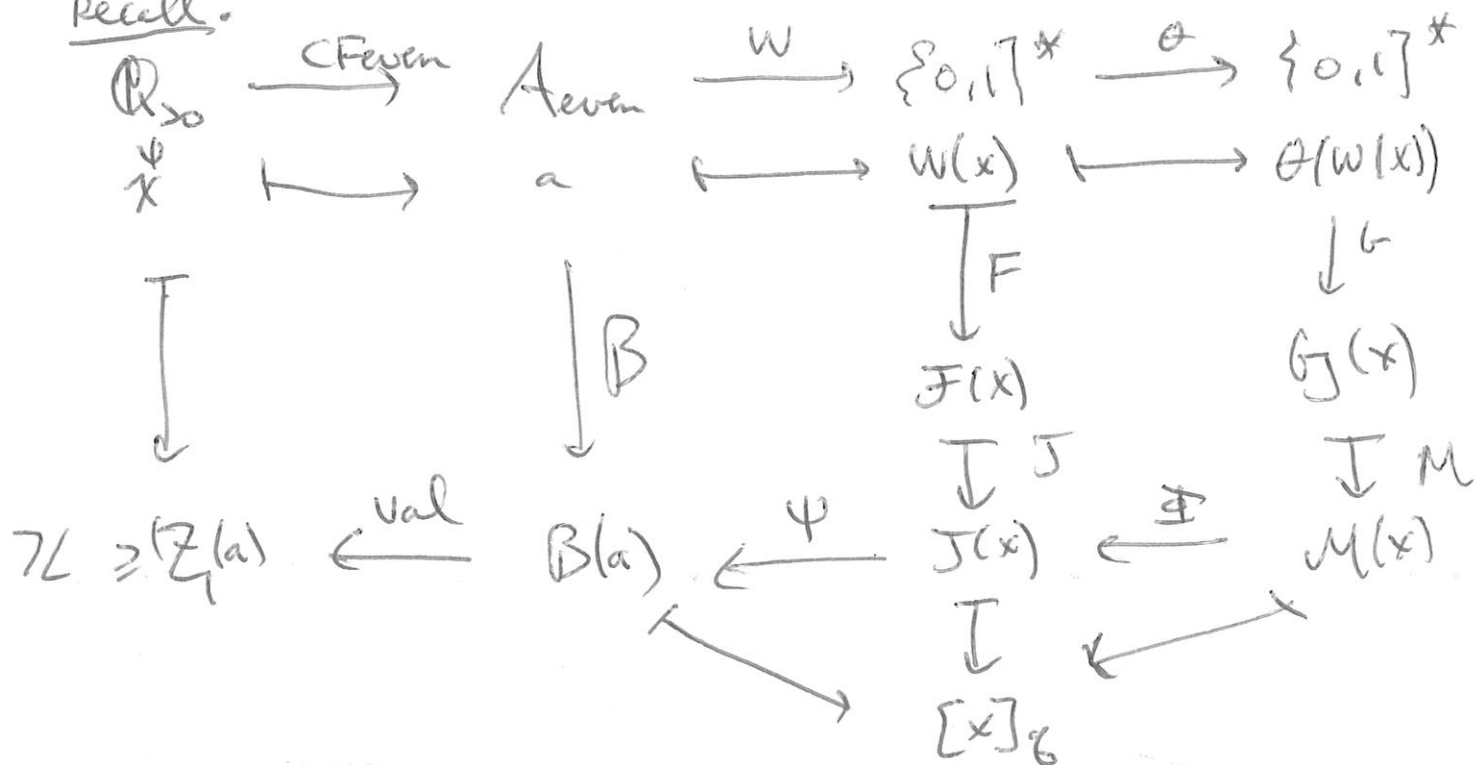


q-analog of rational numbers: from Ostrowski numeration systems to perfect matchings

with JCAval arxiv 2511.11290

Recall:



Ostrowski numeration system and admissible sequences

$\forall x \in \mathbb{R}/\mathbb{Q}$ has a unique CF expansion $[a_0, a_1, \dots]$ $a_0 \in \mathbb{Z}$, $a_i \geq 1, i \neq 0$

Convergent: $\frac{p_i}{q_i} = [a_0, a_1, \dots, a_i]$ $i \geq 0$

$$\begin{aligned} p_i &= a_i p_{i-1} + p_{i-2} & p_{-1} &= 0 & p_0 &= a_0 \\ q_i &= a_i q_{i-1} + q_{i-2} & q_{-1} &= 1 & q_0 &= 1 \end{aligned}$$

Thm (Ostrowski, 1921) $\forall x \in \mathbb{R}/\mathbb{Q} \setminus \{0\}$ $\forall n \geq 0 \exists! k \geq 0 \exists! (b_i)_{i=1}^k$

$$\text{st. } n = \sum_{i=1}^k b_i q_{i-1}$$

where $0 \leq b_i \leq a_i$, $b_i \neq a_i \Rightarrow b_{i-1} = 0$.

