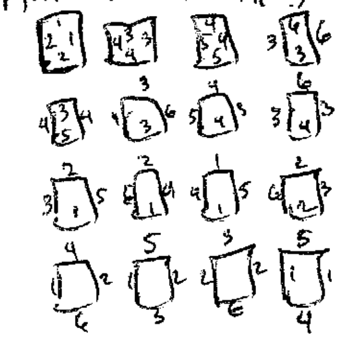
U. Maryland talk 2° partie PLan

Ammann Ke tiles



Goal: Understand and generalize them

$R: \mathbb{Z}^2 \times M \to M$, M compact metric space (today $M = \mathbb{R}^2/\mathbb{Z}^2$)

$\mathcal{P} = \{P_a\}_{a \in A}$ topological partition of M
ie P_a are open disjoint and $M = \bigcup_{a \in A} \overline{P_a}$.

$S \subset \mathbb{Z}^2$, a pattern $w: S \to A$ is allowed for $\mathcal{P}, R$
if $\text{CodingRegion}(w) := \bigcap_{n \in S} R^{-n}(P_{w_n}) \neq \emptyset$

Language $\mathcal{L}_{\mathcal{P},R} = \{w \in A^S : w \text{ allowed for } \mathcal{P}, R\}$

Subshift $\mathcal{X}_{\mathcal{P},R} = \{w \in A^{\mathbb{Z}^2} : \mathcal{L}(w) \subset \mathcal{L}_{\mathcal{P},R}\}$

Def $\mathcal{P}$ gives a symbolic representation of R if $\forall w \in A^{\mathbb{Z}^2}$
$\text{Coding Region}(w) := \bigcap_{n \in \mathbb{Z}^2} R^{-n}(\overline{P_{w_n}})$ contains at most 1 point.

Lemma (Lind Marcus Book) If $\mathcal{P}$ gives a symb. rep. then
$f: \mathcal{X}_{\mathcal{P},R} \to M$
$w \mapsto x \in$
is a factor map

$$\begin{tikzcd} \mathcal{X}_{\mathcal{P},R} \arrow[r, "\sigma"] \arrow[d, "f"'] & \mathcal{X}_{\mathcal{P},R} \arrow[d, "f"] \\ M \arrow[r, "R"] & M \end{tikzcd}$$

Def (LM) $\mathcal{P}$ is a Markov Partition if
- $\mathcal{P}$ gives a symb. rep.
- $\mathcal{X}_{\mathcal{P},R}$ is a SFT, ie. $\exists$ finite set F of patterns s.t. $\mathcal{X}_{\mathcal{P},R} = \mathcal{X}_F := \{w \in A^{\mathbb{Z}^2} :$ $\mathcal{L}(w) \cap F = \emptyset$

Example The Fibonacci crystals wrapped around the cylinder.

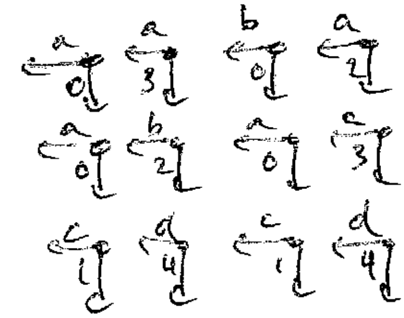
Let $\mathcal{Y} = \{y_v\}_{v\in V}$, $\mathcal{Z} = \{z_h\}_{h\in H}$ be two top partitions of $M = \mathbb{R}^2/\mathbb{Z}^2$. Let $R : \mathbb{Z}^2 \times M \to M$.

$$(n, x) \mapsto x + \varphi n := \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \varphi & 0 \\ 0 & \varphi \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$$

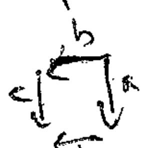
$$\mathcal{Y} \vee \mathcal{Z} \vee R^{e_1}(\mathcal{Y}) \vee R^{e_2}(\mathcal{Z})$$

$V = \{0,1,2,3,4\}$ $\quad$ $H = \{a,b,c,d\}$

0 3 0 2 0 $\quad$ aabaa
0 2 0 3 0 $\quad$ abaaa $\quad \longmapsto$
1 4 1 4 4 $\quad$ cdcdd
$\in \mathcal{L}_{\mathcal{Y},R}$ $\quad$ $\in \mathcal{L}_{\mathcal{Z},R}$



This a tiling by Wang tiles! Which tiles?



