A family of metallic mean Wang tiles

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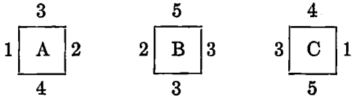
Directions in Aperiodic order
BIRS, Banff
July 27th to August 1st, 2025
http://www.birs.ca/event/25w5437

Outline

1. Kari/Culik, Jeandel–Rao, Ammann sets of Wang tiles
2. A family of metallic mean Wang tiles
3. Some results
4. Conclusion and final remarks

Outline

Wang tiles



Then we can easily find an infinite solution by the following argument. The following configuration satisfies the constraint on the edges:

$$\begin{matrix} A & B & C \\ C & A & B \\ B & C & A \end{matrix}$$

Now the colors on the periphery of the above block are seen to be the following:

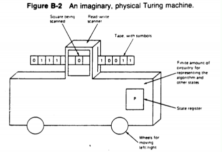
$$\begin{matrix} & 3 & 5 & 4 & \\ 1 & & & & 1 \\ 3 & & & & 3 \\ 2 & & & & 2 \\ & 3 & 5 & 4 & \end{matrix}$$

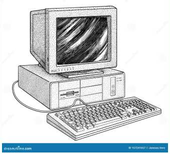
Wang's original question : is it true that a set of Wang tiles tile the plane if and only if there exists such a cyclic rectangle ?

H. Wang. Proving theorems by pattern recognition – II. Bell System Technical Journal*, 40(1) :1–41, January 1961.* doi:10.1002/j.1538-7305.1961.tb03975.x

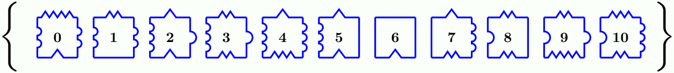
Turing machine reduction to Wang tiles

Berger (1966) : For every Turing machine





there exists a set of Wang tiles



, *e.g.*,

that tiles the plane if and only if the Turing machine does not halt.

- The **domino problem is undecidable** : there exist no algorithm that says whether a finite set of Wang tiles can tile the plane.
- There **exists an aperiodic set** of Wang tiles (*a tile set is **aperiodic** if it tiles the plane, but none of tilings is periodic*).
- Valid Wang tilings are **computing** something.

Aperiodic Wang tile sets

Aperiodic sets of Wang tiles

Positive entropy

- 14 tiles : **Kari** (1996)
- 13 tiles : Culik (1996)
- and their extensions [ENP07]

Matching rules satisfy arithmetic Equations

Substitutive

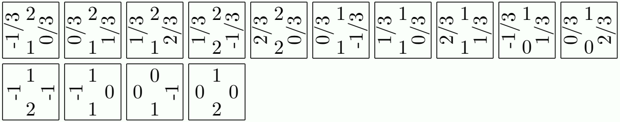
- 104 : Berger (1966)
- 92 : Knuth (1968)
- 56 : Robinson (1971)
- 16 : **Ammann** (1971)
- 11 : **Jeandel-Rao** (2015)

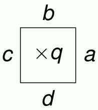
Theorem (Jeandel, Rao, 2015)

All sets of $\leq$ 10 Wang tiles are **periodic** or **don't tile** the plane.

Emmanuel Jeandel and Michaël Rao. An aperiodic set of 11 Wang tiles. Adv. Comb. ***37*** *(2021) Id/No 1.*

Kari's 14 Wang tiles computing $\times\frac{2}{3}$ and $\times 2$



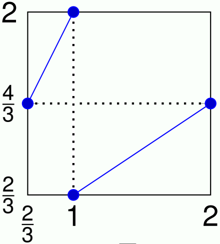


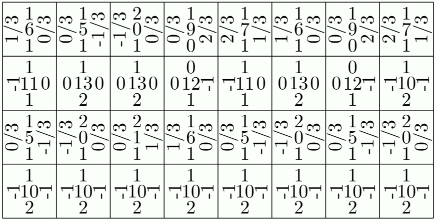
$\iff qb + c = d + a$

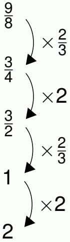
with $q \in \{\frac{2}{3}, 2\}$

$$g(x) = \begin{cases} 2x & \text{if } x \leq 1, \\ \frac{2}{3}x & \text{if } x > 1. \end{cases}$$

