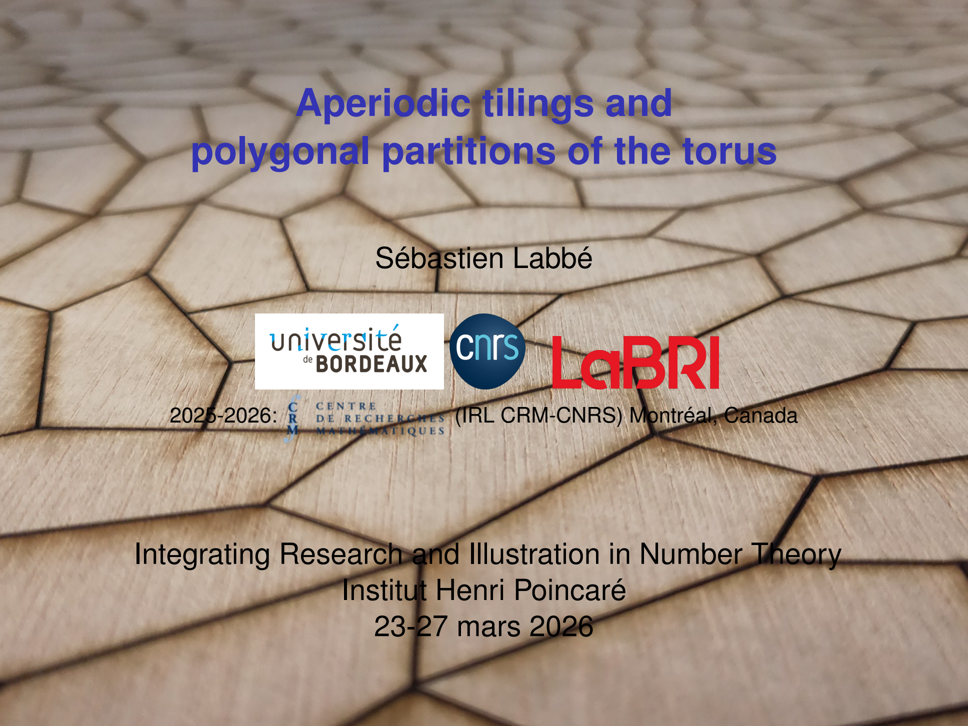

Aperiodic tilings and polygonal partitions of the torus

Sébastien Labbé

université de BORDEAUX · cnrs · LaBRI

2025-2026: CRM CENTRE DE RECHERCHES MATHÉMATIQUES (IRL CRM-CNRS) Montréal, Canada

Integrating Research and Illustration in Number Theory
Institut Henri Poincaré
23-27 mars 2026

- Visiting Scholars program (for sabbatical)

  2019-2020 : Casey Mann & Jennifer McLoud-Mann, University of Seattle
- Cotutelle doctoral program : *Joint supervision of a PhD by the University of Bordeaux and an international university (1 thesis defense, 2 PhD diplomas)*
- Institut de Mathématiques de Bordeaux (IMB) &
  Laboratoire Bordelais de Recherche en Informatique (LaBRI)
- Opportunities for research between Canada and France

  `https://canadafrance.pages.math.cnrs.fr/`

Outline

1. Question
2. Small aperiodic sets of Wang tiles
3. One-dimensional crystallography and golden mean
4. Jeandel-Rao aperiodic tilings
5. Conclusion (incl. a surprise :)

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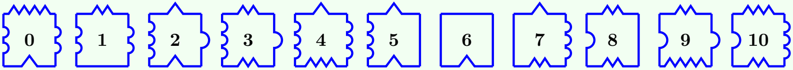
A Fondamental Question

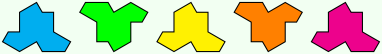
Domino Problem

Given a finite set of tiles,
can you tile the plane with copies of these tiles ?

Exercise (5 minutes)

Cover the largest possible square with the following pieces :





Could you tile the floor of your living room ?

Thanks to Xavier Provençal and ETS for the laser cutting
(École de technologie supérieure, Montréal, décembre 2025).

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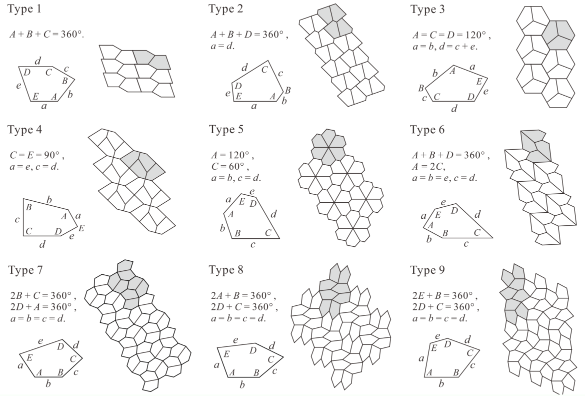
Periodic tiling

A tiling is **periodic** if it is invariant under at least one nonzero translation.
Ex :the 15 types of tilings with convex pentagons are periodic :



Teruhisa SUGIMOTO, Convex Pentagons with Positive Heesch Number, `arXiv:1802.00119v2` ; `https://fr.wikipedia.org/wiki/Pavage_pentagonal`

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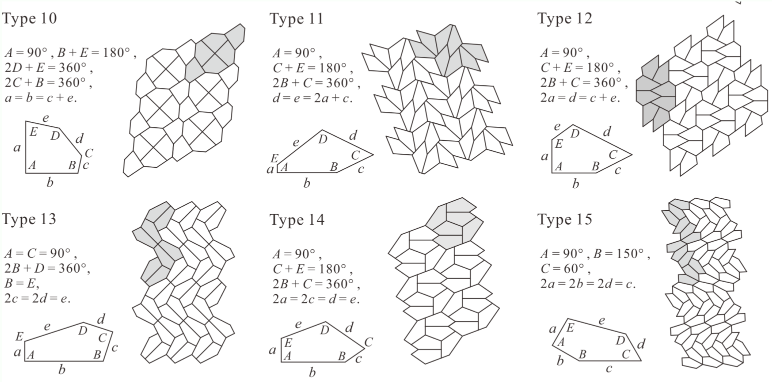


Type 10

$A = 90°$, $B + E = 180°$,
$2D + E = 360°$,
$2C + B = 360°$,
$a = b = c + e$.

