Tuiles de Wang apériodiques associées aux nombres métalliques

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- Visiting Scholars program (for sabbatical)

  2019-2020 : Casey Mann & Jennifer McLoud-Mann, University of Seattle
- Cotutelle doctoral program : *Joint supervision of a PhD by the University of Bordeaux and an international university (1 thesis defense, 2 PhD diplomas)*
- Institut de Mathématiques de Bordeaux (IMB) &
  Laboratoire Bordelais de Recherche en Informatique (LaBRI)
- Opportunities for research between Canada and France

  https://canadafrance.pages.math.cnrs.fr/

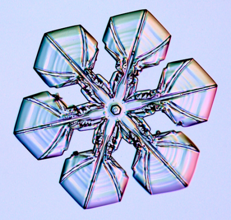
Outline

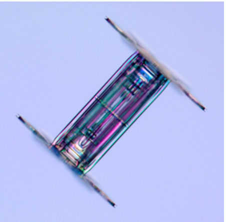
Outline

Motivation

Understand global structure emerging from local rules.

Snow crystals may have many shapes (`https://www.snowcrystals.com/`) :



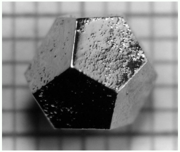
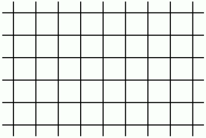
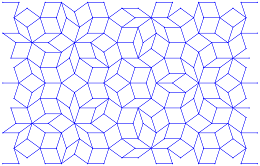



K. G. Libbrecht. Snow Crystals : A Case Study in Spontaneous Structure Formation. Princeton University Press, Dec. 2021. `doi:10.2307/j.ctv1qdqztv`

*Veritasium, **The Snowflake Myth**,* `https://youtu.be/ao2Jfm35XeE`*, 2022*

Crystallography

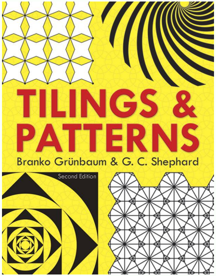
1982 (Shechtman) : observed that aluminium-manganese alloys produced a **quasicrystals structure**.

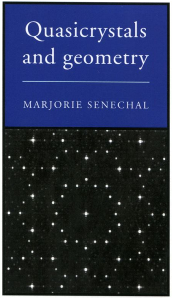
	Pyrite (FeS_2)	A Ho-Mg-Zn quasicrystal
crystals :		
atomic structure :		
	(a square grid)	(a Penrose tiling, 1976)

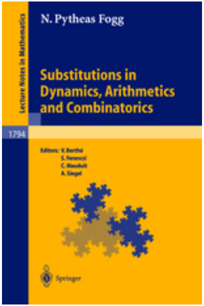
Shechtman received the 2011 **Nobel Prize** in Chemistry :

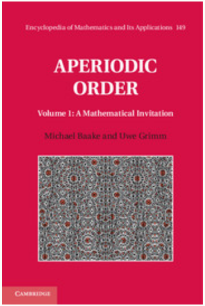
His discovery of quasicrystals revealed a new principle for packing of atoms and molecules [that] *led to a* ***paradigm shift*** *within chemistry*.

Books





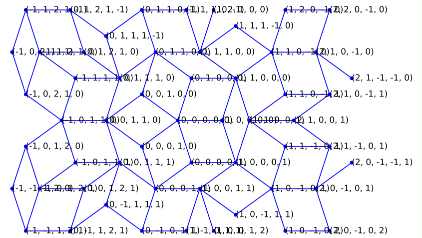




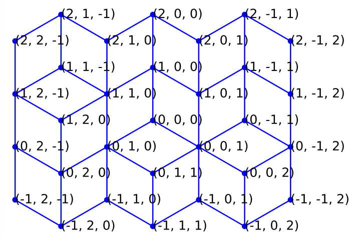
- Tilings and Patterns, by Grünbaum & Shephard, 1987
- Quasicrystals and Geometry, Senechal, 1995
- Pytheas Fogg's book, 2002
- Aperiodic Order, Baake & Grimm, 2013

Cut and project schemes

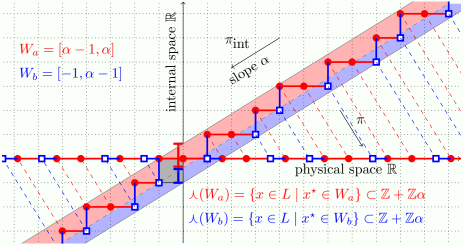
5-to-2



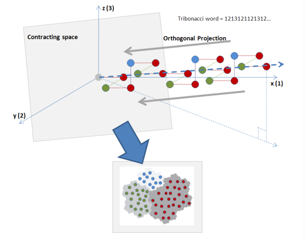
3-to-2



2-to-1



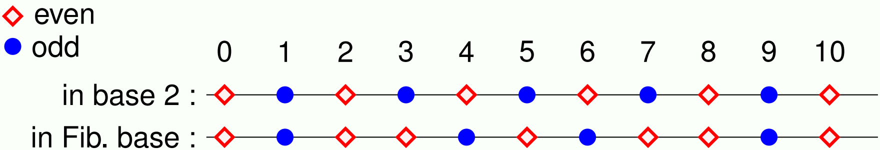
3-to-1



N. G. de Bruijn. "Algebraic theory of Penrose's nonperiodic tilings of the plane. I, II". (1981); *Meyer (1972)*, *Lagarias (1996)*, *Moody (1997)*.

Outline

One-dimensional crystallography

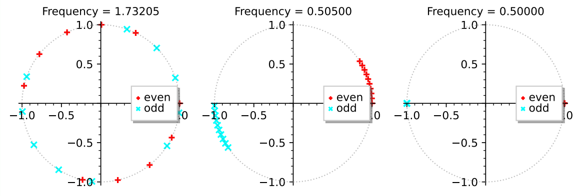


n	$\mathrm{rep}_2(n)$	parity
0=0	`0`	even
1=1	`1`	odd
2=2	`10`	even
3=2+1	`11`	odd
4=4	`100`	even
5=4+1	`101`	odd
6=4+2	`110`	even
7=4+2+1	`111`	odd
8=8	`1000`	even
9=8+1	`1001`	odd
10=8+2	`1010`	even
11=8+2+1	`1011`	odd
12=8+4	`1100`	even
13=8+4+1	`1101`	odd

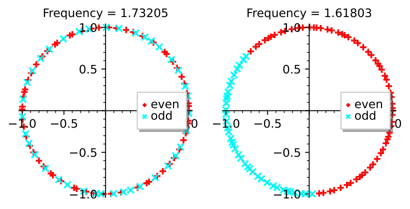
n	$\mathrm{rep}_F(n)$	parity
0=0	`0`	even
1=1	`1`	odd
2=2	`10`	even
3=3	`100`	even
4=3+1	`101`	odd
5=5	`1000`	even
6=5+1	`1001`	odd
7=5+2	`1010`	even
8=8	`10000`	even
9=8+1	`10001`	odd
10=8+2	`10010`	even
11=8+3	`10100`	even
12=8+3+1	`10101`	odd
13=13	`100000`	even

Guessing a frequency (rotation angle)

The odd/even in base 2 has frequency $\frac{1}{2}$



The odd/even in Fibonacci base has frequency $\frac{1}{2}(1+\sqrt{5}) \approx 1.618$:



Remark : there exists $\rho, \alpha \in \mathbb{R}$ such that a point $x \in [0,1]$ is in the red/blue part according to the value $\lfloor x+\alpha+\rho \rfloor - \lfloor x+\rho \rfloor \in \{0,1\}$.