Averages of horizontal labels are orbits of g :

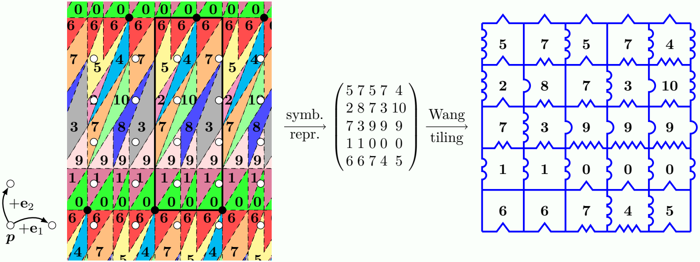






Durand, Gamard, Grandjean (2007) *Kari (2016)*

Jeandel–Rao aperiodic set of 11 Wang tiles



S. Labbé, Aperiodic order : from combinatorics to geometry via symbolic dynamics, number theory and algorithms, Thèse d'habilitation à diriger des recherches (HDR), Université de Bordeaux, June 2025,

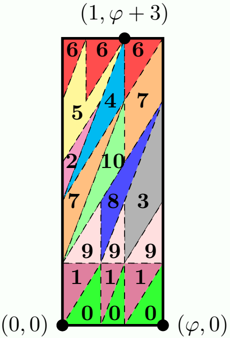
hal.science/tel-05138330

Open question. Find an arithmetical proof of aperiodicity.
Open question. Jeandel–Rao is not alone : characterize the family.

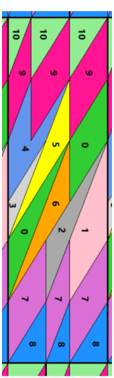
Other Jeandel-Rao aperiodic candidates

Recent progress made on the 33 candidates of aperiodic sets of 11 Wang tiles listed by Jeandel and Rao :

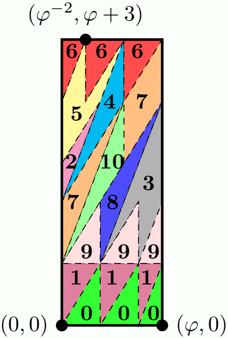
partition for JR



Thompson (2022)



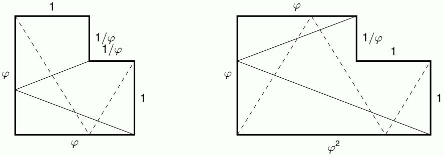
Mann et al. (2024)



R. D. Thompson. "The Jeandel-Rao Aperiodic Wang Tilings of the Plane". MSc in Mathematics. The Open University, Milton Keynes, UK, May 2022

Hults, Jitsukawa, Mann, Zhang, Experimental Results on Potential Markov Partitions for Wang Shifts arXiv:2302.13516

Ammann A2 encoded into 16 Wang tiles



Tilings in the Ammann A2 family can be encoded into 16 Wang tiles :

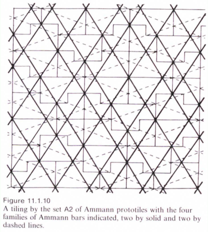


Figure 11.1.10

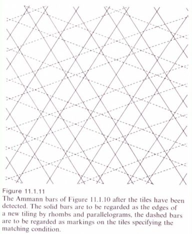


Figure 11.1.11

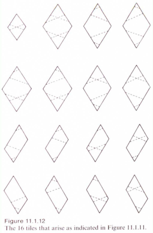


Figure 11.1.12

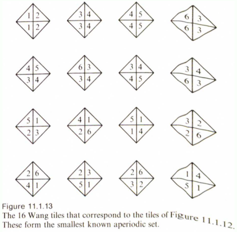


Figure 11.1.13

Branko Grünbaum and G. C. Shephard. Tilings and patterns. *W. H. Freeman and Company, New York, 1987.*

