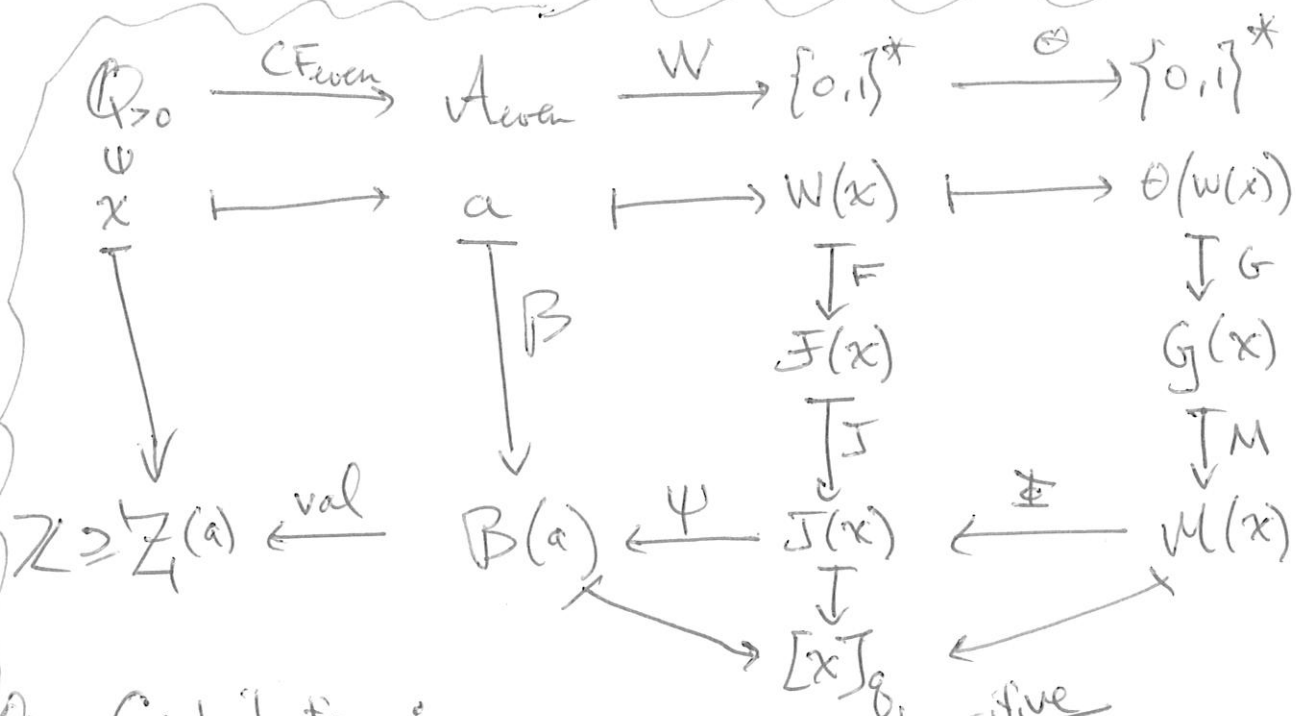


q-analog of rational numbers: from Ostrowski
numeration systems to perfect matchings
with JC Aval arxiv:2511.11290



Our Contribution:

- 3 compatible combinatorial interpretations of all positive rational numbers and their MGD q-analogs
- one of them completely new
- ~~works for all positive rational numbers.~~
- only one combinatorial object $\forall$ rational number
- is the good choice

Def (MG,0) $[\cdot]_q : \mathbb{Q} \cup \{\infty = \frac{1}{0}\} \rightarrow \mathbb{Z}(q)$ unique fct