Ex

$$\frac{29}{12}$$

$$\begin{aligned} 29 &= 2 \cdot 12 + 5 \\ 12 &= 2 \cdot 5 + 2 \\ 5 &= 2 \cdot 2 + 1 \\ 2 &= 2 \cdot 1 + 0 \end{aligned}$$

$$\begin{aligned} \frac{29}{12} &= [2; 2, 2, 2] \\ &= 2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2}}}} \end{aligned}$$

Convergents: $2; 2 + \frac{1}{2} = \frac{5}{2}; 2 + \frac{1}{2 + \frac{1}{2}} = 2 + \frac{2}{5} = \frac{12}{5}$

i	-1	0	1	2	3	4
a_i		2	2	2	2	
p_i	1	2	5	12	29	
q_i	0	1	2	5	12	
r_i		1	3	7	17	41

h	g_0	g_1	g_2	g_3
	1	2	5	12
0	b_0	b_1	b_2	b_3
1	0	0	0	0
2	1	0	0	0
3	0	1	0	0
4	1	1	0	0
5	0	2	0	0
6	0	0	1	0
7	1	0	1	0
8	...	...	...	...
9	0	2	1	0
10	0	0	2	0
11	1	0	2	0
12	0	0	0	1
13	...	...	...	...
24	0	0	0	2
25	...	...	...	...
28	0	2	0	2

DEF Sort $a = [a_0, a_1, \dots, a_{k-1}]$ with $a_0 > 0, a_i \geq 1, i \geq 1$
 A sequence $(b_i)_{i=0}^{k-1}$ is admissible for a if

• $0 \leq b_i \leq a_i \quad \forall i$

• If $i > 0$ odd and $b_i = a_i$, then $b_{i-1} = a_{i-1}$

• If $i > 0$ even and $b_i = 0$, then $b_{i-1} = 0$

Let $B(a) = \{ (b_i)_{i=0}^{k-1} \text{ admissible for } a \}$

Let $(r_i)_{i=0}^k$ st $r_i = p_{i-1} + b_{i-1}$

Thm Let $Z(a) = \begin{cases} \mathbb{Z} \cap [0, r_k) & k \text{ odd} \\ \mathbb{Z} \cap [r_{k-1}, r_k) & k \text{ even} \end{cases}$

The function $\text{val}_a : B(a) \rightarrow Z(a)$
 $(b_i)_{i=0}^{k-1} \mapsto \sum_{i=0}^{k-1} (-1)^i b_i r_i$ is a bijection.

EX $\frac{3}{4} = [0, 1, 2, 1]$ ^{even}

i	-1	0	1	2	3	4
a_i		0	1	2	1	
p_i	1	0	1	2	3	
q_i	0	1	1	3	4	
r_i		1	1	2	5	7

n	r_0	$-r_1$	r_2	$-r_3$	r_4
	1	-1	2	-5	7
0	0	0	0	0	0
1	0	1	1	0	0
2	0	0	1	0	0
3	0	1	2	0	0
4	0	0	2	0	0
-1	0	0	2	1	0
-2	0	1	2	1	0

$\rightarrow B^0 \text{ vs } B^1$

$\frac{3}{4} = [0, 1, 3]$ ^{odd}

i	-1	0	1	2	3
a_i		0	1	3	
p_i	1	0	1	3	
q_i	0	1	1	4	
r_i		1	1	2	7

n	r_0	$-r_1$	r_2
	1	-1	2
0	0	0	0
1	0	1	1
2	0	0	1
3	0	1	2
4	0	0	2
5	0	1	3
6	0	0	3

Disjoint union $B(a) = B^+(a) \cup B^-(a)$

$$B^+(a) = \{ (b_i)_{i=0}^{k-1} \in B(a) \mid b_0 > 0 \text{ or } a_0 = 0 < b_1 = a_1 \}$$

$$B^-(a) = \{ (b_i)_{i=0}^{k-1} \in B(a) \mid b_0 = 0 < a_0 \text{ or } a_0 = 0 \leq b_1 < a_1 \}$$

Thm

$$[X]_q = q^{-1} \frac{\sum_{b \in B^+(a)} q^{\|b\|_1}}{\sum_{b \in B^-(a)} q^{\|b\|_1}} \quad \text{on } a = (F_{\text{even}}(X))$$

$$\Rightarrow \left[\frac{3}{4} \right]_q = \frac{q^{-1} (q^2 + q^3 + q^4)}{q^0 + q^1 + q^2 + q^3}$$

Thm • $B(a)$ is convex, i.e. $\text{conv}(B(a)) \cap \mathbb{Z}^k = B(a)$

$$\bullet \exists \text{ half-space } H \text{ s.t. } \begin{cases} B^+(a) = H \cap B(a) \\ B^-(a) = H^c \cap B(a) \end{cases}$$

Thm $\psi : J(X) \rightarrow B(a)$ is a poset isomorphism