Outline

The θ_n-computer chip

For every integer $n \geq 1$, we define the finite set of vectors

$$V_n = \{(v_0, v_1, v_2) \in \mathbb{N}^3 \colon 0 \leq v_0 \leq v_1 \leq 1 \text{ and } v_1 \leq v_2 \leq n+1\}$$

with **nondecreasing** entries. Let

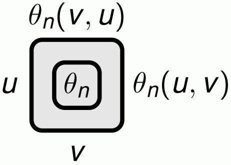
$$\begin{array}{rccl} \boldsymbol{\theta_n} : & V_n \times V_n & \to & \mathbb{Z}^3 \\ & (u_0, u_1, u_2), (v_0, v_1, v_2) & \mapsto & (r_0, r_1, r_2), \end{array}$$

be the **map** defined by the rule

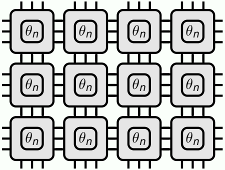
$$r_0 = u_0, \quad r_1 = \begin{cases} v_2 - n & \text{if } u_0 = 0, \\ 1 & \text{if } u_0 = 1, \end{cases}, \quad r_2 = \begin{cases} v_1 + u_0 & \text{if } v_0 = 0, \\ u_2 + 1 & \text{if } v_0 = 1. \end{cases}$$

Let

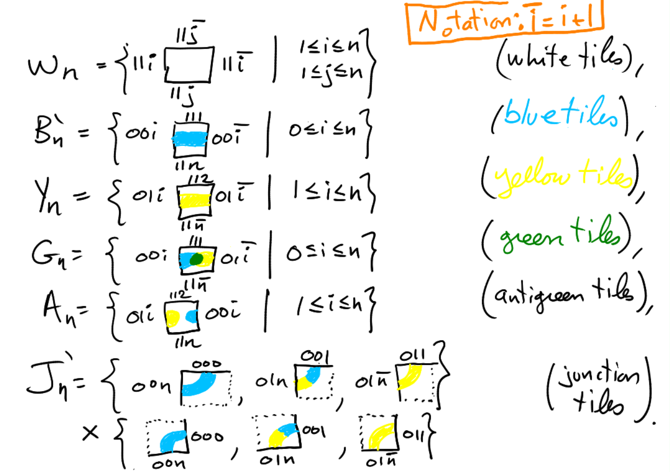
$\mathcal{C}_n = \{$



$\mid u, v \in V_n$ and $\theta_n(u, v)$, $\theta_n(v, u) \in V_n \}$



be the set of instances of the θ_n-chip with outputs **restricted** to V_n.



An **extended set** $\mathcal{T}'_n$ of metallic mean Wang tiles :

$$\mathcal{T}'_n = W_n \cup B'_n \cup G_n \cup Y_n \cup A_n \cup \widehat{B'_n} \cup \widehat{G_n} \cup \widehat{Y_n} \cup \widehat{A_n} \cup J'_n.$$

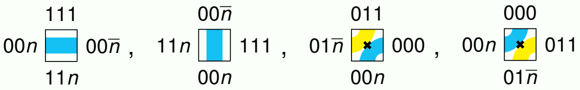
Theorem : $\mathcal{C}_n = \mathcal{T}'_n$.

Metallic mean Wang tiles II : the dynamics of an aperiodic computer chip.
arXiv:2403.03197

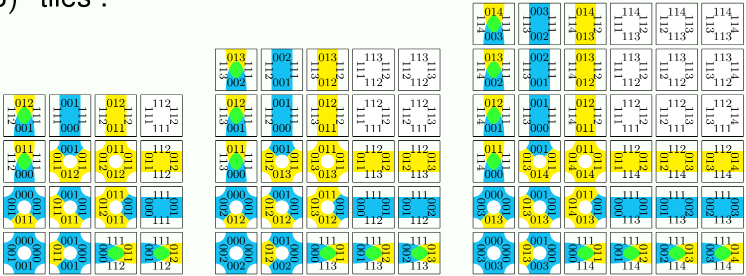
A family $\{\mathcal{T}_n\}_{n\geq 1}$ of metallic mean Wang tiles

The following tiles are **non-extendible** (proof is not trivial) :

$$\mathcal{D} = A_n \cup \widehat{A_n} \cup \{ \ldots \}.$$



The **subset $\mathcal{T}_n = \mathcal{T}'_n \setminus \mathcal{D}$ of metallic mean Wang tiles** contains $(n+3)^2$ tiles :