Sturmian sequences

slope $\alpha \in [0,1]$, intercept $\rho \in \mathbb{R}$,

lower mechanical sequence :

$$s_{\alpha,\rho}(n) = \lfloor \alpha(n+1) + \rho \rfloor - \lfloor \alpha n + \rho \rfloor,$$

upper mechanical sequence :

$$s'_{\alpha,\rho}(n) = \lceil \alpha(n+1) + \rho \rceil - \lceil \alpha n + \rho \rceil.$$

Factor (pattern) complexity :

$x = \cdots 1010010100100101001010 0101 \cdots$

n	$\mathcal{L}_n(x)$	$\#\mathcal{L}_n(x)$
0	ε	1
1	0, 1	2
2	00, 01, 10	3
3	001, 010, 100, 101	4
4	0010, 0100, 0101, 1001, 1010	5

Theorem (Morse, Hedlund, 1940 & Coven, Hedlund, 1970)

Let $w \in \{0,1\}^{\mathbb{Z}}$ be a non-ultimately periodic sequence.
There exists $\alpha \in [0,1] \setminus \mathbb{Q}$ and $\rho \in [0,1)$ s.t. $w = s_{\alpha,\rho}$ or $w = s'_{\alpha,\rho}$

if and only if

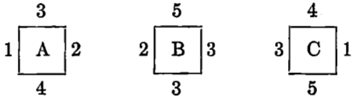
the sequence w **has factor complexity** $n+1$.

Proof ($\Longrightarrow$) : Easy part. ($\Longleftarrow$) : Harder. Desubstitute + Rauzy induction + Continued fractions + Ostrowki num. syst.

P. Arnoux. Sturmian sequences. In : Substitutions in dynamics, arithmetics and combinatorics. 2002, pp. 143-198. doi:10.1007/3-540-45714-3_6

Outline

Wang tiles



Then we can easily find an infinite solution by the following argument. The following configuration satisfies the constraint on the edges:

A	*B*	*C*
C	*A*	*B*
B	*C*	*A*

Now the colors on the periphery of the above block are seen to be the following:

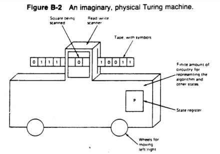
```
    3 5 4
1           1
3           3
2           2
    3 5 4
```

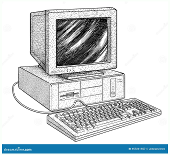
Wang's original question : is it true that a set of Wang tiles tile the plane if and only if there exists such a cyclic rectangle ?

H. Wang. Proving theorems by pattern recognition – II. Bell System Technical Journal*, 40(1) :1–41, January 1961.* doi:10.1002/j.1538-7305.1961.tb03975.x

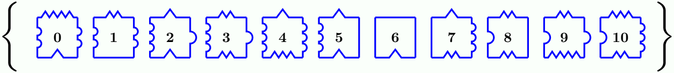
Turing machine reduction to Wang tiles

Berger (1966) : For every Turing machine





there exists a set of Wang tiles



, *e.g.*,

that tiles the plane if and only if the Turing machine does not halt.

- The **domino problem is undecidable** : there exist no algorithm that says whether a finite set of Wang tiles can tile the plane.
- There **exists an aperiodic set** of Wang tiles (*a tile set is **aperiodic** if it tiles the plane, but none of the tilings is periodic*).
- Valid Wang tilings are **computing** something.

Aperiodic Wang tile sets

Aperiodic sets of Wang tiles

Positive entropy

Matching rules satisfy arithmetic Equations

- 14 tiles : **Kari** (1996)
- 13 tiles : Culik (1996)
- and their extensions [ENP07]

Substitutive

- 104 : Berger (1966)
- 92 : Knuth (1968)
- 56 : Robinson (1971)
- 16 : **Ammann** (1971)
- 11 : **Jeandel-Rao** (2015)

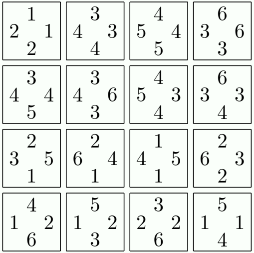
Theorem (Jeandel, Rao, 2015)

All sets of ≤ 10 Wang tiles are **periodic** or **don't tile** the plane.

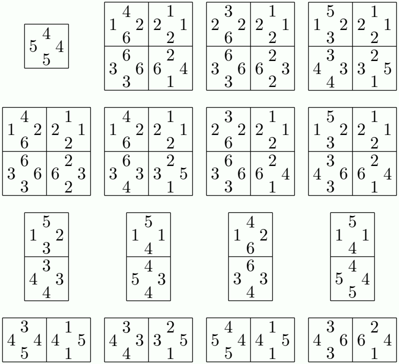
Emmanuel Jeandel and Michaël Rao. An aperiodic set of 11 Wang tiles.
Adv. Comb. ***37*** *(2021) Id/No 1.*

Ammann 16 Wang tiles are self-similar

16 tiles :



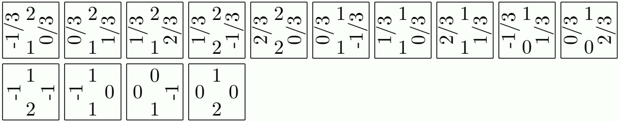
16 equivalent supertiles :

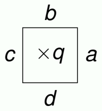


Branko Grünbaum and G. C. Shephard. Tilings and patterns. *W. H. Freeman and Company, New York, 1987.*

M. Senechal. The mysterious Mr. Ammann. Math. Intelligencer*, 26(4) :10–21, 2004.* doi:10.1007/BF02985414

Kari's 14 Wang tiles computing $\times\frac{2}{3}$ and $\times 2$



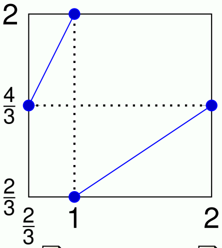


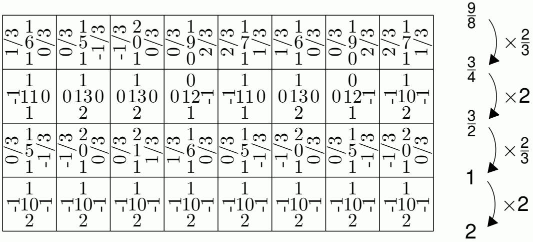
$$\iff qb + c = d + a$$

with $q \in \{\frac{2}{3}, 2\}$

$$g(x) = \begin{cases} 2x & \text{if } x \le 1, \\ \frac{2}{3}x & \text{if } x > 1. \end{cases}$$

Averages of horizontal labels are orbits of g :





Kari (1996) Durand, Gamard, Grandjean (2007) Kari (2016)

Jeandel-Rao's set of 11 Wang tiles

Theorem (Jeandel, Rao, 2015)

The following set of 11 Wang tiles is **aperiodic** :

A geometrical encoding of the Jeandel-Rao tiles and its Wang shift :

$\mathcal{T}_0 = \Bigg\{$  $\Bigg\}$

$$\Omega_0 := \Omega_{\mathcal{T}_0} := \left\{ w : \mathbb{Z}^2 \to \{0, 1, \ldots, 10\} \,\middle|\, w \text{ is a valid tiling with } \mathcal{T}_0 \right\}$$

on which the shift $\mathbb{Z}^2 \overset{\sigma}{\curvearrowright} \Omega_0$ acts naturally as

$$\begin{array}{rccl} \sigma : & \mathbb{Z}^2 \times \Omega_0 & \to & \Omega_0 \\ & (\mathbf{k}, w) & \mapsto & \sigma^{\mathbf{k}}(w) := (\boldsymbol{n} \mapsto w_{\boldsymbol{n}+\mathbf{k}}). \end{array}$$

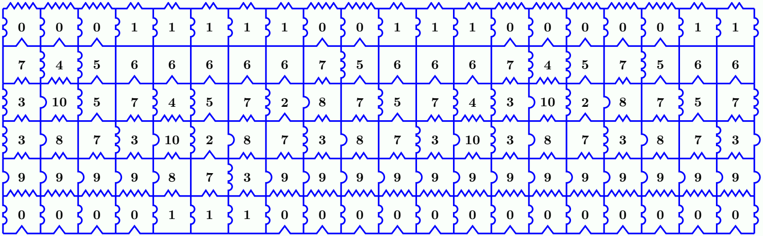
Question : what is $\mathbb{Z}^2 \overset{\sigma}{\curvearrowright} \Omega_0$ computing ?