Type 11

$A = 90°$,
$C + E = 180°$,
$2B + C = 360°$,
$d = e = 2a + c$.

Type 12

$A = 90°$,
$C + E = 180°$,
$2B + C = 360°$,
$2a = d = c + e$.

Type 13

$A = C = 90°$,
$2B + D = 360°$,
$B = E$,
$2c = 2d = e$.

Type 14

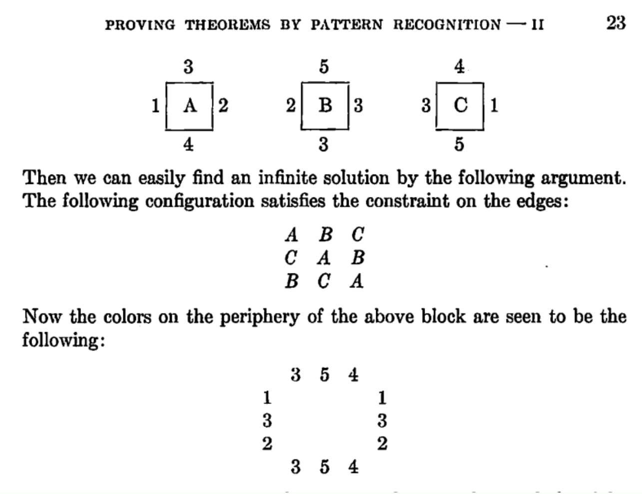
$A = 90°$,
$C + E = 180°$,
$2B + C = 360°$,
$2a = 2c = d = e$.

Type 15

$A = 90°$, $B = 150°$,
$C = 60°$,
$2a = 2b = 2d = c$.

Teruhisa SUGIMOTO, Convex Pentagons with Positive Heesch Number,
`arXiv:1802.00119v2`; `https://fr.wikipedia.org/wiki/Pavage_pentagonal`

Wang tiles



Then we can easily find an infinite solution by the following argument. The following configuration satisfies the constraint on the edges:

A B C
C A B
B C A

Now the colors on the periphery of the above block are seen to be the following:

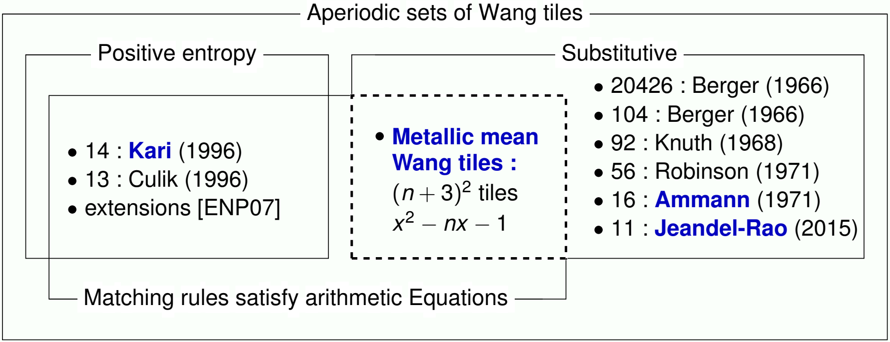
3 5 4
1 1
3 3
2 2
3 5 4

Wang's original question : is it true that a set of Wang tiles tile the plane if and only if there exists such a cyclic rectangle ?

H. Wang. Proving theorems by pattern recognition – II. Bell System Technical Journal*, 40(1) :1–41, January 1961.* doi:10.1002/j.1538-7305.1961.tb03975.x

Aperiodic Wang tile sets

- Berger (1966) proved the domino problem is undecidable.
- Berger provided aperiodic sets of 20426 and 104 (or 105 ?) tiles.
- The search for a **smallest aperiodic set** ended in 2015.



Aperiodic sets of Wang tiles

Positive entropy

- 14 : **Kari** (1996)
- 13 : Culik (1996)
- extensions [ENP07]

Substitutive

- 20426 : Berger (1966)
- 104 : Berger (1966)
- 92 : Knuth (1968)
- 56 : Robinson (1971)
- 16 : **Ammann** (1971)
- 11 : **Jeandel-Rao** (2015)

- **Metallic mean Wang tiles :**
  $(n+3)^2$ tiles
  x^2-nx-1

Matching rules satisfy arithmetic Equations

Metallic mean Wang tiles I : self-similarity, aperiodicity and minimality.
Forum of Mathematics, Sigma 13 (2025) e133. `doi:10.1017/fms.2025.10069`

Metallic mean Wang tiles II : the dynamics of an aperiodic computer chip.
Forum of Mathematics, Sigma 13 (2025) e155. `doi:10.1017/fms.2025.10098`

Jeandel-Rao's set of 11 Wang tiles

Théorème (Jeandel, Rao, 2015)

All sets of $\leq$ 10 Wang tiles are **periodic** or **don't tile** the plane.