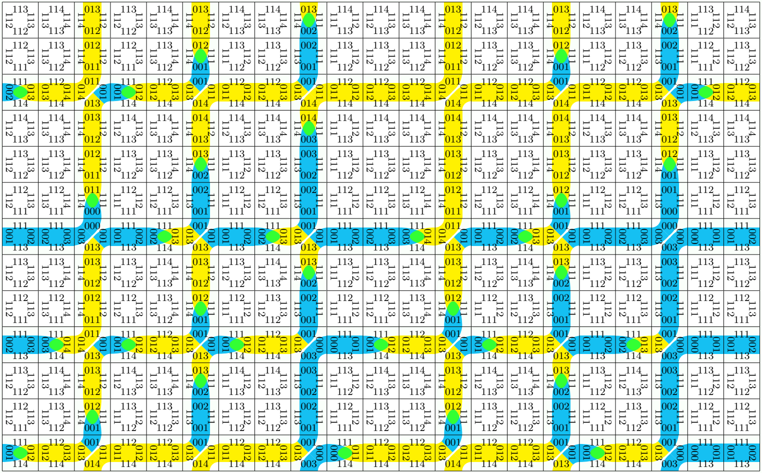


$\mathcal{T}_1$ $\mathcal{T}_2$ $\mathcal{T}_3$

Metallic mean Wang shift

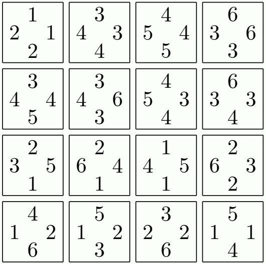
The ***n*-th metallic mean Wang shift** is $\mathbb{Z}^2 \overset{\sigma}{\curvearrowright} \Omega_n$ where

$$\Omega_n := \Omega_{\mathcal{T}_n} = \{w : \mathbb{Z}^2 \to \mathcal{T}_n \colon w \text{ is a valid configuration}\}.$$

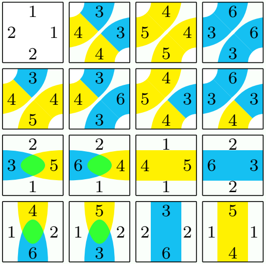


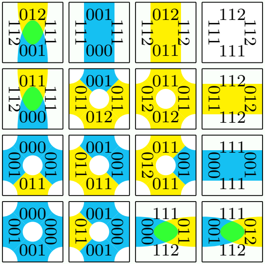
(a 21×13 valid patch with $\mathcal{T}_3$)

Ammann $\equiv \mathcal{T}_1$



Ammann





$\mathcal{T}_1$

Theorem

The Ammann set of 16 Wang tiles is equivalent to $\mathcal{T}_1$.

Proof : the bijection of the tile labels is

$$1 \mapsto 112, \quad 2 \mapsto 111, \quad 3 \mapsto 001, \quad 4 \mapsto 011, \quad 5 \mapsto 012, \quad 6 \mapsto 000.$$

Outline

Self-similarity, aperiodicity and minimality

Theorem

For every integer $n \geq 1$, the metallic mean Wang shift Ω_n is

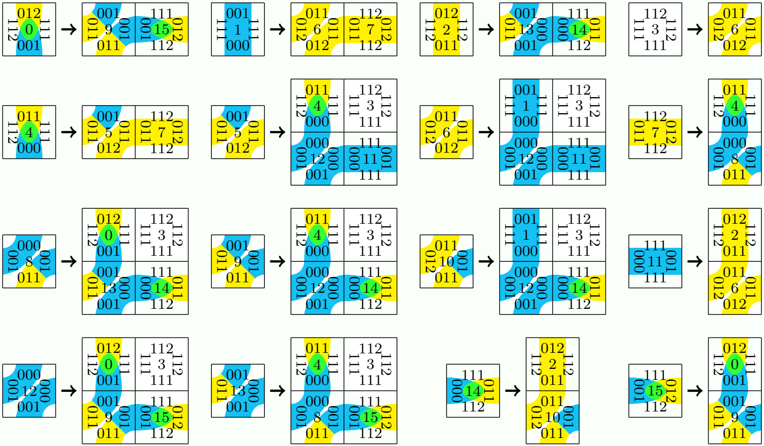
- **self-similar**,
- **aperiodic** and
- **minimal**.

The inflation factor of the self-similarity of Ω_n is the n-th metallic mean, that is, the positive root of $x^2 - nx - 1$.

Self-similarity proof (main idea) : the set of **return blocks** to the junction tiles J_n is in bijection with the **extended set** $\mathcal{T}_n'$.

