

Sort $V: I \rightarrow \mathbb{R}^n$ et $H: J \rightarrow \mathbb{R}^n$ et

$f: T \rightarrow \mathbb{R}^n: \begin{array}{|c|c|} \hline a & b \\ \hline c & d \\ \hline \end{array} \mapsto V(a) - V(c) + H(b) - H(d)$

Fact 1 Sort $w: \mathbb{Z}^2 \rightarrow T$ a valid tiling $w(\vec{n} + (k, 0)) = w(\vec{n}) = w(\vec{n} + (0, k))$
 and suppose that w is doubly periodic $\exists K \in \mathbb{N}_{>0}$ st $\mathbb{Z}$
 On the domain $S = \{0, \pm 1, \dots, K-1\}^2$, we have

$\frac{1}{|S|} \sum_{\vec{n} \in S} f(w(\vec{n})) = 0$

Def T has horizontal stripes if $I = I_0 \cup I_1$ and $T \subseteq (I_0 \times J \times I_0 \times J) \cup (I_1 \times J \times I_1 \times J)$

T has vertical stripes if $J = J_0 \cup J_1$ and $T \subseteq (I \times J_0 \times I \times J_0) \cup (I \times J_1 \times I \times J_1)$

EX Ammann $I_0 = J_0 = \{1, 2\}$ (no stripe)
 $I_1 = J_1 = \{3, 4, 5, 6\}$ (stripe)

Def

A set of Wang tiles $T \subset I \times S \times I \times S$ with horizontal and vertical stripes $T = T_0 \cup T_1 \cup T_2 \cup T_3$ determines the quadrilateral $ABDC \in \mathbb{R}^2$ if

$$f(t) = \begin{cases} A & \text{if } t \in T_0 \\ B & \text{if } t \in T_1 \\ C & \text{if } t \in T_2 \\ D & \text{if } t \in T_3 \end{cases} \quad \forall t \in T.$$

for some

~~depends on the~~ choice of maps $V: I \rightarrow \mathbb{R}^2, H: J \rightarrow \mathbb{R}^2$.

Fact 2

If α (β , resp) is the density of vertical (horiz, resp) stripes in the doubly-periodic configuration $W: \mathbb{Z}^2 \rightarrow T$, then

$$0 = \frac{1}{|S|} \sum_{n \in S} f(w_n) = \alpha \beta f(\text{H}) + \alpha(1-\beta) f(\text{I}) + (1-\alpha)\beta f(\text{J}) + (1-\alpha)(1-\beta) f(\text{K})$$

$$(*) \quad 0 = (1-\alpha)[(1-\beta)A + \beta B] + \alpha[(1-\beta)C + \beta D]$$

If $ABDC \in \mathbb{Z}^2$ ~~has integer coordinates~~ are in generic position, then solutions (α, β) are quadratic irrationals.

Tangent lines to a parabola circumscribed to the quadrilateral $ABDC$.
See geogebra.org/m/a46mxgft

EX

Ammann

$$V = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

$$H = \begin{pmatrix} 1 & 0 & 1 & 1 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

yield the quadrilateral

$$ABCD = \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix}$$

in $\mathbb{R}^2$

$$f\left(\begin{smallmatrix} 2 \\ 6 \end{smallmatrix} \begin{smallmatrix} \blacksquare \\ 4 \end{smallmatrix}\right) = v(4) - v(6) + h(2) - h(1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = B$$

Thm A

$$\text{If } \det \begin{pmatrix} A_1 & D_1 \\ A_2 & D_2 \end{pmatrix} = 0, \det \begin{pmatrix} B_1 & D_1 \\ B_2 & D_2 \end{pmatrix} = -\det \begin{pmatrix} C_1 & D_1 \\ C_2 & D_2 \end{pmatrix} \neq 0$$

and $\exists i \in \{1, 2\}$ s.t. $(B_i + C_i)^2 - 4A_i D_i > 0$ is not a perfect square, then T admits no periodic tiling.

Corollary

The Ammann set admits no periodic tiling

Proof

$$\det(AD) = 0, \det(BD) = 1, \det(CD) = -1$$

$$\Delta = (B_1 + C_1)^2 - 4A_1 D_1 = 1^2 + 4 = 5 > 0 \text{ not perfect square.}$$

Works also for Penrose encoded as 24 Wang tiles, metallic mean Wang tiles, and more.

How to find the quadrilateral and maps H, V?

Short answer: linear algebra

Let

$$S_T = \begin{array}{c} \begin{array}{c} \square \quad \square \quad \square \quad \square \\ \begin{array}{c} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{array} \end{array} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$M_T = \begin{array}{c} \begin{array}{c} I_0 \quad I_1 \quad I_2 \quad I_3 \\ \begin{array}{c} 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \end{array} \end{array} \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 1 & 0 & 0 & -1 \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix} \begin{array}{c} \begin{array}{c} I_0 \quad I_1 \\ \begin{array}{c} 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \end{array} \end{array} \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix} \begin{array}{c} t_1 = \frac{1}{2} \\ t_2 \\ t_3 \\ t_4 = \frac{2}{2} \end{array} \end{array}$$

We want ~~solutions to equation~~ ~~(V|H)~~ ~~and (ABCD)~~
to equation $M_T \cdot (V|H)^T = S_T \cdot (ABCD)^T$

$$\Leftrightarrow (-S_T | M_T) \cdot (ABCD | V | H)^T = 0$$

Solutions where (ABCD) has full rank are in the nullspace of $\begin{pmatrix} -S_T | M_T \\ 0 | K \end{pmatrix}$ where K rows of K generate the nullspace of M_T .

In many examples, $\dim \ker \begin{pmatrix} -S_T | M_T \\ 0 | K \end{pmatrix} = 2$.

Theorem B $\forall \alpha, \beta \in [0, 1]$ in the same quadratic number field $\mathbb{Q}(\sqrt{d})$ (both rational or both irrational), $\exists T = T_{\square} \cup T_{\square} \cup T_{\square} \cup T_{\square}$

a set of Wang tiles with stripes and $\exists A, B, C, D \in \mathbb{Z}^2$ s.t.

(i) (α, β) solution to (*)

(ii) T determines the quadrilateral $ABDC$, whose opposite sides are not parallel,

(iii) T admits a tiling $\omega: \mathbb{Z}^2 \rightarrow T$ where vertical (resp. horizontal) stripes have density α (resp. β)

(iv) in every tiling $\omega \in \Omega_T$, the densities of H/V stripes exist and are solutions of (*)

In particular, if α and β are irrational, then T is aperiodic.