s.t. $[0]_q = 0$, $[x+1]_q = q[x]_q + 1$, $[x]_q [x^{-1}]_q = \frac{1}{q}$

EX

$$[1]_q = q[0]_q + 1 = 1$$

$$[2]_q = q[1]_q + 1 = q + 1$$

$$[\frac{1}{2}]_q = \frac{1}{q} [2]_q^{-1} = \frac{1}{q^2 + q}, \text{ etc.}$$

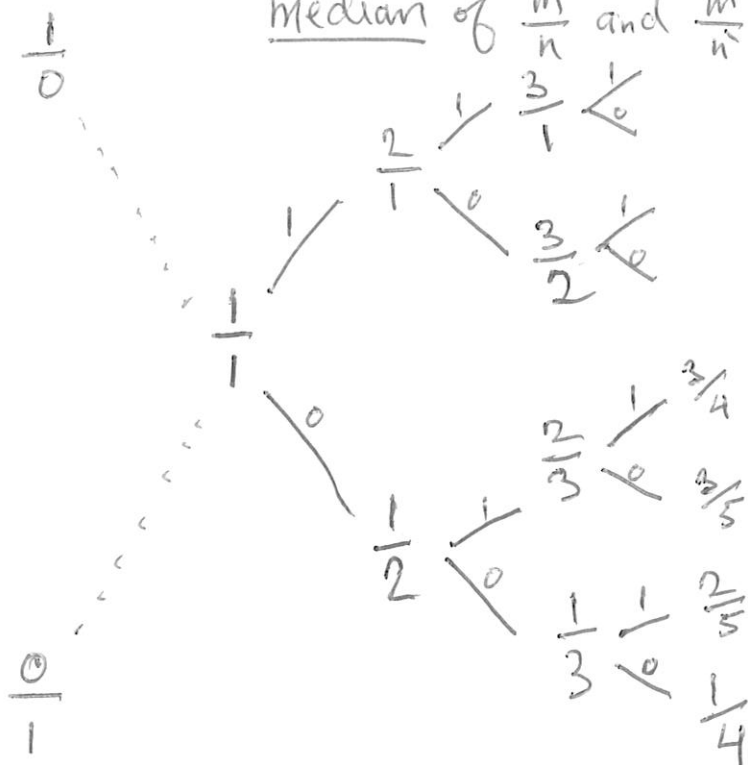
optional

Arbre de Stern-Brocot (1858)

Gauss
Stern
Piemann

Brocot: French
clock maker
gear ratio...

median of $\frac{m}{n}$ and $\frac{m'}{n'}$ is $\frac{m+m'}{n+n'}$



Théorème Every positive rational number appears exactly once in the binary tree.

Recent Ref = Concrete Math.
Graham, Knuth, Patashnik

Using $\{0,1\}$ to encode down and up steps, we get a bijection $W: \mathbb{Q}_{>0} \rightarrow \{0,1\}^*$

EX $W(1) = \epsilon$, $W(3/4) = 011$

A Formula: $W: \mathbb{Q}_{>0} \rightarrow \{0,1\}^*$

$$W(x) = 1^{a_0} 0^{a_1} \dots 1^{a_{2l-2}} 0^{a_{2l-1}}$$

where $[a_0, a_1, \dots, a_{2l-2}, a_{2l-1}]$ is the even-length CF expansion of x .

optional

EX $\frac{3}{4} = 0 + \frac{1}{4/3} = 0 + \frac{1}{1 + \frac{1}{3}} = 0 + \frac{1}{1 + \frac{1}{2 + \frac{1}{1}}} = [0; 1, 2, 1]$

Thus $W(3/4) = 1^0 0^1 1^2 0^1 = 011$

Fence posets

For every word $w = w_1 \dots w_n \in \{0,1\}^*$, let $F(w) = (\{y_0, \dots, y_n\}, \triangleleft)$ the partial order (fence poset) whose cover relations are

$$\begin{cases} y_{i-1} \triangleleft y_i & \text{if } w_i = 1 \text{ (up-step)} \\ y_{i-1} \triangleright y_i & \text{if } w_i = 0 \text{ (down-step)} \end{cases}$$

EX $F(011) = \begin{array}{c} y_0 \quad y_2 \quad y_3 \\ \quad \diagdown \quad \diagup \\ \quad y_1 \end{array} = F \circ W(50; 12, 1] = F(\frac{3}{4})$

Remark: $F: \{0,1\}^* \rightarrow \{\text{fence posets}\}$ is a bijection

$F = F \circ W: \mathbb{Q}_{>0} \rightarrow \{\text{fence posets}\}$ is a bijection

Let $J(x)$ be the set of order ideals (downsets) of $F(x)$.