Metallic mean Wang tiles I : self-similarity, aperiodicity and minimality.
arXiv:2312.03652

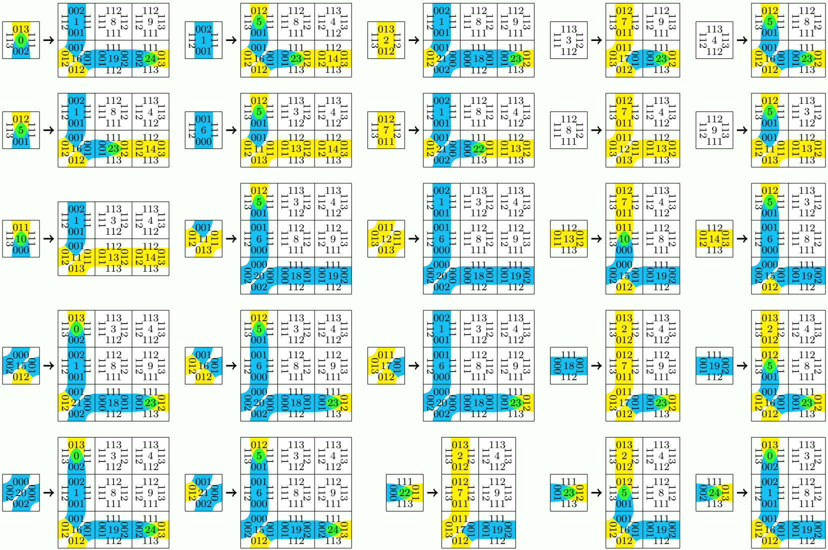
Substitution $\omega_1 : \Omega_1 \to \Omega_1$



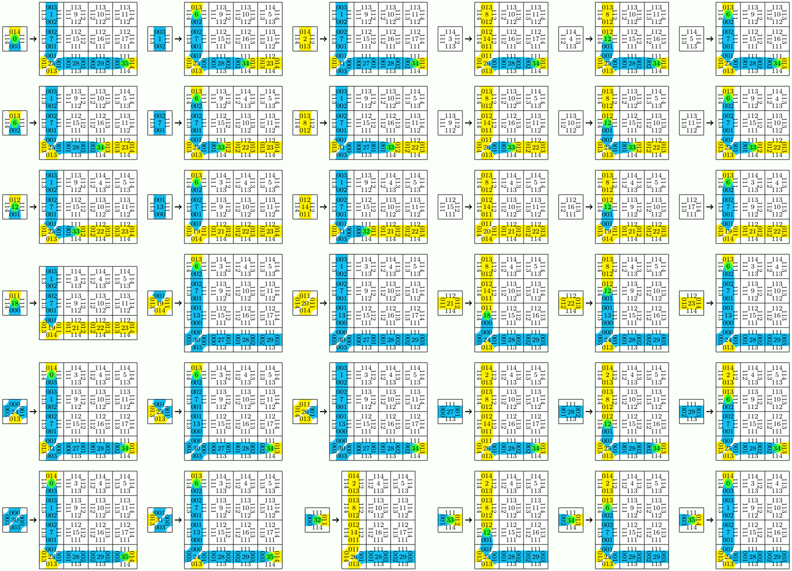
The self-similarity is a non-uniform rectangular 2-dimensional substitution as in Mozes (1989).

Shahar Mozes. Tilings, substitution systems and dynamical systems generated by them. J. Analyse Math., *53 :139–186, 1989.*