Outline

Downloading a 100×100 patch

```
sage: # using "https://members.loria.fr/EJeandel/research/100.txt"
sage: with open("Data/100.txt", "r") as f: content = f.read()
sage: J = [line for line in content.splitlines()]
sage: len(J), len(J[0])
(100, 100)
sage: J[0][:70]
0001111100111000001110001100000111000111110001100000111000111110011100
sage: J[1][:70]
7456666675666745756667456674575666745666667456674575666745666667566674
sage: J[2][:70]
3a574572875743a2875743a5743a2875743a5745743a5743a2875743a574572875743a
```

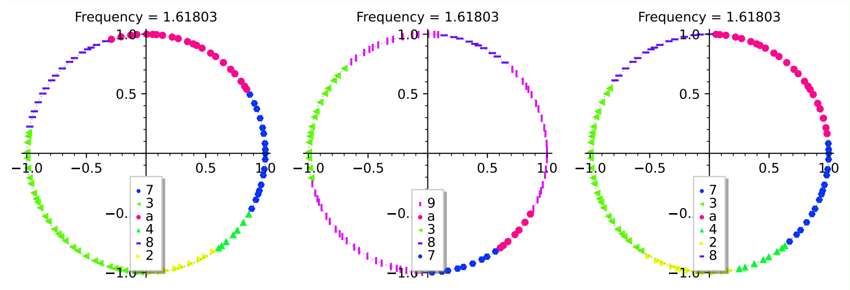
The first 6 rows (limited to 20 columns) are :



Wrapping the rows on a circle

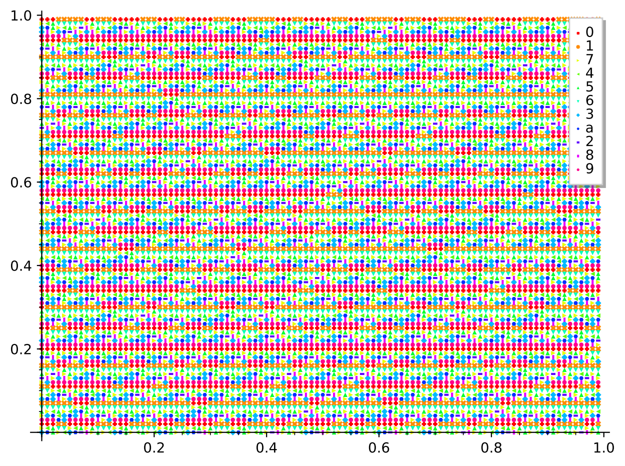
```
sage: J[35][:70]
73a43a3873a2873a43a3873a2873873a2873a43a3873a2873a43a2873a2873873a2873
sage: J[36][:70]
999a399999873999a39999987399999873999a399999873999a3873998739999987399
sage: J[58][:70]
73a43a2873a43a3873a2873a43a2873a2873873a2873a43a2873a2873a43a2873a43a3
```

Wrapping these rows on a circle using the golden ratio frequency gives :



Experiment (step 1)

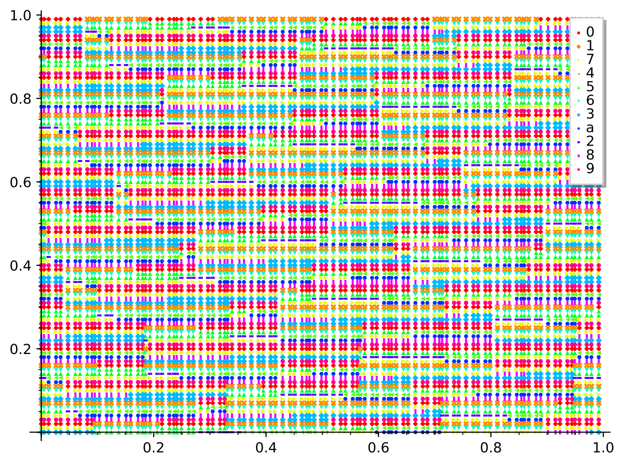
Wrapping on the 2-torus with frequency $\begin{pmatrix} 100 & 0 \\ 0 & 100 \end{pmatrix}^{-1}$:



Using frequency $\frac{1}{100}$ horizontally and vertically is a trick to make the points represent the tiling itself.

Experiment (step 2)

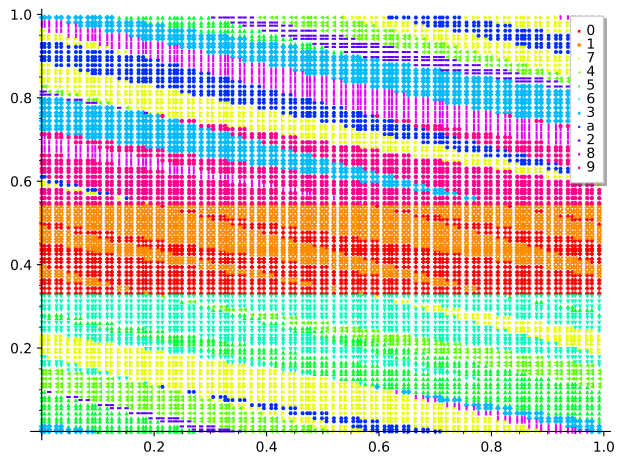
Wrapping on the 2-torus with frequency $\begin{pmatrix} \varphi & 0 \\ 0 & 100 \end{pmatrix}^{-1}$:



This makes each row in the patch to wrap around a circle (shown horizontally on the image below) with golden mean frequency.

Experiment (step 3)

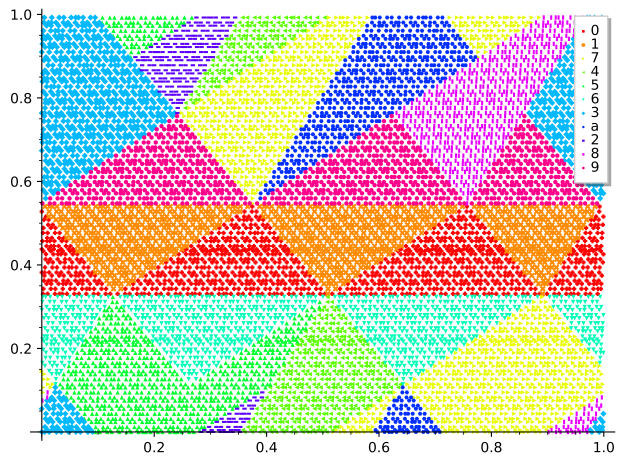
Wrapping on the 2-torus with frequency $\begin{pmatrix} \varphi & 0 \\ 0 & \varphi+3 \end{pmatrix}^{-1}$.



This makes sense because the vertical distance (or return time) between rows involving tiles labeled #0 and #1 is 4 or 5 with an average of $\varphi + 3$ as noticed already by Jeandel and Rao.

Experiment (step 4)

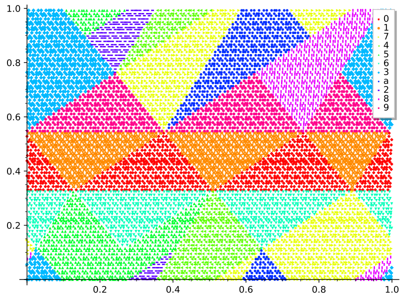
Wrapping on the 2-torus with frequency $\begin{pmatrix} \varphi & 1 \\ 0 & \varphi+3 \end{pmatrix}^{-1}$.



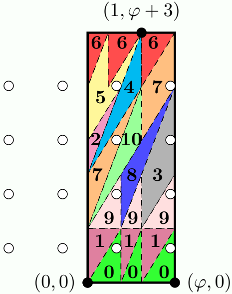
A shear is happening in Jeandel-Rao tilings.
This is one of the reasons that makes the description of Jeandel-Rao tilings more difficult, but certainly very interesting !