Let also $J^\circ(x) = \{I \in J(x) \mid y_0 \in I\}$

$J^\circ(x) = \{I \in J(x) \mid y_0 \notin I\}$

Thm (Aval, L.) (our interpretation of MG-0)

$$[x]_q = q^{-1} \frac{\sum_{I \in J^\circ(x)} q^{|I|}}{\sum_{I \in J(x)} q^{|I|}} \quad \forall x \in \mathbb{Q}_{>0}$$

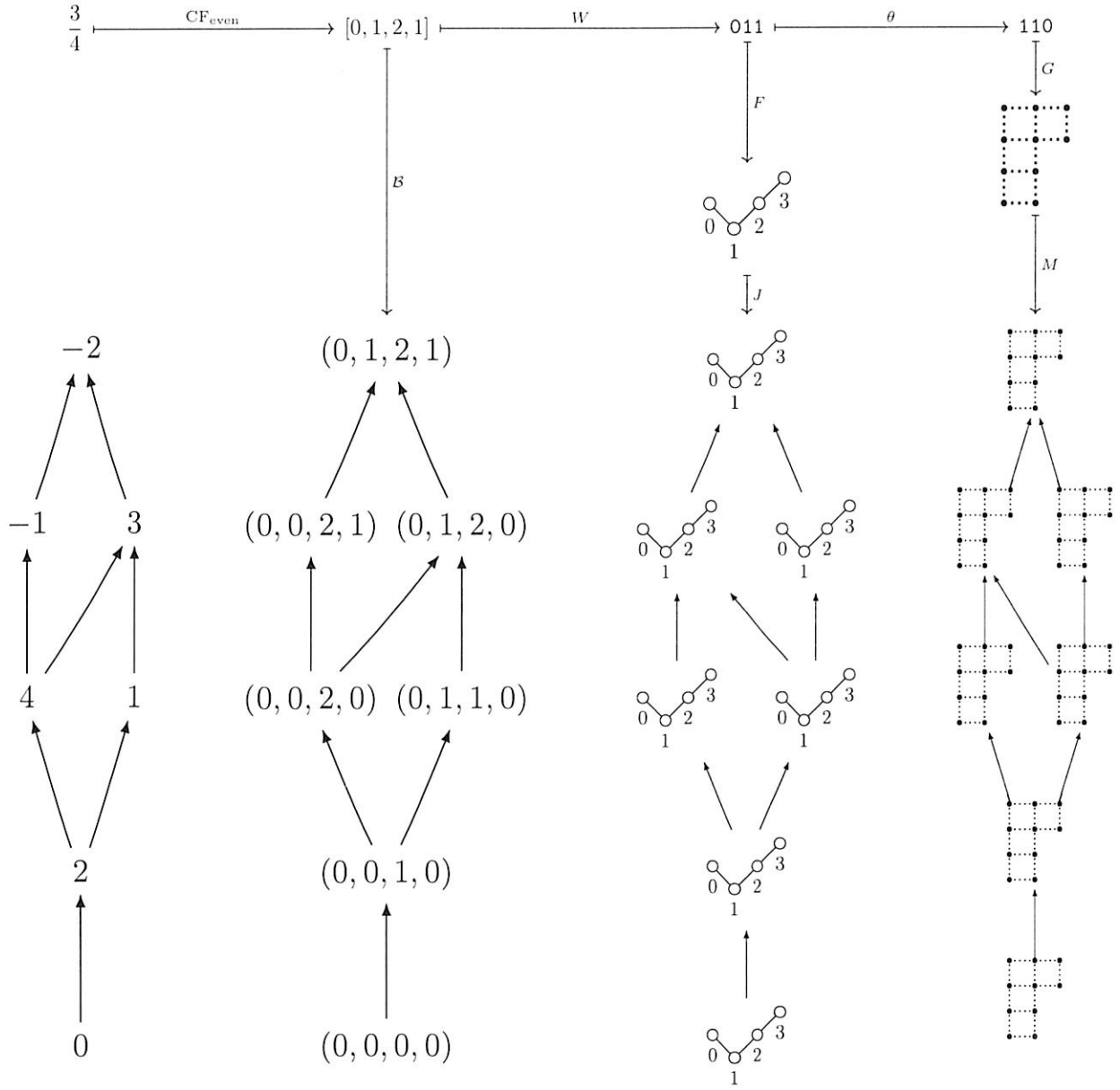
EX Directly on the poster.

Why is the good correspondence?