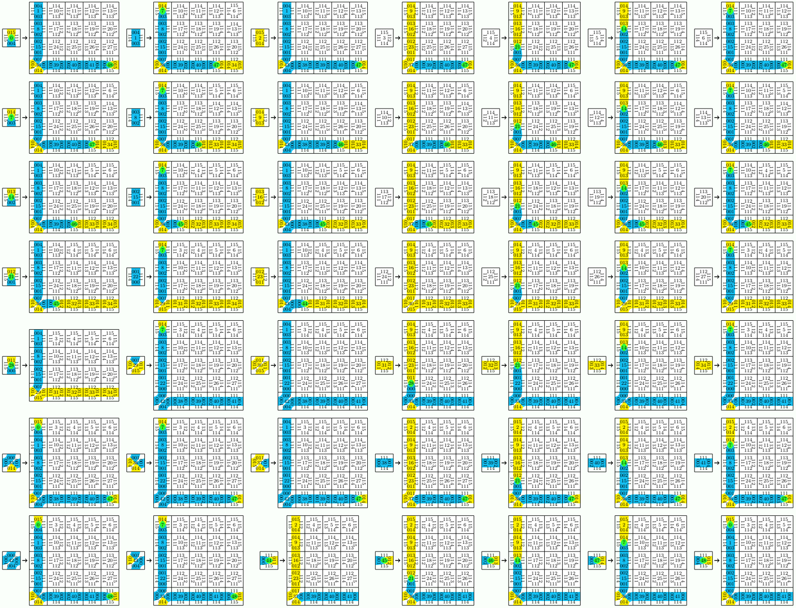
Substitution $\omega_2 : \Omega_2 \to \Omega_2$



Substitution $\omega_3 : \Omega_3 \to \Omega_3$



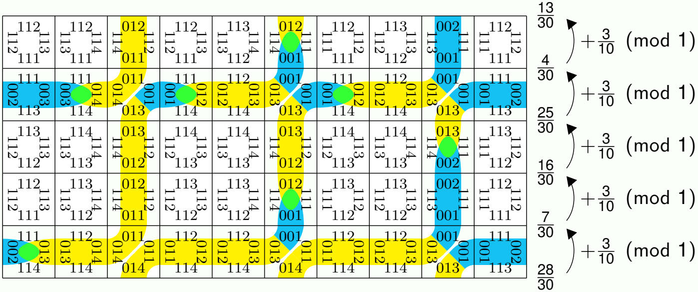
Substitution $\omega_4 : \Omega_4 \to \Omega_4$



An explicit factor map (example)

A 10×5 valid rectangular tiling with the set $\mathcal{T}_n$ with $n = 3$.

The numbers indicated in the right margin are the average of the inner products $\langle \frac{1}{n} d, v \rangle$ over the vectors v appearing as top (or bottom) labels of a horizontal row of tiles and where $d = (0, -1, 1)$.



We observe that these numbers increase by $\frac{3}{10} \pmod 1$ from row to row. The number $\frac{3}{10}$ is equal to the frequency of columns containing junction tiles (a junction tile is a tile whose labels all start with 0).

An explicit factor map

Theorem

Let $d = (0, -1, 1)$, $n \geq 1$ be an integer and Ω_n be the n^{th} metallic mean Wang shift. The map

$$\begin{array}{rccl} \Phi_n : & \Omega_n & \to & \mathbb{T}^2 \\ & w & \mapsto & \displaystyle\lim_{k\to\infty} \frac{1}{2k+1} \sum_{i=-k}^{k} \begin{pmatrix} \langle \frac{1}{n} d, \textsc{Right}(w_{0,i}) \rangle \\ \langle \frac{1}{n} d, \textsc{Top}(w_{i,0}) \rangle \end{pmatrix} \end{array}$$

is a factor map commuting the shift $\mathbb{Z}^2 \overset{\sigma}{\curvearrowright} \Omega_n$ with $\mathbb{Z}^2 \overset{R_n}{\curvearrowright} \mathbb{T}^2$ by the equation $\Phi_n \circ \sigma^k = R_n^k \circ \Phi_n$ for every $k \in \mathbb{Z}^2$ where

$$\begin{array}{rccl} R_n & : \mathbb{Z}^2 \times \mathbb{T}^2 & \to & \mathbb{T}^2 \\ & (k, x) & \mapsto & R_n^k(x) := x + \beta k \end{array}$$

and $\beta = \frac{n+\sqrt{n^2+4}}{2}$ is the n^{th} metallic mean, that is, the positive root of the polynomial $x^2 - nx - 1$.

Proof uses **Weyl equidistribution thm** (Kuipers, Niederreiter, 1974).
Remark : Φ_n satisfies $\Phi_n(c_{(x,y)}) = (x, y)$.