Théorème (Jeandel, Rao, 2015)

The following set of 11 Wang tiles is **aperiodic** :

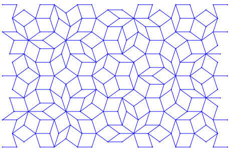
An equivalent geometrical encoding :

Emmanuel Jeandel and Michaël Rao. An aperiodic set of 11 Wang tiles. Adv. Comb. ***37*** *(2021) Id/No 1.*

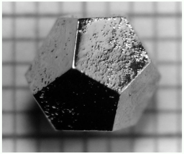
Quasicrystals

1982 (Shechtman) : observed that aluminium-manganese alloys produced a **quasicrystals structure**. 2011 **Nobel Prize** :

"*His discovery of quasicrystals revealed a new principle for packing of atoms and molecules* [that] *led to a* ***paradigm shift*** *within chemistry*."



Penrose tiling (1976)



A Ho-Mg-Zn quasicrystal

Smith (2023)

Darb-e Imam, Isfahan, Iran (1453)

en.wikipedia.org/wiki/Quasicrystal, en.wikipedia.org/wiki/Darb-e_Imam

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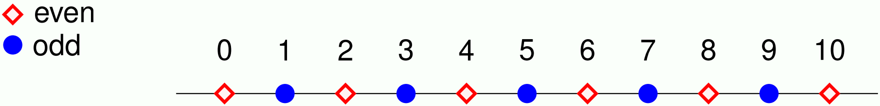
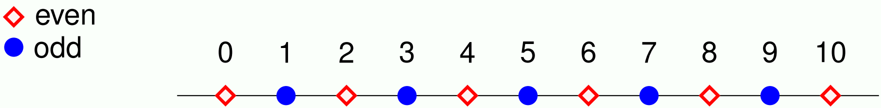
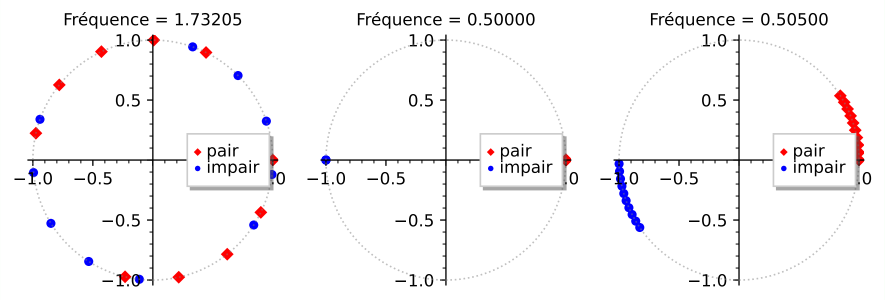
1D Crystal of even and odd numbers



1D Crystal of even and odd numbers



The odd/even in base 2 has **frequency** $\frac{1}{2}$:



Decimal, binary, hexadecimal and Fibonacci Syst

$$\begin{aligned}2026 &= 2 \times 1000 + 2 \times 10 + 6 \times 1\\ &= 2 \cdot 10^3 + 0 \cdot 10^2 + 2 \cdot 10^1 + 6 \cdot 10^0\\ &= (\mathtt{2026})_{10} \qquad \text{(used by certain } \textbf{Homo Sapiens}\text{)}\end{aligned}$$

$$\begin{aligned}2026 &= 1024 + 512 + 256 + 128 + 64 + 32 + 8 + 2\\ &= 2^{10} + 2^9 + 2^8 + 2^7 + 2^6 + 2^5 + 2^3 + 2^1\\ &= (\mathtt{11111101010})_2\\ 2026 &= 7 \cdot 16^2 + 14 \cdot 16^1 + 10 \cdot 16^0\\ &= (\mathtt{7EA})_{hex} \qquad \text{(used by certain } \textbf{computer}\text{)}\end{aligned}$$

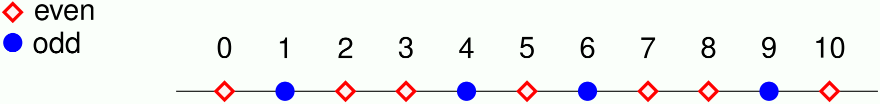
$$\begin{aligned}2026 &= 1597 + 377 + 34 + 13 + 5\\ &= F_{15} + F_{12} + F_7 + F_5 + F_3\\ &= (\mathtt{1001000010101000})_{Fibo} \qquad \text{(used by some } \textbf{crystals/tilings}\text{)}\end{aligned}$$

... using $F_0 = 1$, $F_1 = 2$ and $F_{n+1} = F_n + F_{n-1}$

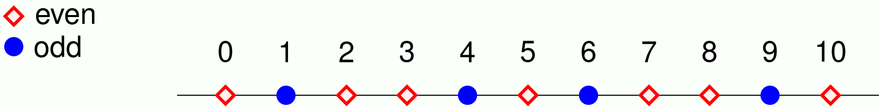
One-dimensional crystallography

n	$(n)_2$	parity	$(n)_{Fibo}$	Fibo-parity
0	`0`	◇	`0`	◇
1	`1`	●	`1`	●
2	`10`	◇	`10`	◇
3	`11`	●	`100`	◇
4	`100`	◇	`101`	●
5	`101`	●	`1000`	◇
6	`110`	◇	`1001`	●
7	`111`	●	`1010`	◇
8	`1000`	◇	`10000`	◇
9	`1001`	●	`10001`	●
10	`1010`	◇	`10010`	◇
11	`1011`	●	`10100`	◇
12	`1100`	◇	`10101`	●
13	`1101`	●	`100000`	◇
14	`1110`	◇	`100001`	●
15	`1111`	●	`100010`	◇