Two involutions on $\mathbb{Q}_{>0}$ ($x \mapsto \frac{1}{x}$ and $\frac{r}{s} \mapsto \frac{s^*}{r^*}$ where r^* is the mult. inv. of $r \bmod (r+s)$) correspond to horiz/vert symmetries of the fence poset $F(x)$.

EX $\left\{ \begin{array}{c} y_2 \quad y_3 \\ \diagdown \quad \diagup \\ y_1 \end{array}, \begin{array}{c} y_3 \quad y_4 \\ \diagdown \quad \diagup \\ y_2 \end{array}, \begin{array}{c} y_4 \quad y_5 \\ \diagdown \quad \diagup \\ y_3 \end{array} \right\} = J^\circ(\frac{3}{4})$

$\left\{ \begin{array}{c} y_0 \quad y_1 \quad y_2 \\ \diagdown \quad \diagup \\ y_1 \end{array}, \begin{array}{c} y_1 \quad y_2 \quad y_3 \\ \diagdown \quad \diagup \\ y_2 \end{array}, \begin{array}{c} y_2 \quad y_3 \quad y_4 \\ \diagdown \quad \diagup \\ y_3 \end{array}, \begin{array}{c} y_3 \quad y_4 \quad y_5 \\ \diagdown \quad \diagup \\ y_4 \end{array} \right\} = J^\circ(\frac{3}{4})$



$$\begin{array}{c}
 \mathcal{Z}([0, 1, 2, 1]) \xleftarrow{\text{val}} \mathcal{B}([0, 1, 2, 1]) \xleftarrow{\Psi} \mathcal{J}\left(\frac{3}{4}\right) \xleftarrow{\Phi} \mathcal{M}\left(\frac{3}{4}\right) \\
 \begin{array}{l} (r_0, -r_1, r_2, -r_3) = (1, -1, 2, -5) \\ r_4 = 7 \end{array} \quad \downarrow \quad \downarrow \quad \downarrow \\
 \left[\frac{3}{4} \right]_q = q^{-1} \frac{q^4 + q^3 + q^2}{q^3 + q^2 + q + 1}
 \end{array}$$

Snake graphs

$$\theta: \{0,1\}^* \rightarrow \{0,1\}^* \\ a w \mapsto \begin{cases} \bar{a} \theta(w) & \text{if } |w| \text{ even} \\ a \theta(w) & \text{if } |w| \text{ odd} \end{cases}$$

$$\frac{3}{4} \mapsto [0;1,2,1] \xrightarrow{w} 011 \xrightarrow{\theta} 110$$

$\downarrow F$

$\downarrow G$

Let $G = G \circ \theta \circ W : \mathbb{Q}_{>0} \rightarrow \{\text{snake graphs}\}$ bijection

Perfect matchings of $G(\frac{3}{4})$ are $\mathcal{M}(\frac{3}{4})$

$$\mathcal{M}(\frac{3}{4}) = \left\{ \begin{array}{c} \text{diagram 1} \\ \text{diagram 2} \\ \text{diagram 3} \end{array} \right\} \cup \left\{ \begin{array}{c} \text{diagram 4} \\ \text{diagram 5} \\ \text{diagram 6} \end{array} \right\}$$

last edge is vertical

$$\mathcal{M}^+(x) = \left\{ \begin{array}{l} \text{p.m. s.t. the} \\ \text{first edge is} \\ \perp \text{ to the} \\ \text{first edge of} \\ \text{basic p.m.} \end{array} \right\}$$

$$\mathcal{M}''(x) = \left\{ \begin{array}{l} \text{p.m. s.t. the} \\ \text{first edge} \\ \text{is } \parallel \text{ to the} \\ \text{first edge} \\ \text{of the basic} \end{array} \right\}$$

Thm $(Aval, L) \quad \forall x \in \mathbb{Q}_{>0}$

$$[x]_q = q^{-1} \frac{\sum_{m \in \mathcal{M}^+(x)} q^{\text{area}(m \Delta b)}}{\sum_{m \in \mathcal{M}''(x)} q^{\text{area}(m \Delta b)}}$$

where b is the basic perfect matching of $G(x)$.

EX

$$[\frac{3}{4}]_q = q^{-1} \frac{(q^4 + q^3 + q^2)}{q^0 + q^1 + q^2 + q^3} = \frac{q + q^2 + q^3}{1 + q + q^2 + q^3}$$