Rescaling to get $\mathbb{Z}^2$-action R_0 and partition $\mathcal{P}_0$

Step 4 of the experiment	Rescaling
$\mathbb{Z}^2 \curvearrowright \mathbb{R}^2/\mathbb{Z}^2$	$\mathbb{Z}^2 \overset{R_0}{\curvearrowright} \mathbb{R}^2/\begin{pmatrix}\varphi & 1\\ 0 & \varphi+3\end{pmatrix}\mathbb{Z}^2$
$(\mathbf{k},\mathbf{x}) \mapsto \mathbf{x} + \begin{pmatrix}\varphi & 1\\ 0 & \varphi+3\end{pmatrix}^{-1}\mathbf{k}.$	$R_0 : (\mathbf{k},\mathbf{x}) \mapsto \mathbf{x} + \mathbf{k}.$



$\mathcal{P}_0 =$

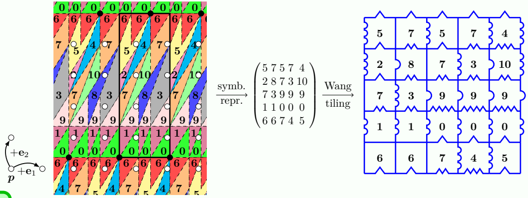


$$\begin{pmatrix}5 & 7\\ 7 & 3\\ 9 & 9\\ 0 & 0\end{pmatrix} \in \mathcal{L}_{\mathcal{P}_0,R_0} = \left\{ w : S \to \mathcal{A} \,\middle|\, S \subset \mathbb{Z}^2 \text{ and } w \text{ is } \textbf{allowed} \right\}$$

Jeandel–Rao aperiodic set of 11 Wang tiles

Coding $\mathbb{Z}^2 \overset{R_0}{\curvearrowright} \mathbb{R}^2/\Gamma_0$ by partition $\mathcal{P}_0$ defines a **symb. dyn. system** :

$$\mathcal{X}_{\mathcal{P}_0,R_0} = \left\{ w : \mathbb{Z}^2 \to \{0,1,\ldots,10\} \,\middle|\, \mathcal{L}(w) \subset \mathcal{L}_{\mathcal{P}_0,R_0} \right\}.$$



Theorem

- $\mathcal{X}_{\mathcal{P}_0,R_0}$ is a **minimal, aperiodic and uniquely ergodic** subshift of the Jeandel-Rao Wang shift, i.e., $\mathcal{X}_{\mathcal{P}_0,R_0} \subset \Omega_0$.
- Occurences of patterns in $\mathcal{X}_{\mathcal{P}_0,R_0}$ is a **4-to-2 C&P set**.

A Wang shift $\Omega_{\mathcal{T}}$ is **minimal** if every orbit by the shift is dense in Ω.

Markov partitions for toral $\mathbb{Z}^2$-rotations featuring Jeandel-Rao Wang shift and model sets. Ann. H. Lebesgue *4 (2021) 283–324.* doi:10.5802/ahl.73

Substitutive structure of Ω_0 and $\mathcal{X}_{\mathcal{P}_0,R_0}$

Using **algorithms** `FindMarkers` and `FindSubstitution` :

$$
\begin{array}{ccccccccccccccccc}
\Omega_0 & \xleftarrow{\omega_0} & \Omega_1 & \xleftarrow{\omega_1} & \Omega_2 & \xleftarrow{\omega_2} & \Omega_3 & \xleftarrow{\omega_3} & \Omega_4 & & & & & & & & \\
\cup & & \cup & & \cup & & \cup & & \cup & & & & & & & & \circlearrowleft \omega_{\mathcal{U}} \\
X_0 & \xleftarrow{\omega_0} & X_1 & \xleftarrow{\omega_1} & X_2 & \xleftarrow{\omega_2} & X_3 & \xleftarrow{\omega_3} & X_4 & \xleftarrow{\pi} & \Omega_5 & \xleftarrow{\eta} & \Omega_6 & \xleftarrow{\omega_6} & \Omega_7 & \xleftarrow{\omega_7\omega_8\omega_9\omega_{10}\omega_{11}} \Omega_{12} \xleftarrow{\rho} & \Omega_{\mathcal{U}}
\end{array}
$$

Using **algorithms** `induced_partition` and `induced_transformation` :

$$
\mathcal{X}_{\mathcal{P}_0,R_0} \xleftarrow{\beta_0} \mathcal{X}_{\mathcal{P}_1,R_1} \xleftarrow{\beta_1} \mathcal{X}_{\mathcal{P}_2,R_2} \xleftarrow{\beta_2} \mathcal{X}_{\mathcal{P}_3,R_3} \xleftarrow{\beta_3} \mathcal{X}_{\mathcal{P}_4,R_4} \xleftarrow{\beta_4} \mathcal{X}_{\mathcal{P}_5,R_5} \xleftarrow{\beta_5} \mathcal{X}_{\mathcal{P}_6,R_6} \xleftarrow{\beta_6} \mathcal{X}_{\mathcal{P}_7,R_7} \xleftarrow{\beta_7} \mathcal{X}_{\mathcal{P}_8,R_8} \xleftarrow{\rho} \mathcal{X}_{\mathcal{P}_{\mathcal{U}},R_{\mathcal{U}}} \circlearrowleft \omega_{\mathcal{U}}
$$

Theorem

$X_0 \subsetneq \Omega_0$ and $\mathcal{X}_{\mathcal{P}_0,R_0}$ have **the same** substitutive structure :

$$\omega_0\omega_1\omega_2\omega_3 = \beta_0, \quad \pi\eta\omega_6 = \beta_1\beta_2, \quad \omega_7\omega_8\omega_9\omega_{10}\omega_{11} = \beta_3\beta_4\beta_5\beta_6\beta_7$$

thus are **equal**.

Substitutive structure of Jeandel-Rao aperiodic tilings.
Discrete Comput. Geom., *65 (2021) 800–855.* doi:10.1007/s00454-019-00153-3

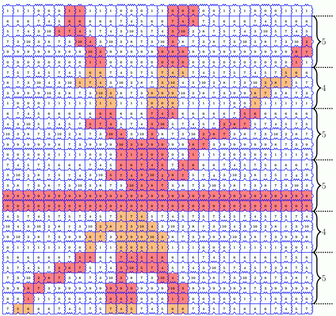
Rauzy induction of polygon partitions and toral $\mathbb{Z}^2$-rotations
Journal of Modern Dynamics 17 (2021) 481–528. doi:10.3934/jmd.2021017

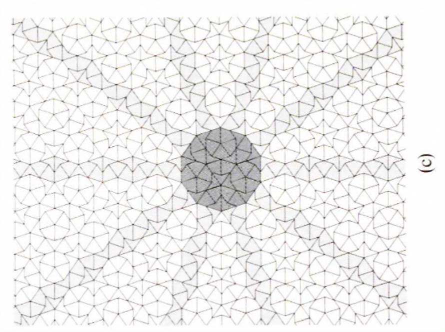
4 slopes of Conway worms in Jeandel-Rao WS

Theorem (L., Mann, McLoud-Mann, 2023)

The minimal subshift X_0 of the Jeandel-Rao Wang shift contains exactly **4 nonexpansive directions** whose slopes are

$$\left\{0, \quad \varphi + 3, \quad -3\varphi + 2, \quad -\varphi + \frac{5}{2}\right\}.$$





It reminds of the cartwheel tiling in the context of Penrose tilings.

Outline

Golden mean omnipresence

Golden mean (ratio) $\varphi = \frac{1+\sqrt{5}}{2}$ is everywhere !

- **Penrose** tilings : two-tile frequency ratio is $\varphi = \frac{1+\sqrt{5}}{2}$.
- Patch frequencies in **Jeandel-Rao** tilings are in $\mathbb{Q}[\varphi]$.
- Patch frequencies in **Ammann** tilings are in $\mathbb{Q}[\varphi]$.
- In tilings with **David Smith's aperiodic monotile** (the **hat**), the frequency of  versus its mirror image  is φ^4.

In fact, not really :

D. Frettlöh. More inflation tilings. In Aperiodic order. Vol. 2*, volume 166 of* Encyclopedia Math. Appl.*, pages 1–37. Cambridge Univ. Press, Cambridge, 2017.*

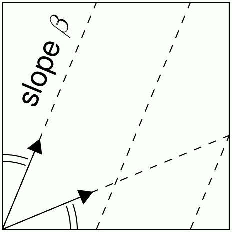
D. Frettlöh, A. Garber, and N. Mañibo. Substitution tilings with transcendental inflation factor, December 2022. `arXiv:2208.01327`

but still, golden mean is very **omnipresent** in aperiodic tilings.