An isomorphism (mod 0)

Theorem

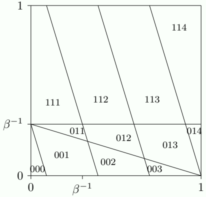
The Wang shift Ω_n and the $\mathbb{Z}^2$-action R_n have the following additional properties :

- $\mathbb{Z}^2 \overset{R_n}{\curvearrowright} \mathbb{T}^2$ is the **maximal equicontinuous factor** of $\mathbb{Z}^2 \overset{\sigma}{\curvearrowright} \Omega_n$,
- the factor map $\Phi_n : \Omega_n \to \mathbb{T}^2$ is **almost one-to-one** and its **set of fiber cardinalities** is $\{1, 2, 8\}$,
- the shift-action $\mathbb{Z}^2 \overset{\sigma}{\curvearrowright} \Omega_n$ on the metallic mean Wang shift is **uniquely ergodic**,
- the measure-preserving dynamical system $(\Omega_n, \mathbb{Z}^2, \sigma, \nu)$ is **isomorphic** to $(\mathbb{T}^2, \mathbb{Z}^2, R_n, \lambda)$ where ν is the unique shift-invariant probability measure on Ω_n and λ is the Haar measure on $\mathbb{T}^2$.

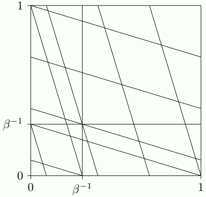
A Markov partition

$\mathcal{P}_n = \{\Phi_n([t])\}_{t \in \mathcal{T}_n}$ partitions the unit square into $(n+3)^2$ polygons.

$\{\Lambda_3^{-1}(v)\}_{v \in V_3}$



$\mathcal{P}_3$



Theorem

For every integer $n \geq 1$, the symbolic dynamical system $\mathcal{X}_{\mathcal{P}_n, R_n}$ corresponding to $\mathcal{P}_n$, R_n **is the metallic mean Wang shift** Ω_n :

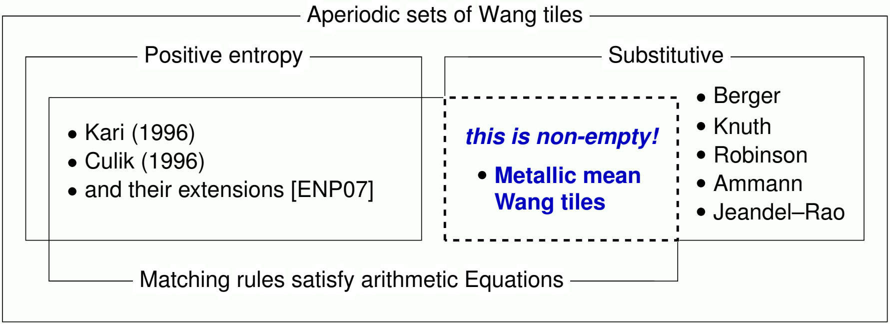
$$\Omega_n = \mathcal{X}_{\mathcal{P}_n, R_n}.$$

In particular, $\mathcal{P}_n$ is a **Markov partition** for $\mathbb{Z}^2 \overset{R_n}{\curvearrowright} \mathbb{T}^2$.

(this is the partition of the window in the internal space of a 4-to-2 CAP)

Outline

Venn Diagram again



Question

Which other aperiodic sets of Wang tiles have matching rules satisfying arithmetic equations ?

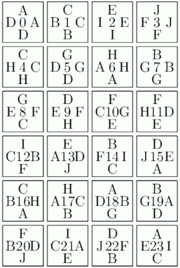
Jang and Robinson's Bifurcation diagram

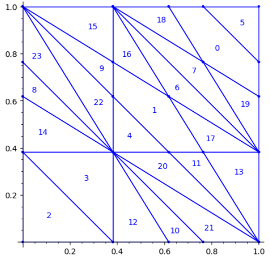
Jang and Robinson's studied the set of 24 Wang tiles deduced from Penrose tilings and obtained what they call a **bifurcation diagram**.

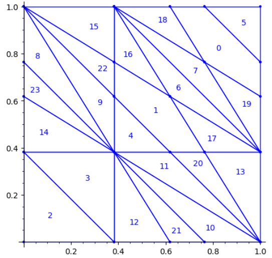
Hyeeun Jang, Directional Expansivenss, PhD Thesis, 2021.

