q-analog of rational numbers: from Ostrowski numeration systems to perfect matchings   avec JC-Aval, arxiv 2511.11290

Euler $\simeq 1760$: $[n]_q = 1 + q + \cdots + q^{n-1}$

q-factorial : $[n]_q! = [1]_q \cdot [2]_q \cdots [n]_q = \sum_{\pi \in S_n} q^{inv(\pi)}$

Gauss $\simeq 1808$ q-binomial

$$\binom{n}{m}_q = \frac{[n]_q!}{[n-m]_q!\,[m]_q!} = \sum_{\substack{w \in \{0,1\}^* \\ \vec{w} = (|w|_0, |w|_1) = (m, n-m)}} q^{area(w)}$$

Sylvester (1878) : polynomials $\binom{n}{m}_q$ are unimodal (conjectured by Cayley 25 years before)

Morier-Genoud, Ovsienko (2020) : $[x]_q$ $\forall x \in \mathbb{Q}$, $\forall x \in \mathbb{R}$

31 citations + 2025 AMS David P. Robbins Prize

Gave a combinatorial interpretation for all $x \in \mathbb{Q}_{>1}$

Def $[x]_q : \mathbb{Q} \cup \{\infty = \frac{1}{0}\} \longrightarrow \mathbb{Z}(q)$ is the unique fct

s.t. $[0]_q = 0$, $[x+1]_q = q[x]_q + 1$, $[x]_q [-x^{-1}]_q = -q^{-1}$

$\boxed{EX}$ $[1]_q = q[0]_q + 1 = 1$ $\quad [\frac{-1}{2}]_q = \frac{-1}{q} \cdot [2]_q^{-1} = \frac{-1}{q^2+q}$, etc.

$[2]_q = q[1]_q + 1 = q+1$

Our contribution : 3 comb. interpretations $\forall x \in \mathbb{Q}_{>0}$

$$\begin{tikzcd}
\mathbb{Q}_{>0} \arrow[r, "CFeven"] & A_{even} \arrow[r, "w"] & \{0,1\}^* \arrow[r, "\theta"] & \{0,1\}^* \\
x \arrow[r, mapsto] \arrow[ddd, mapsto] & a \arrow[r, mapsto] \arrow[dd, mapsto, "B"] & w(x) \arrow[r, mapsto] \arrow[d, mapsto, "F"] & \theta(w(x)) \arrow[d, mapsto, "G"] \\
 & & \mathcal{F}(x) \arrow[d, mapsto, "J"] & G(x) \arrow[d, mapsto, "M"] \\
\mathbb{Z} \supseteq Z_1(a) & B(a) \arrow[l, "val"'] \arrow[dr] & J(x) \arrow[l, "\Psi"'] \arrow[d, mapsto] & M(x) \arrow[l, "\Phi"'] \arrow[dl, mapsto] \\
 & & [x]_q &
\end{tikzcd}$$

Continued fractions and fence posets

$$\frac{22}{7} = 3+\frac{1}{7} = 3+\cfrac{1}{6+\cfrac{1}{1}} = \overset{\text{even}}{[3;7]} = [3;6,1]$$

$$\frac{7}{22} = 0+\frac{1}{22/7} = 0+\cfrac{1}{3+\cfrac{1}{7}} = 0+\cfrac{1}{3+\cfrac{1}{6+\cfrac{1}{1}}} = [0;3,7] = \overset{\text{even}}{[0;3,6,1]}$$

$\forall x \in \mathbb{Q}$, let $CF_{even}(x) = [a_0; a_1, \dots, a_{2\ell-1}]$ be its even-length CF expansion $a_0 \in \mathbb{Z}$, $a_i \geq 1$, $\forall i \geq 1$.

We define
$$W : \mathbb{Q}_{>0} \longrightarrow \{0,1\}^*$$
$$x \mapsto 1^{a_0} 0^{a_1} \cdots 1^{a_{2\ell-2}} 0^{a_{2\ell-1}-1}$$

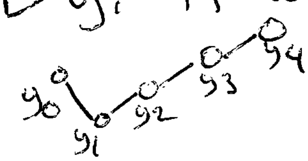
__Lemma__ $W : \mathbb{Q}_{>0} \to \{0,1\}^*$ is a bijection.