Metallic means

Definition

The ***n*-th metallic mean** is the positive root of $x^2 - nx - 1$.



$$\beta = \frac{n + \sqrt{n^2+4}}{2} = n + \cfrac{1}{n + \cfrac{1}{n + \cfrac{1}{\cdots}}}$$

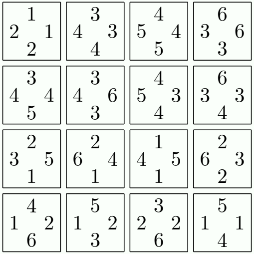
https://oeis.org/wiki/Metallic_means

V. W. de Spinadel. The family of metallic means. Vis. Math., *1(3) :1 HTML document ; approx. 16, 1999.*

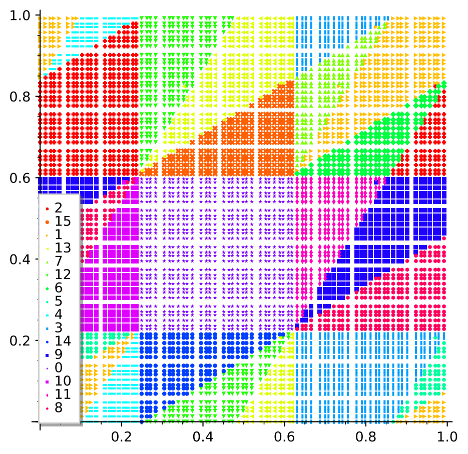
*Also called **silver means** (Schroeder 1991) or **noble means** (Baake, Grimm, 2013).*

Wrapping Ammann tilings

Wrapping Ammann tilings on the 2-torus with frequency $\begin{pmatrix} \varphi & 0 \\ 0 & \varphi \end{pmatrix}^{-1}$:

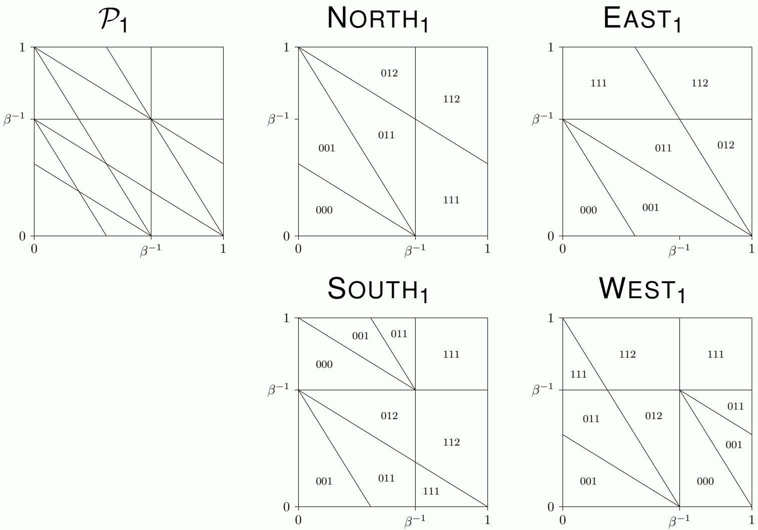


gives



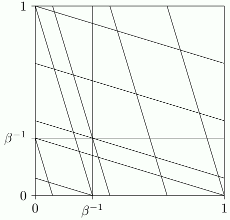
Understanding the Ammann partition

Unfold the partition into 4 partitions, one for each tile label :

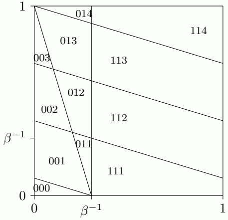


Replace golden ratio by other metallic mean

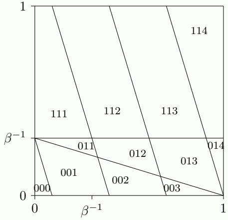
$\mathcal{P}_3$



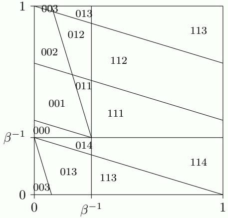
NORTH$_3$



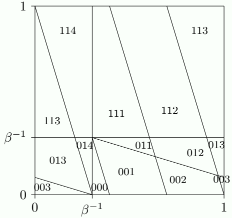
EAST$_3$



SOUTH$_3$



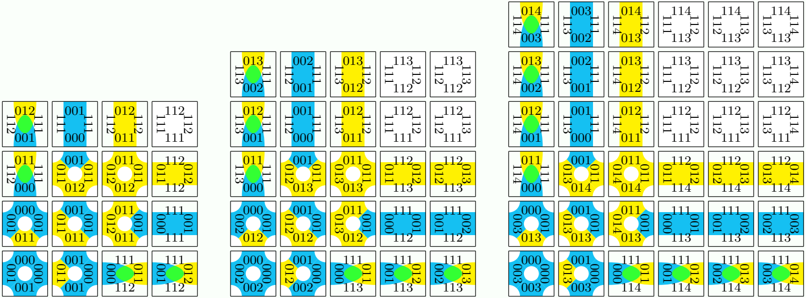
WEST$_3$



Works for all metallic means : Golden, Silver, Bronze, Copper, Nickel, etc.

A family $\{\mathcal{T}_n\}_{n\geq 1}$ of metallic mean Wang tiles

For every integer $n \geq 1$, $\mathcal{T}_n$ is made of n^2 white tiles, $2n$ blue stripe tiles, $2n$ yellow stripe tiles, $2(n+1)$ green tiles and 7 junction tiles. Total : $n^2+6n+9 = (n+3)^2$ tiles.



$\mathcal{T}_1 \equiv$ **Ammann** $\qquad \mathcal{T}_2 \qquad \mathcal{T}_3$

Tile labels are vectors in the finite set

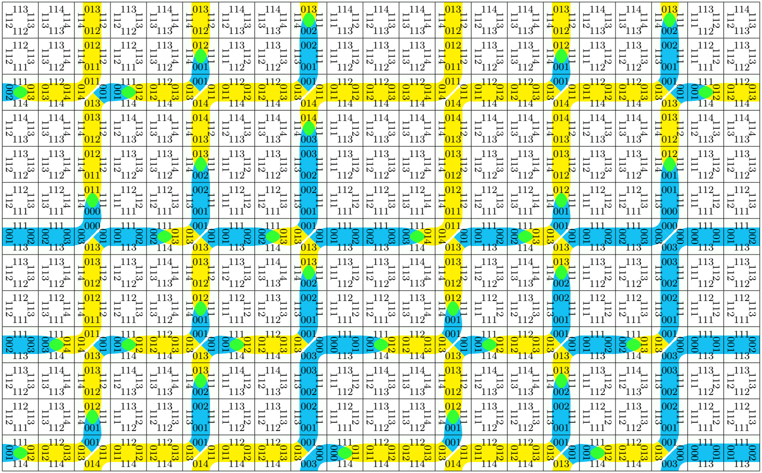
$$V_n = \{(v_0, v_1, v_2) \in \mathbb{N}^3 : 0 \leq v_0 \leq v_1 \leq 1 \text{ and } v_1 \leq v_2 \leq n+1\}$$

that we represent compactly as words, e.g., $113 := (1, 1, 3)$.

Metallic mean Wang shift

The ***n*-th metallic mean Wang shift** is $\mathbb{Z}^2 \overset{\sigma}{\curvearrowright} \Omega_n$ where

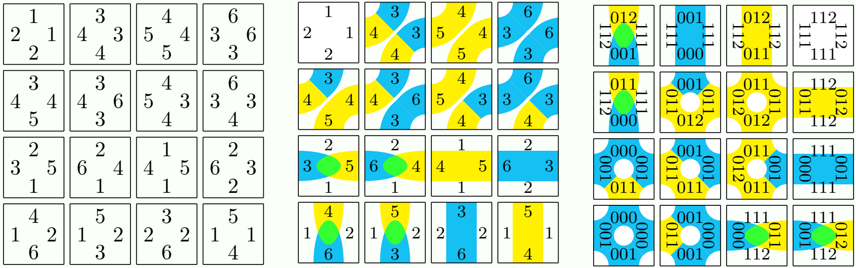
$$\Omega_n := \Omega_{\mathcal{T}_n} = \{w : \mathbb{Z}^2 \to \mathcal{T}_n \colon w \text{ is a valid configuration}\}.$$



(a 21×13 valid patch with $\mathcal{T}_3$)

Outline

Ammann $\equiv \mathcal{T}_1$



Ammann $\qquad\qquad$ $\mathcal{T}_1$

Theorem

The Ammann set of 16 Wang tiles is equivalent to $\mathcal{T}_1$.