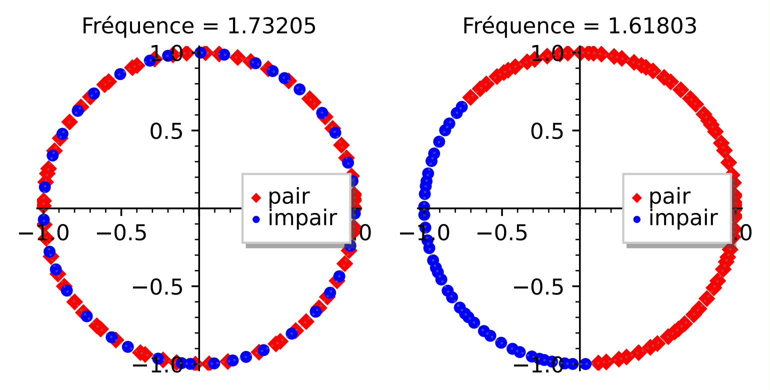
1D Crystal 1D of even-Fibo and odd-Fibo numbers



1D Crystal 1D of even-Fibo and odd-Fibo numbers



The odd/even in Fibonacci base has frequency $\frac{1}{2}(1+\sqrt{5}) \approx 1.618$:



This crystal is not periodic, it is a **quasicrystal** !

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A 100 × 100 patch

The file `https://members.loria.fr/EJeandel/research/100.txt` contains :

ligne 0 : `0001111100111000001110001100000111000111110001100...`

ligne 1 : `7456666675666745756667456674575666745666667456674...`

ligne 2 : `3a574572875743a2875743a5743a2875743a5745743a5743a...`

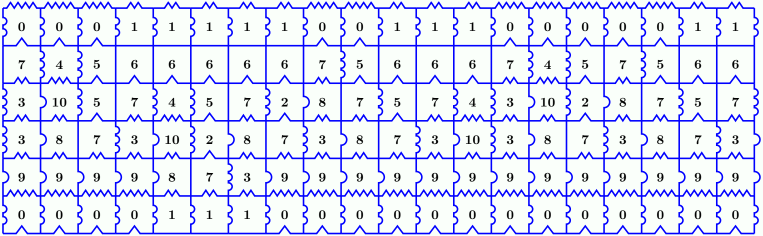
ligne 3 : `3873a2873873a3873873a3873a3873873a3873a43a3873a38...`

ligne 4 : `99998739999999999999999999999999999999a399999999...`

ligne 5 : `0000111000000000000000000000000000000001100000000...`

... : ...

The first 6 rows (limited to 20 columns) are :



0	0	0	1	1	1	1	1	0	0	1	1	1	0	0	0	0	0	1	1
7	4	5	6	6	6	6	6	7	5	6	6	6	7	4	5	7	5	6	6
3	10	5	7	4	5	7	2	8	7	5	7	4	3	10	2	8	7	5	7
3	8	7	3	10	2	8	7	3	8	7	3	10	3	8	7	3	8	7	3
9	9	9	9	8	7	3	9	9	9	9	9	9	9	9	9	9	9	9	9
0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0

Wrapping the rows on a circle

...

ligne 35 : `73a43a3873a2873a43a3873a2873873a2873a43a3873a2873...`

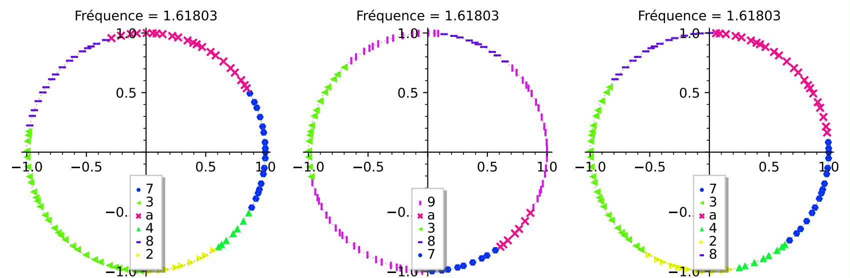
ligne 36 : `999a399999873999a3999998739999873999a39999987399...`

...

ligne 58 : `73a43a2873a43a3873a2873a43a2873a2873873a2873a43a2...`

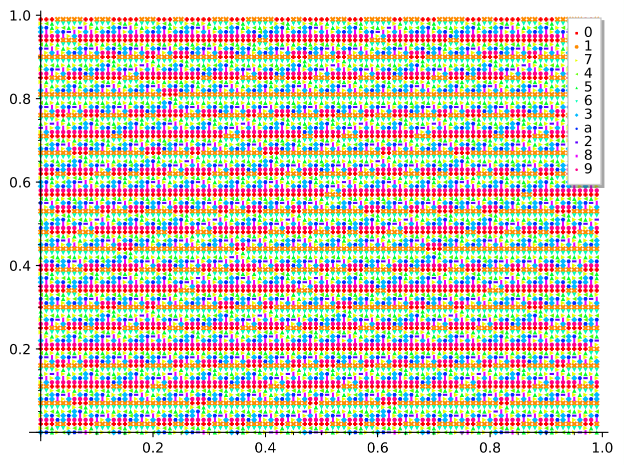
...