__Remark__: Si $[x]_q = \frac{R(q)}{S(q)}$ alors $\begin{pmatrix} R(q) \\ S(q) \end{pmatrix} = H_q(W(x)) \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ avec $H_q : \{0,1\}^* \to SL_2(\mathbb{Z}[$

$0 \mapsto L_q = \begin{pmatrix} q & 0 \\ q & 1 \end{pmatrix}$

$1 \mapsto R_q = \begin{pmatrix} q & 1 \\ 0 & 1 \end{pmatrix}$

$\forall$ word $w = w_1 \cdots w_n \in \{0,1\}^*$ let $F(w) = (\{y_0, \dots, y_n\}, \lhd)$ the partial order (__fence poset__) whose cover relations are
$$\begin{cases} y_{i-1} \lhd y_i & \text{if } w_i = 1 \\ y_{i-1} \rhd y_i & \text{if } w_i = 0 \end{cases}$$

$\boxed{\text{EX}}$ $F(0111) =$ 

$= F \circ W([0;1,3,1]) = \mathcal{F}\left(\frac{4}{5}\right)$

__Remark__ $F : \{0,1\}^* \longrightarrow \{\text{fence posets}\}$ is a bijection

$\mathcal{F} = F \circ W : \mathbb{Q}_{>0} \longrightarrow \{\text{fence posets}\}$ is a bijection

Let $J(x)$ be the set of __order ideals__ (downsets) of $\mathcal{F}(x)$.

Also let $J^1(x) = \{ I \in J(x) \mid y_0 \in I \}$

$J^0(x) = \{ I \in J(x) \mid y_0 \notin I \}$

$\boxed{\text{Theorem}}$
$$[x]_q = q^{-1} \frac{\sum_{I \in J^1(x)} q^{|I|}}{\sum_{I \in J^0(x)} q^{|I|}}$$

after

Ostrowski numeration system and admissible sequences

$\forall x \in \mathbb{R}/\mathbb{Q}$ has a unique CF expansion $[a_0; a_1 \dots]$ $\quad a_0 \in \mathbb{Z}$, $a_i \geq 1, i \neq 0$

Convergent: $\dfrac{p_i}{q_i} = [a_0; a_1, \dots, a_i]$ $\quad i \geq 0$

$p_i = a_i p_{i-1} + p_{i-2} \qquad p_{-1} = 1 \qquad p_0 = a_0$

$q_i = a_i q_{i-1} + q_{i-2} \qquad q_{-1} = 0 \qquad q_0 = 1$

Thm (Ostrowski, 1921) $\forall n \geq 0 \ \exists! \ k \geq 0 \ \exists! \ (b_i)_{i=1}^{k}$

s.t. $\quad n = \sum_{i=1}^{k} b_i q_{i-1}$

where $0 \leq b_i \leq a_i$, $b_1 \neq a_1$, $\quad b_i = a_i \Rightarrow b_{i-1} = 0$.

$\boxed{\text{DEF}}$ Soit $a = [a_0; a_1, \dots, a_{k-1}]$ with $a_0 \geq 0$, $a_i \geq 1$, $i \geq 1$

A sequence $(b_i)_{i=0}^{k-1}$ is admissible for a if

- $0 \leq b_i \leq a_i \quad \forall i$
- If $i > 0$ odd and $b_i = a_i$, then $b_{i-1} = a_{i-1}$
- if $i > 0$ even and $b_i = 0$, then $b_{i-1} = 0$

Let $\mathcal{B}(a) = \{ (b_i)_{i=0}^{k-1}$ admissible for $a \}$

Let $(r_i)_{i=0}^{k}$ st $r_i = p_{i-1} + q_{i-1}$

$\boxed{\text{Thm}}$ Let $\mathcal{Z}(a) = \begin{cases} \mathbb{Z} \cap [0, r_k) & k \text{ odd} \\ \mathbb{Z} \cap [r_{k-1} - r_k, r_{k-1}) & k \text{ even} \end{cases}$

The function $\mathrm{val}_a : \mathcal{B}(a) \longrightarrow \mathcal{Z}(a)$ is a bijection.

$$(b_i)_{i=0}^{k-1} \longmapsto \sum_{i=0}^{k-1} (-1)^i b_i r_i$$

Disjoint union $\mathcal{B}(a) = \mathcal{B}^{\bullet}(a) \cup \mathcal{B}^{\circ}(a)$

$\mathcal{B}^{\bullet}(a) = \{ (b_i)_{i=0}^{k-1} \in \mathcal{B}(a) \mid b_0 > 0 \text{ or } a_0 = 0 < b_1 = a_1 \}$

$\mathcal{B}^{\circ}(a) = \{ \quad \text{—}\|\text{—} \quad \mid b_0 = 0 < a_0 \text{ or } a_0 = 0 \leq b_1 < a_1 \}$

$\boxed{\text{Thm}}$ $$[x]_q = q^{-1} \frac{\sum_{b \in \mathcal{B}^{\bullet}(a)} q^{\|b\|_1}}{\sum_{b \in \mathcal{B}^{\circ}(a)} q^{\|b\|_1}} \quad \text{où } a = CF_{\text{even}}(x)$$