Proof : the bijection of the tile labels is

$$1 \mapsto 112, \quad 2 \mapsto 111, \quad 3 \mapsto 001, \quad 4 \mapsto 011, \quad 5 \mapsto 012, \quad 6 \mapsto 000.$$

Self-similarity, aperiodicity and minimality

Theorem

For every integer $n \geq 1$, the metallic mean Wang shift Ω_n is

- **self-similar**,
- **aperiodic** and
- **minimal** (if X subshift and $\varnothing \neq X \subset \Omega_n$, then $X = \Omega_n$).

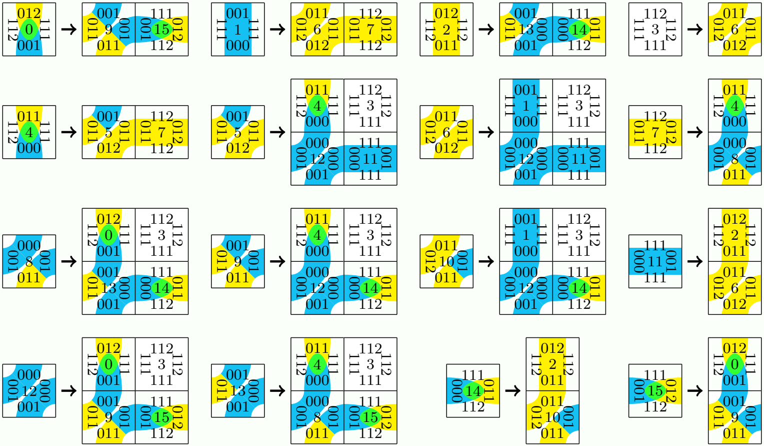
The inflation factor of the self-similarity of Ω_n is the n-th metallic mean, that is, the positive root of $x^2 - nx - 1$.

Self-similarity proof (main idea) : the set of **return blocks** to the junction tiles J_n is in bijection with some **extended set** $\mathcal{T}_n' \supseteq \mathcal{T}_n$.

2D structure associated with 1D **substitution** $a \mapsto ab^n, b \mapsto ab^{n-1}$.

Metallic mean Wang tiles I : self-similarity, aperiodicity and minimality. Forum of Mathematics, Sigma 13 (2025) e133. doi:10.1017/fms.2025.10069

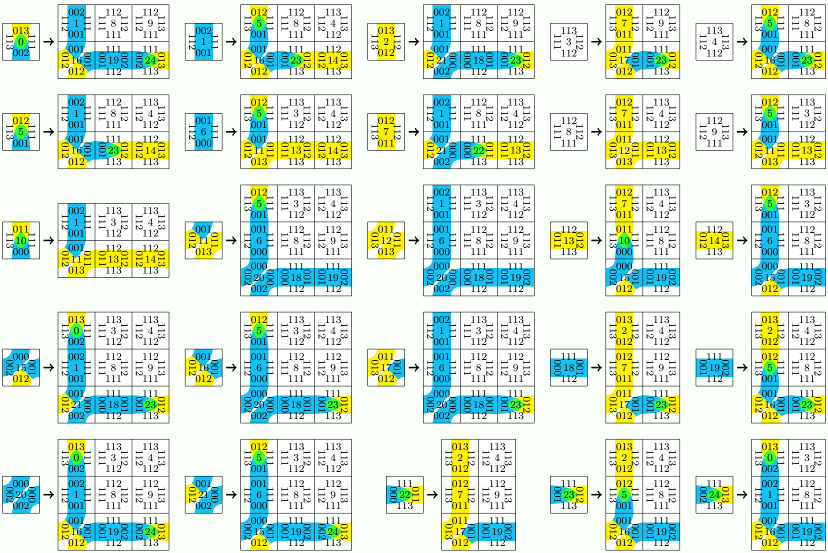
Substitution $\omega_1 : \Omega_1 \to \Omega_1$



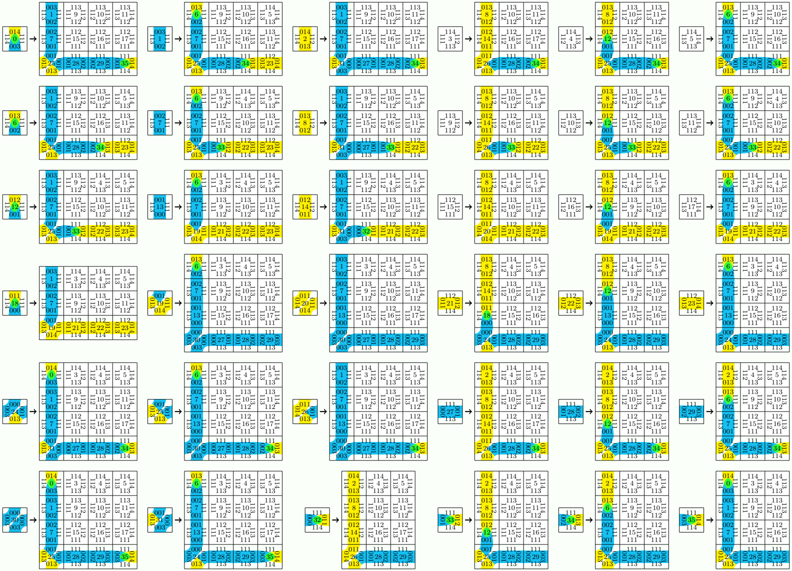
The self-similarity is a non-uniform rectangular 2-dimensional substitution as in Mozes (1989).

Shahar Mozes. Tilings, substitution systems and dynamical systems generated by them. J. Analyse Math., *53 :139–186, 1989.*

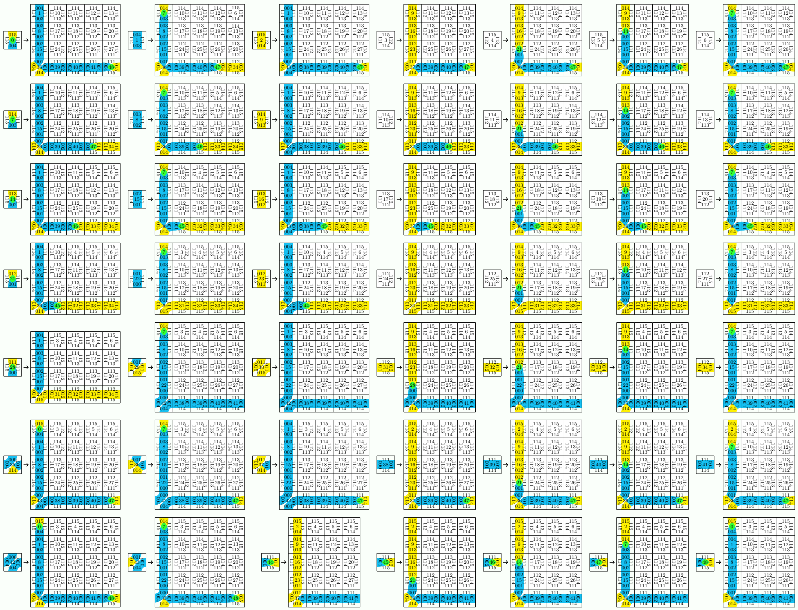
Substitution $\omega_2 : \Omega_2 \to \Omega_2$



Substitution $\omega_3 : \Omega_3 \to \Omega_3$



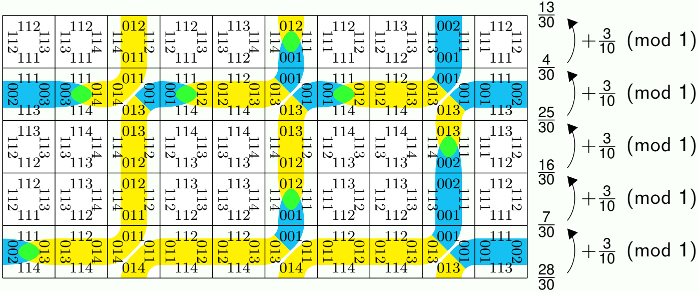
Substitution $\omega_4 : \Omega_4 \to \Omega_4$



An explicit factor map (example)

A 10×5 valid rectangular tiling with the set $\mathcal{T}_n$ with $n = 3$.