Wrapping these rows on a circle using the golden ratio frequency gives :



Experiment (step 1)

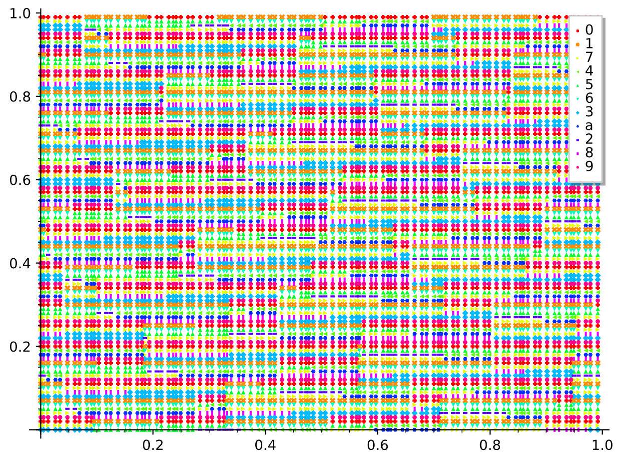
Wrapping on the 2-torus with frequency $\begin{pmatrix} 100 & 0 \\ 0 & 100 \end{pmatrix}^{-1}$:



Using frequency $\frac{1}{100}$ horizontally and vertically is a trick to make the points represent the tiling itself.

Experiment (step 2)

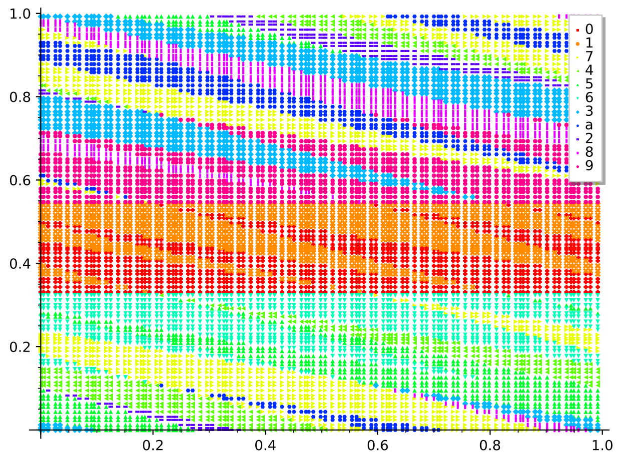
Wrapping on the 2-torus with frequency $\begin{pmatrix} \varphi & 0 \\ 0 & 100 \end{pmatrix}^{-1}$:



This makes each row in the patch to wrap around a circle (shown horizontally on the image above) with golden mean frequency.

Experiment (step 3)

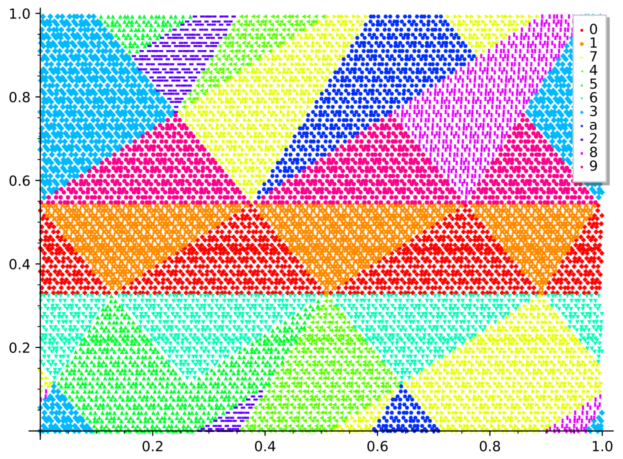
Wrapping on the 2-torus with frequency $\begin{pmatrix} \varphi & 0 \\ 0 & \varphi+3 \end{pmatrix}^{-1}$.



This makes sense because the vertical distance (or return time) between rows involving tiles labeled #0 and #1 is 4 or 5 with an average of $\varphi + 3 \approx 4.618$ as noticed already by Jeandel and Rao.

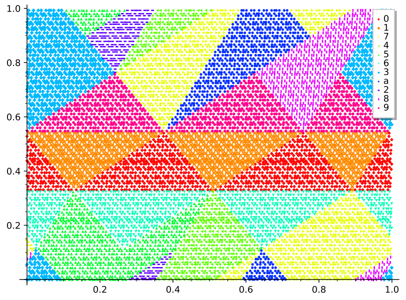
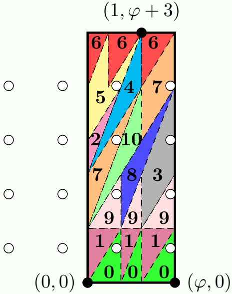
Experiment (step 4)

Wrapping on the 2-torus with frequency $\begin{pmatrix} \varphi & 1 \\ 0 & \varphi+3 \end{pmatrix}^{-1}$.



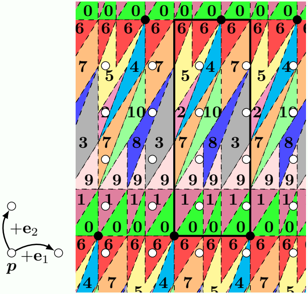
A shear is happening in Jeandel-Rao tilings.
This is one of the reasons that makes the description of Jeandel-Rao tilings more difficult, but certainly very interesting !