Hyeeun Jang, E. Arthur Robinson Jr, Directional Expansiveness for $\mathbb{R}^d$*-Actions and for Penrose Tilings,* `arXiv:2504.10838v2` .

the 24 tiles and their (incorrect) diagram | the correct top. partition







`http://www.slabbe.org/blogue/2025/05/on-the-bifurcation-diagram-proposed-by-jang-and-robinson/`

S. Labbé, Three characterizations of a self-similar aperiodic 2-dimensional subshift, `arXiv:2012.03892`

Open questions

- Describe Jeandel-Rao tiles as the instances of a **computer chip**.
- Which **other algebraic numbers** arise in aperiodic tilings ?
- Describe a **Tribonacci** set of Wang tiles
  and its 4-dimensional Rauzy fractal

  (a Tribonacci set of Wang tiles must exist following Mozes 1989)
- Find **geometric shapes** with Ammann bars on them associated with metallic-mean Wang tiles for $n > 2$.

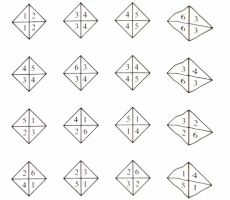


Figure 11.1.13
The 16 Wang tiles that correspond to the tiles of Figure 11.1.12.
These form the smallest known aperiodic set.

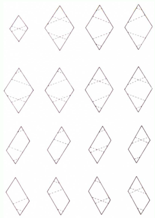


Figure 11.1.12
The 16 tiles that arise as indicated in Figure 11.1.11.

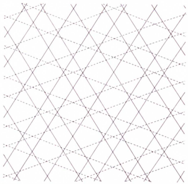


Figure 11.1.11
The Ammann bars of Figure 11.1.10 after the tiles have been detected. The solid bars are to be regarded as the edges of a new tiling by rhombs and parallelograms, the dashed bars are to be regarded as markings on the tiles specifying the matching condition.

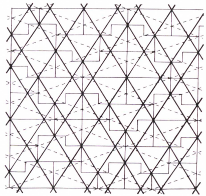


Figure 11.1.10
A tiling by the set A2 of Ammann prototiles with the four families of Ammann bars indicated, two by solid and two by dashed lines.

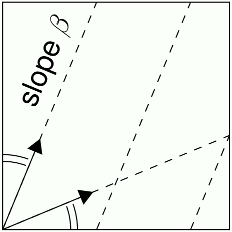
Figure 11.1.13 Figure 11.1.12 Figure 11.1.11 Figure 11.1.10

Branko Grünbaum and G. C. Shephard. 1987

Metallic means

Definition

The ***n*-th metallic mean** is the positive root of $x^2 - nx - 1$.



$$\beta = \frac{n + \sqrt{n^2+4}}{2} = n + \cfrac{1}{n + \cfrac{1}{n + \cfrac{1}{\cdots}}}$$

https://oeis.org/wiki/Metallic_means

V. W. de Spinadel. The family of metallic means. Vis. Math., *1(3) :1 HTML document ; approx. 16, 1999.*

*Also called **silver means** (Schroeder 1991) or **noble means** (Baake, Grimm, 2013).*

Existence of valid tilings

For every $(x, y) \in [0,1)^2$, let $\Lambda_n(x,y) = \begin{pmatrix} \lfloor y - \beta^{-1} + 1 \rfloor \\ \lfloor \beta^{-1} x + y - \beta^{-1} + 1 \rfloor \\ \lfloor \beta x + y - \beta^{-1} + 1 \rfloor \end{pmatrix} \in \mathbb{N}^3$

where β is the positive root of the polynomial $x^2 - nx - 1$ and

$$\Lambda_n(y, x)$$

$$t_n(x,y) = \quad \Lambda_n(\{x - \beta^{-1}\}, y) \; \boxed{} \; \Lambda_n(x, y)$$ be a Wang tile

$$\Lambda_n(\{y - \beta^{-1}\}, x)$$

where $\{x\} = x - \lfloor x \rfloor$ is the fractional part of a number $x \in \mathbb{R}$.

Theorem

For every integer $n \geq 1$ and every $(x, y) \in [0,1)^2$,

$$\begin{array}{rccl} c_{(x,y)} : & \mathbb{Z}^2 & \rightarrow & \mathcal{T}_n \\ & (i,j) & \mapsto & t_n\left(\{x + i\beta^{-1}\}, \{y + j\beta^{-1}\}\right) \end{array}$$

is a **valid tiling** with the **metallic mean** Wang tiles $\mathcal{T}_n$.