The numbers indicated in the right margin are the average of the inner products $\langle \frac{1}{n} d, v \rangle$ over the vectors v appearing as top (or bottom) labels of a horizontal row of tiles and where $d = (0, -1, 1)$.



We observe that these numbers increase by $\frac{3}{10} \pmod 1$ from row to row. The number $\frac{3}{10}$ is equal to the frequency of columns containing junction tiles (a junction tile is a tile whose labels all start with 0).

An explicit factor map

Theorem

Let $d = (0, -1, 1)$, $n \geq 1$ be an integer and Ω_n be the n^{th} metallic mean Wang shift. The map

$$\begin{array}{rccl} \Phi_n : & \Omega_n & \to & \mathbb{T}^2 \\ & w & \mapsto & \displaystyle\lim_{k\to\infty} \frac{1}{2k+1} \sum_{i=-k}^{k} \begin{pmatrix} \langle \frac{1}{n} d, \textsc{Right}(w_{0,i}) \rangle \\ \langle \frac{1}{n} d, \textsc{Top}(w_{i,0}) \rangle \end{pmatrix} \end{array}$$

is a factor map commuting the shift $\mathbb{Z}^2 \overset{\sigma}{\curvearrowright} \Omega_n$ with $\mathbb{Z}^2 \overset{R_n}{\curvearrowright} \mathbb{T}^2$ by the equation $\Phi_n \circ \sigma^k = R_n^k \circ \Phi_n$ for every $k \in \mathbb{Z}^2$ where

$$\begin{array}{rccl} R_n & : \mathbb{Z}^2 \times \mathbb{T}^2 & \to & \mathbb{T}^2 \\ & (k, x) & \mapsto & R_n^k(x) := x + \beta k \end{array}$$

and $\beta = \frac{n+\sqrt{n^2+4}}{2}$ is the n^{th} metallic mean, that is, the positive root of the polynomial $x^2 - nx - 1$.

Metallic mean Wang tiles II : the dynamics of an aperiodic computer chip.
Forum of Mathematics, Sigma 13 (2025) e155. doi:10.1017/fms.2025.10098

An isomorphism (mod 0)

Theorem

The Wang shift Ω_n and the $\mathbb{Z}^2$-action R_n have the following additional properties :

- $\mathbb{Z}^2 \overset{R_n}{\curvearrowright} \mathbb{T}^2$ is the **maximal equicontinuous factor** of $\mathbb{Z}^2 \overset{\sigma}{\curvearrowright} \Omega_n$,
- the factor map $\Phi_n : \Omega_n \to \mathbb{T}^2$ is **almost one-to-one** and its **set of fiber cardinalities** is $\{1, 2, 8\}$,
- the shift-action $\mathbb{Z}^2 \overset{\sigma}{\curvearrowright} \Omega_n$ on the metallic mean Wang shift is **uniquely ergodic**,
- the measure-preserving dynamical system $(\Omega_n, \mathbb{Z}^2, \sigma, \nu)$ is **isomorphic** to $(\mathbb{T}^2, \mathbb{Z}^2, R_n, \lambda)$ where ν is the unique shift-invariant probability measure on Ω_n and λ is the Haar measure on $\mathbb{T}^2$.

The θ_n-computer chip

For every integer $n \geq 1$, we define the finite set of vectors

$$V_n = \{(v_0, v_1, v_2) \in \mathbb{N}^3 \colon 0 \leq v_0 \leq v_1 \leq 1 \text{ and } v_1 \leq v_2 \leq n+1\}$$

with **nondecreasing** entries. Let

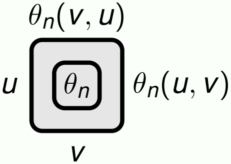
$$\begin{array}{rccc} \theta_{\mathbf{n}} : & V_n \times V_n & \to & \mathbb{Z}^3 \\ & (u_0, u_1, u_2), (v_0, v_1, v_2) & \mapsto & (r_0, r_1, r_2), \end{array}$$

be the **map** defined by the rule

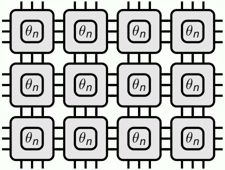
$$r_0 = u_0, \quad r_1 = \begin{cases} v_2 - n & \text{if } u_0 = 0, \\ 1 & \text{if } u_0 = 1, \end{cases}, \quad r_2 = \begin{cases} v_1 + u_0 & \text{if } v_0 = 0, \\ u_2 + 1 & \text{if } v_0 = 1. \end{cases}$$

Let

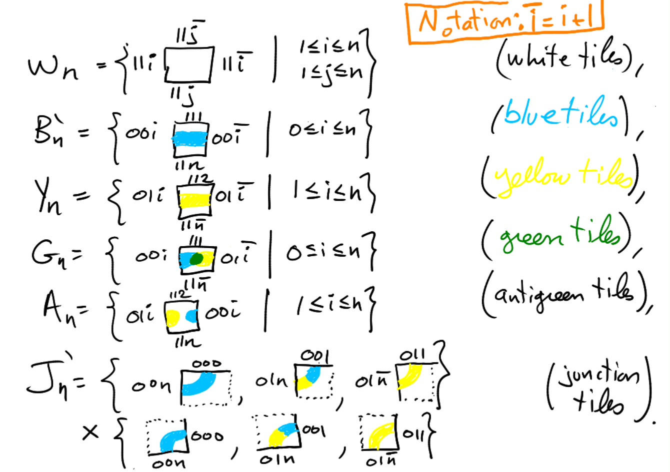
$\mathcal{C}_n = \{$



$\mid u, v \in V_n$ and $\theta_n(u, v)$, $\theta_n(v, u) \in V_n \}$



be the set of instances of the θ_n-chip with outputs **restricted** to V_n.



An **extended set** $\mathcal{T}'_n$ of metallic mean Wang tiles :

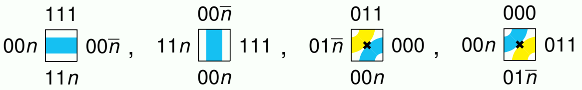
$$\mathcal{T}'_n = W_n \cup B'_n \cup G_n \cup Y_n \cup A_n \cup \widehat{B'_n} \cup \widehat{G_n} \cup \widehat{Y_n} \cup \widehat{A_n} \cup J'_n.$$

Theorem

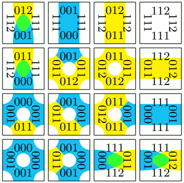
For every integer $n \geq 1$, $\mathcal{C}_n = \mathcal{T}'_n$.

A family $\{\mathcal{T}_n\}_{n\geq 1}$ of metallic mean Wang tiles

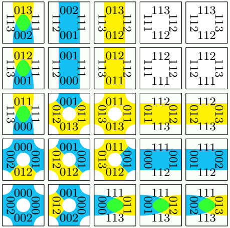
The following tiles are **non-extendible** (proof is not trivial) :

$\mathcal{D} = A_n \cup \widehat{A_n} \cup \Big\{$  $\Big\}$.