Rescaling to get $\mathbb{Z}^2$-action R_0 and partition $\mathcal{P}_0$

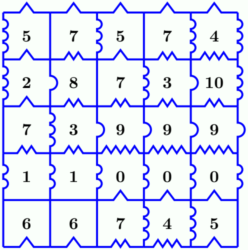
Step 4 of the experiment	Rescaling
$\mathbb{Z}^2 \curvearrowright \mathbb{R}^2/\mathbb{Z}^2$	$\mathbb{Z}^2 \overset{R_0}{\curvearrowright} \mathbb{R}^2/\begin{pmatrix}\varphi & 1\\ 0 & \varphi+3\end{pmatrix}\mathbb{Z}^2$
$(\mathbf{k},\mathbf{x}) \mapsto \mathbf{x} + \begin{pmatrix}\varphi & 1\\ 0 & \varphi+3\end{pmatrix}^{-1}\mathbf{k}.$	$R_0 : (\mathbf{k},\mathbf{x}) \mapsto \mathbf{x} + \mathbf{k}.$
	$\mathcal{P}_0 =$ 

$$\begin{pmatrix}5 & 7\\ 7 & 3\\ 9 & 9\\ 0 & 0\end{pmatrix} \in \mathcal{L}_{\mathcal{P}_0,R_0} = \left\{ w : S \to \mathcal{A} \,\middle|\, S \subset \mathbb{Z}^2 \text{ and } w \text{ is } \textbf{allowed} \right\}$$

Jeandel–Rao aperiodic set of 11 Wang tiles



$$\xrightarrow[\text{repr.}]{\text{symb.}} \begin{pmatrix} 5 & 7 & 5 & 7 & 4 \\ 2 & 8 & 7 & 3 & 10 \\ 7 & 3 & 9 & 9 & 9 \\ 1 & 1 & 0 & 0 & 0 \\ 6 & 6 & 7 & 4 & 5 \end{pmatrix} \xrightarrow[\text{tiling}]{\text{Wang}}$$



S. Labbé, Aperiodic order : from combinatorics to geometry via symbolic dynamics, number theory and algorithms, Thèse d'habilitation à diriger des recherches (HDR), Université de Bordeaux, June 2025,

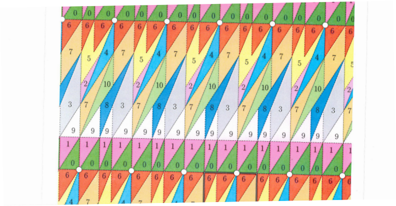
`hal.science/tel-05138330`

Jeandel-Rao tiling solver tutorial

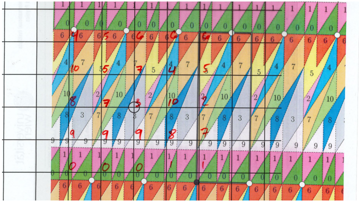
Sébastien Labbé, `http://www.slabbe.org/blogue/2024/04/a-do-it-yourself-polygonal-partition-to-construct-jeandel-rao-tilings/`, May 2024

Step 1. Create copies of a geometric version of the 11 Jeandel-Rao Wang tiles:

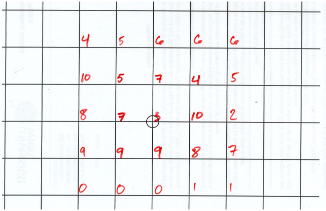
Step 2. Print the following polygonal partition (1 unit = 3cm):



Step 3. Choose a random starting point on the partition and place the origin of a lattice (1 unit = 3cm) at this point (rotation is not allowed):



Step 4. Encode each lattice point by the label of the atom of the partition it falls in:



4	5	6	6	6
10	5	7	4	5
8	7	3	10	2
9	9	9	8	7
0	0	0	1	1

Step 5. The above pattern encodes a valid patch using Jeandel-Rao tiles:

Step 6. Repeat the process and try creating larger patches.

Note. *If the tile size is the same as the lattice unit (3cm), then a tiling can be constructed without the lattice by placing tiles on top of the partition according to some arbitrary point on each tile (for example the lower left corner). This is a cut-and-project setup where the physical space and internal space unite.*

S. Labbé. Markov partitions for toral $\mathbb{Z}^2$-rotations featuring Jeandel-Rao Wang shift and model sets. *Ann. H. Lebesgue*, 4:283–324, 2021. `doi:10.5802/ahl.73`.

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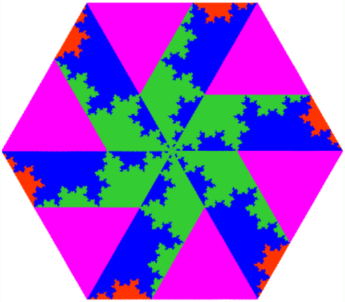
The Baake, Gähler, Sadun C&P window

A partition of the window of a 4-to-2 cut and project scheme  :



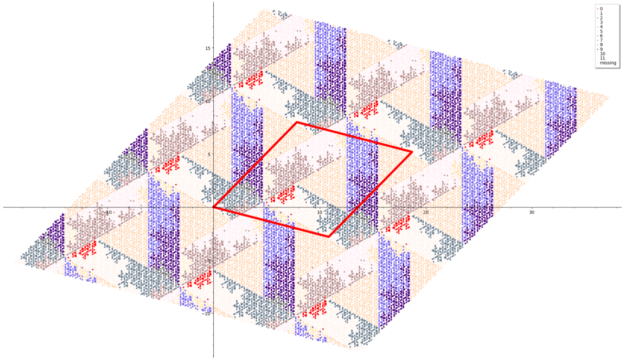
Colors are associated to the four meta-pieces denoted T, H, P, F :

> *"The color code refers to the type of the tile whose control point is plotted (orange for T-, blue for H-, green for P- and purple for F-tiles)."*

Michael Baake, Franz Gähler, Lorenzo Sadun,
Israel J. Math. 270 (2025) 449–485, `doi:10.1007/s11856-025-2780-8`

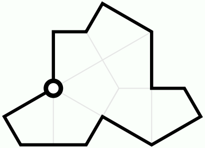
Surprise

Last September 2025, in Montreal, Peter Selinger (U. Dalhousie, Halifax, Canada) showed me a picture. Here is a reproduction of it :