$\forall (a,b,c,d) \in V \times H \times V \times H$ let $P_{(a,b,c,d)} = y_a \cap z_b \cap R^{e_1}(y_c) \cap R^{e_2}(z_d)$

Set of Wang tiles: $T = \{ \text{tile}(a,b,c,d) : P_{(a,b,c,d)} \neq \phi \}$

Top. partition : $\mathcal{P} = \{ P_t : t \in T \}$

A Wang shift: $\Omega_T = \{ w : \mathbb{Z}^2 \to T : w \text{ valid tiling} \}$

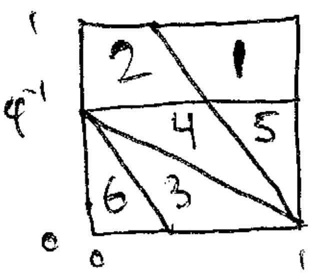
<u>Proposition</u> $\mathcal{X}_{\mathcal{P},R} \subseteq \Omega_T$

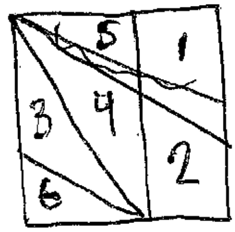
<u>Remark</u> If Ω_T is minimal, then $\mathcal{X}_{\mathcal{P},R} = \Omega_T$ and $\mathcal{P}$ is a Markov partition for $\mathbb{Z}^2 \overset{R}{\curvearrowright} M$.

<u>Def</u> X subshift, X is <u>minimal</u> if $\forall$ subshift $Y \neq \phi$

$$Y \subseteq X \Rightarrow Y = X.$$

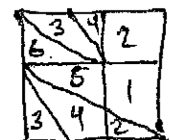
$\mathcal{Y}$ $\qquad$ $\mathcal{Z}$





$$R : \mathbb{Z}^2 \times M \to M$$
$$(n, x) \mapsto x + \varphi n = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \varphi & 0 \\ 0 & \varphi \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$$

$$\mathcal{P} = \underbrace{\mathcal{Y}}_{\text{(east)}} \vee \underbrace{\mathcal{Z}}_{\text{(north)}} \vee \underbrace{R^{e_1}(\mathcal{Y})}_{\text{(west)}} \vee \underbrace{R^{e_2}(\mathcal{Z})}_{\text{(south)}}$$
$$= \{P_t\}_{t \in T}$$



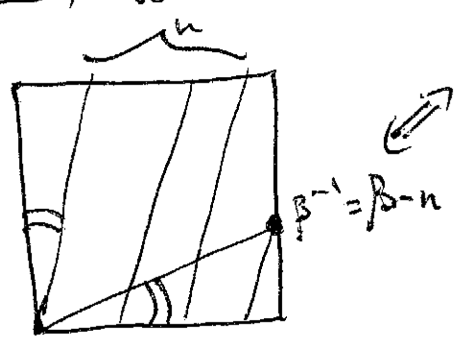
Theorem ~~Lemma~~

- T is the Ammann set of 16 Wang tiles.
- $\mathcal{X}_{\mathcal{P}, R} \subset \Omega_T$
- Ω_T is minimal, thus $\mathcal{X}_{\mathcal{P}, R} = \Omega_T$
- $\mathcal{P}$ is a Markov partition for R.

It works for all metallic mean.

$$\beta^2 = n\beta + 1 \iff \beta \text{ root of } x^2 - nx - 1 = 0.$$
$$\iff \beta = n + \cfrac{1}{n + \cfrac{1}{n + \cfrac{1}{n + \dots}}}$$

$n = 1 \Rightarrow \beta$ golden mean
$n = 2 \Rightarrow \beta$ silver mean
etc



$\beta^{-1} = \beta - n$

For $\mathcal{Y}$:
$n = 3$

recovering
Amann tiles