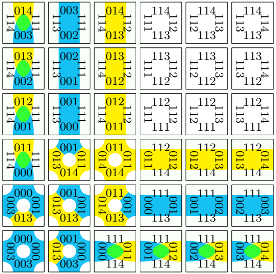
The **subset $\mathcal{T}_n = \mathcal{T}'_n \setminus \mathcal{D}$ of metallic mean Wang tiles** contains $(n+3)^2$ tiles :



$\mathcal{T}_1$



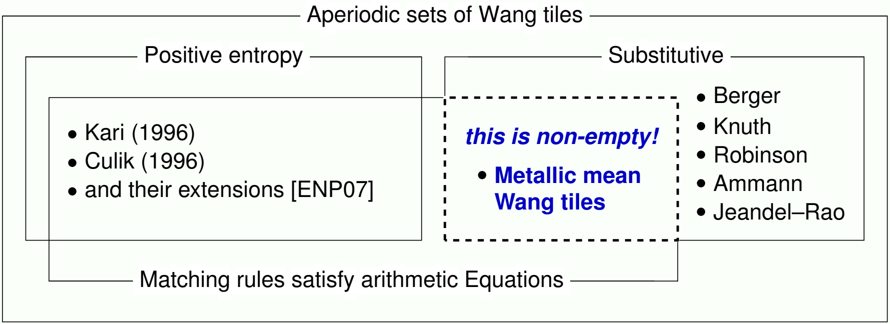
$\mathcal{T}_2$



$\mathcal{T}_3$

Outline

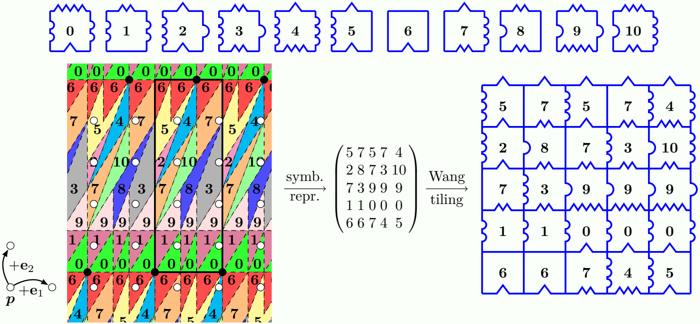
Venn Diagram again



Question

Which other aperiodic sets of Wang tiles have matching rules satisfying arithmetic equations ?

Jeandel–Rao aperiodic set of 11 Wang tiles



S. Labbé, Aperiodic order : from combinatorics to geometry via symbolic dynamics, number theory and algorithms, Thèse d'habilitation à diriger des recherches (HDR), Université de Bordeaux, June 2025,

`hal.science/tel-05138330`

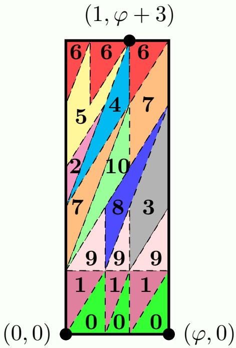
Open question. Find an arithmetical proof of aperiodicity.
Open question. Jeandel–Rao is not alone : characterize the family.
Open question. Wang tiles involving an alg. number of degree 3 ?

Open questions on Jeandel-Rao tilings

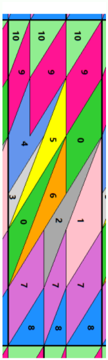
- Describe all of the 33 candidates of 11 Wang tiles listed by JR

  recent progress was made by Thompson (2022) and Mann (2024)

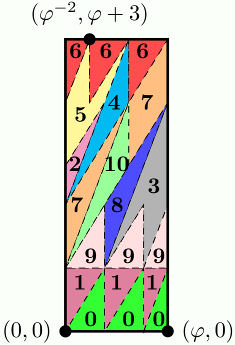
partition for JR



Thompson (2022)



Mann et al. (2024)

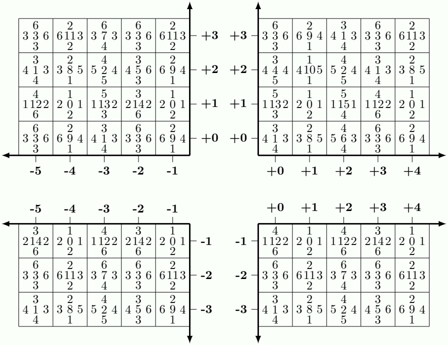


R. D. Thompson. "The Jeandel-Rao Aperiodic Wang Tilings of the Plane". MSc in Mathematics. The Open University, Milton Keynes, UK, May 2022

Hults, Jitsukawa, Mann, Zhang, Experimental Results on Potential Markov Partitions for Wang Shifts arXiv:2302.13516

Jana Lepšová's Ph. D. thesis

Let $\mathrm{rep}_{\mathcal{F}c}$ be the (padded) **Fibonacci comp. num. syst.** for $\mathbb{Z}^2$.
There exists a **deterministic finite automaton with output** (DFAO) $\mathcal{A}$ such that $\binom{n_1}{n_2} \mapsto \mathcal{A}\left(\mathrm{rep}_{\mathcal{F}c}\binom{n_1}{n_2}\right)$ is a **valid tiling** with Ammann tiles

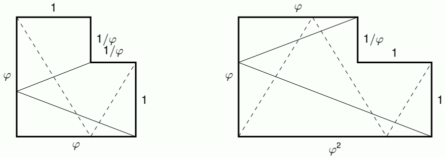


L., Lepšová. *A Fibonacci analogue of the two's complement numeration system. RAIRO - Theoretical Informatics and Applications 57 (2023) 12.*

L., Lepšová. *Dumont-Thomas complement numeration systems for $\mathbb{Z}$. Integers 24 (2024) Paper No. A112, 27 pages.* doi:10.5281/zenodo.14340125

Appendix

Ammann A2 encoded into 16 Wang tiles



Tilings in the Ammann A2 family can be encoded into 16 Wang tiles :

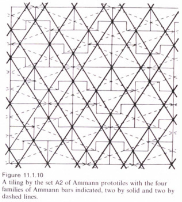


Figure 11.1.10

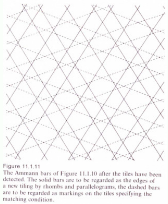


Figure 11.1.11

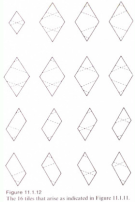


Figure 11.1.12

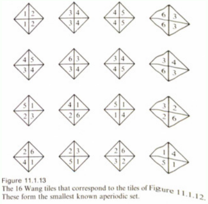


Figure 11.1.13

- Find **geometric shapes** with Ammann bars on them associated with metallic-mean Wang tiles for $n \geq 2$.

Branko Grünbaum and G. C. Shephard. 1987

Existence of valid tilings

For every $(x, y) \in [0,1)^2$, let $\Lambda_n(x,y) = \begin{pmatrix} \lfloor y - \beta^{-1} + 1 \rfloor \\ \lfloor \beta^{-1} x + y - \beta^{-1} + 1 \rfloor \\ \lfloor \beta x + y - \beta^{-1} + 1 \rfloor \end{pmatrix} \in \mathbb{N}^3$

where β is the positive root of the polynomial $x^2 - nx - 1$ and

$$t_n(x,y) = \quad \begin{array}{ccc} & \Lambda_n(y,x) & \\ \Lambda_n(\{x - \beta^{-1}\}, y) & \boxed{} & \Lambda_n(x,y) \\ & \Lambda_n(\{y - \beta^{-1}\}, x) & \end{array} \quad \text{be a Wang tile}$$

where $\{x\} = x - \lfloor x \rfloor$ is the fractional part of a number $x \in \mathbb{R}$.

Theorem

For every integer $n \geq 1$ and every $(x, y) \in [0,1)^2$,

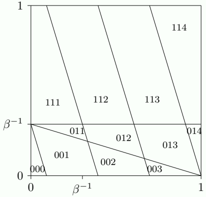
$$\begin{array}{rccl} c_{(x,y)}: & \mathbb{Z}^2 & \to & \mathcal{T}_n \\ & (i,j) & \mapsto & t_n\left(\{x + i\beta^{-1}\}, \{y + j\beta^{-1}\}\right) \end{array}$$

is a **valid tiling** with the **metallic mean** Wang tiles $\mathcal{T}_n$.

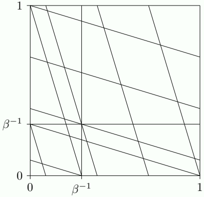
A Markov partition

$\mathcal{P}_n = \{\Phi_n([t])\}_{t\in\mathcal{T}_n}$ partitions the unit square into $(n+3)^2$ polygons.

$\{\Lambda_3^{-1}(v)\}_{v\in V_3}$



$\mathcal{P}_3$



Theorem

For every integer $n \geq 1$, the symbolic dynamical system $\mathcal{X}_{\mathcal{P}_n,R_n}$ corresponding to $\mathcal{P}_n$, R_n **is the metallic mean Wang shift** Ω_n :

$$\Omega_n = \mathcal{X}_{\mathcal{P}_n,R_n}.$$

In particular, $\mathcal{P}_n$ is a **Markov partition** for $\mathbb{Z}^2 \overset{R_n}{\curvearrowright} \mathbb{T}^2$.

(this is the partition of the window in the internal space of a 4-to-2 CAP)