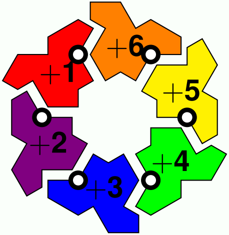


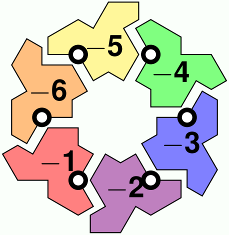
... that he obtained studying the hat using the same procedure we have done with Jeandel-Rao tilings by wrapping on the torus $\mathbb{C}/\Lambda$ with lattice $\Lambda = \langle \phi^2 + \xi, \xi(\phi^2 + \xi) \rangle_{\mathbb{Z}} \subset \mathbb{Z}[\phi, \xi]$ where $\phi = \frac{1+\sqrt{5}}{2}$ and $\xi = \exp(\pi i/3)$. Our preprint will on arxiv/our website soon (today ?).

Selinger's choice for the anchors



Key observation : these points live on a lattice.





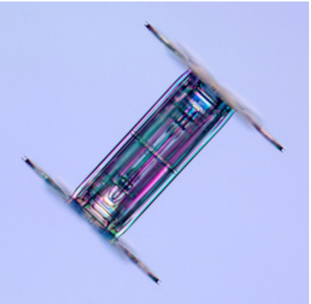
Thus every tiling by the hat can be described as a map

$$\mathbb{Z}^2 \to \{-6, -5, \dots, 5, 6\}$$

where 0 denotes **no tile is anchored at that vertex**.

End

casserole à induction



flocon de neige

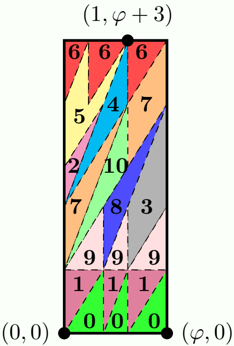
K. G. Libbrecht. Snow Crystals : A Case Study in Spontaneous Structure Formation. Princeton University Press, Dec. 2021. `doi:10.2307/j.ctv1qdqztv`

Open questions on Jeandel-Rao tilings

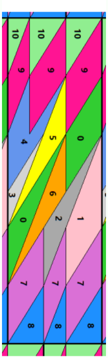
- Describe all of the 33 candidates of 11 Wang tiles listed by JR

  recent progress was made by Thompson (2022) and Mann (2024)

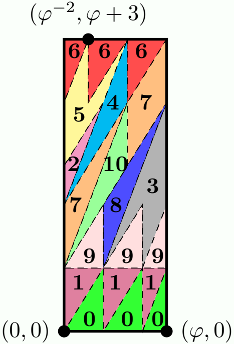
partition for JR



Thompson (2022)



Mann et al. (2024)



R. D. Thompson. "The Jeandel-Rao Aperiodic Wang Tilings of the Plane". MSc in Mathematics. The Open University, Milton Keynes, UK, May 2022

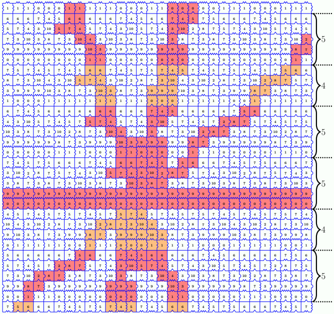
Hults, Jitsukawa, Mann, Zhang, Experimental Results on Potential Markov Partitions for Wang Shifts `arXiv:2302.13516`

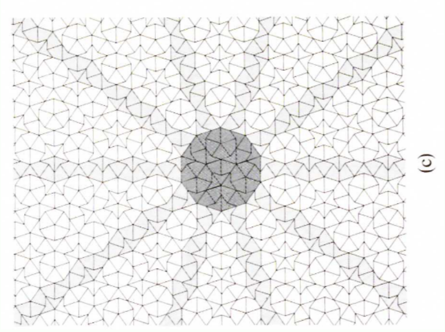
4 slopes of Conway worms in Jeandel-Rao WS

Theorem (L., Mann, McLoud-Mann, 2023)

The minimal subshift X_0 of the Jeandel-Rao Wang shift contains exactly **4 nonexpansive directions** whose slopes are

$$\left\{0, \quad \varphi + 3, \quad -3\varphi + 2, \quad -\varphi + \frac{5}{2}\right\}.$$

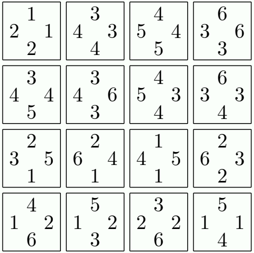




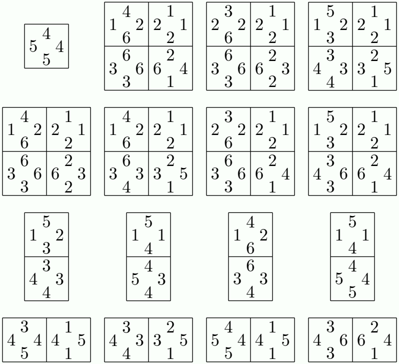
It reminds of the cartwheel tiling in the context of Penrose tilings.

Ammann 16 Wang tiles are self-similar

16 tiles :



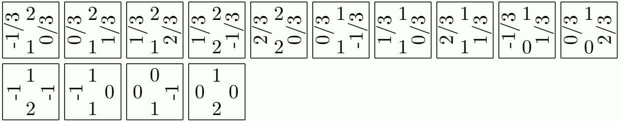
16 equivalent supertiles :

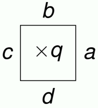


Branko Grünbaum and G. C. Shephard. Tilings and patterns. *W. H. Freeman and Company, New York, 1987.*

M. Senechal. The mysterious Mr. Ammann. Math. Intelligencer*, 26(4) :10–21, 2004.* `doi:10.1007/BF02985414`

Kari's 14 Wang tiles computing $\times\frac{2}{3}$ and $\times 2$



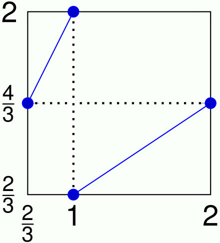


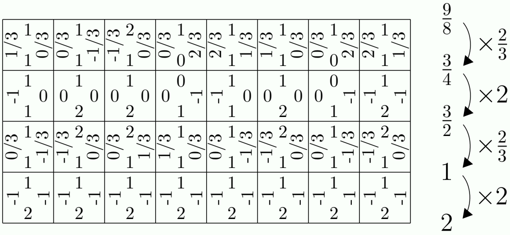
$$\iff qb + c = d + a$$

with $q \in \{\frac{2}{3}, 2\}$

$$g(x) = \begin{cases} 2x & \text{if } x \leq 1, \\ \frac{2}{3}x & \text{if } x > 1. \end{cases}$$

Averages of horizontal labels are orbits of g :





Kari (1996) *Durand, Gamard, Grandjean (2007)* *Kari (2016)*

Théorème fondamental de l'arithmétique

Théorème fondamental de l'arithmétique

Tout entier strictement positif **peut être écrit** comme un produit de nombres premiers d'une **unique façon**, à l'ordre près des facteurs.

$$2025 = 3^4 \times 5^2 \qquad 2027 = 2027$$

$$2026 = 2 \times 1013 \qquad 2028 = 2^2 \cdot 3 \cdot 13^2$$

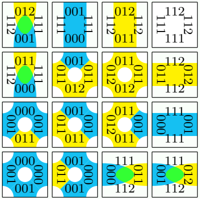
Preuve de non-périodicité de Kari : Supposons qu'il existe un rectangle cyclique d'hauteur $n > 0$. Soit $x \in [\frac{2}{3}, 2]$ la moyenne tout en du haut du rectangle. On doit avoir $g^n(x) = x$. Donc, il existe $k, \ell \geq 0$ tel que $k + \ell = n$ et

$$x = g^n(x) = \left(\frac{2}{3}\right)^k \cdot 2^\ell \cdot x.$$

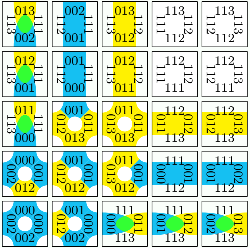
Donc, $3^k = 2^{k+\ell}$. On déduit que $k = \ell = n = 0$ par le théorème. Contradiction. Les tuiles de Kari n'admettent donc pas de rectangle cyclique.

A family $\{\mathcal{T}_n\}_{n\geq 1}$ of metallic mean Wang tiles

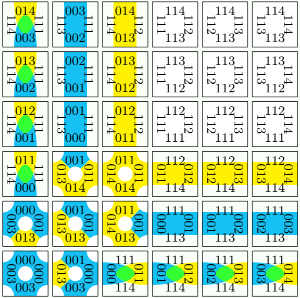
For every integer $n \geq 1$, $\mathcal{T}_n$ is made of n^2 white tiles, $2n$ blue stripe tiles, $2n$ yellow stripe tiles, $2(n{+}1)$ green tiles and 7 junction tiles. Total : $n^2{+}6n{+}9 = (n+3)^2$ tiles.



$\mathcal{T}_1 \equiv$ **Ammann**



$\mathcal{T}_2$



$\mathcal{T}_3$

Tile labels are vectors in the finite set

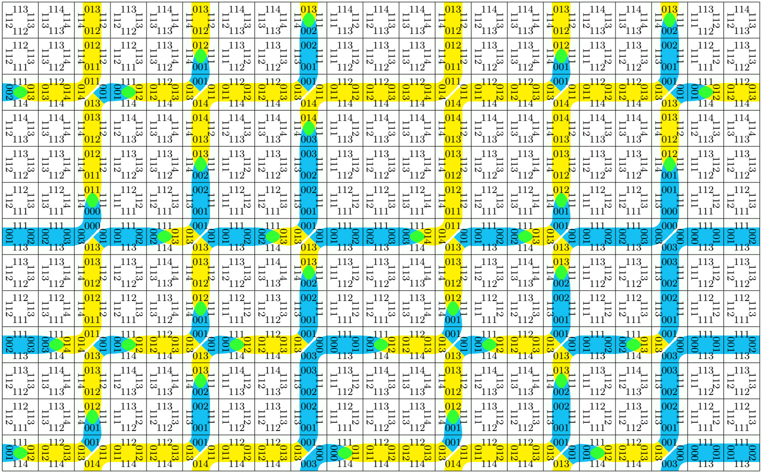
$$V_n = \{(v_0, v_1, v_2) \in \mathbb{N}^3 \colon 0 \leq v_0 \leq v_1 \leq 1 \text{ and } v_1 \leq v_2 \leq n+1\}$$

that we represent compactly as words, e.g., $113 := (1, 1, 3)$.

Metallic mean Wang shift

The ***n*-th metallic mean Wang shift** is $\mathbb{Z}^2 \overset{\sigma}{\curvearrowright} \Omega_n$ where

$$\Omega_n := \Omega_{\mathcal{T}_n} = \{w : \mathbb{Z}^2 \to \mathcal{T}_n \colon w \text{ is a valid configuration}\}.$$



(a 21×13 valid patch with $\mathcal{T}_3$)