A family of metallic mean Wang tiles

Sébastien Labbé

IRL CRM-CNRS, Montréal, Canada

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https://www.math.mcgill.ca/dcl/DDC-Seminar/

- Visiting Scholars program (for sabbatical)

  2019-2020 : Casey Mann & Jennifer McLoud-Mann, University of Seattle

- Cotutelle doctoral program : *Joint supervision of a PhD by the University of Bordeaux and an international university (1 thesis defense, 2 PhD diplomas)*

- Institut de Mathématiques de Bordeaux (IMB) &
  Laboratoire Bordelais de Recherche en Informatique (LaBRI)

Outline

Outline

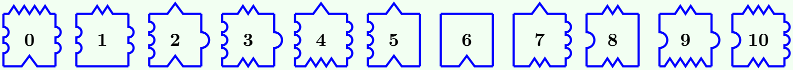
A Fondamental Question

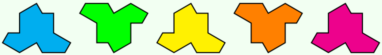
Domino Problem

Given a finite set of tiles,
can you tile the plane with copies of these tiles ?

Exercise (5 minutes)

Cover the largest possible square with the following pieces :





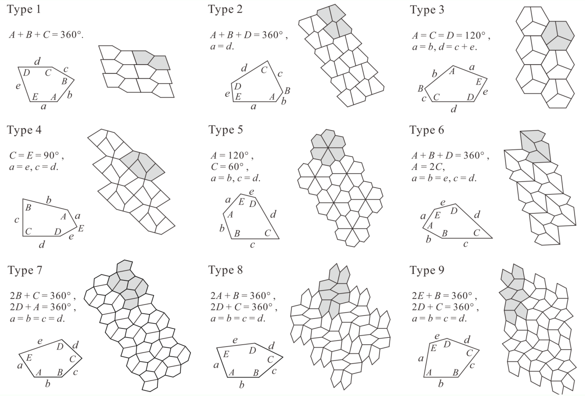
Could you tile the floor of your living room ?

Thanks to Xavier Provençal and ETS for the laser cutting (École de technologie supérieure, Montréal, décembre 2025).

Outline

Periodic tiling

A tiling is **periodic** if it is invariant under at least one nonzero translation.
Ex :the 15 types of tilings with convex pentagons are periodic :



Teruhisa SUGIMOTO, Convex Pentagons with Positive Heesch Number,
`arXiv:1802.00119v2` ; `https://fr.wikipedia.org/wiki/Pavage_pentagonal`

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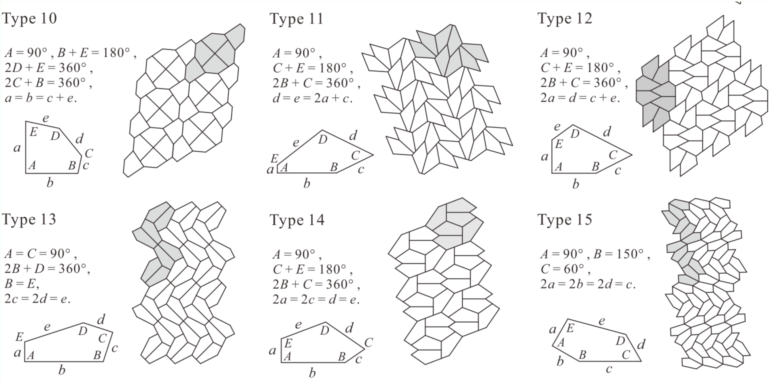
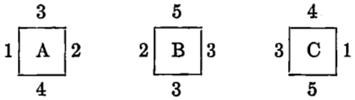


Teruhisa SUGIMOTO, Convex Pentagons with Positive Heesch Number, `arXiv:1802.00119v2` ; `https://fr.wikipedia.org/wiki/Pavage_pentagonal`

Wang tiles



Then we can easily find an infinite solution by the following argument. The following configuration satisfies the constraint on the edges:

A	*B*	*C*
C	*A*	*B*
B	*C*	*A*

Now the colors on the periphery of the above block are seen to be the following:

```
   3 5 4
1         1
3         3
2         2
   3 5 4
```

Wang's original question : is it true that a set of Wang tiles tile the plane if and only if there exists such a cyclic rectangle ?

H. Wang. Proving theorems by pattern recognition – II. Bell System Technical Journal, *40(1) :1–41, January 1961.* doi:10.1002/j.1538-7305.1961.tb03975.x

Small sets of aperiodic Wang tiles

Aperiodic sets of Wang tiles

Positive entropy / **Matching rules satisfy arithmetic Equations**

- 14 tiles : **Kari** (1996)
- 13 tiles : Culik (1996)
- and their extensions [ENP07]

Substitutive

- 104 : Berger (1966)
- 92 : Knuth (1968)
- 56 : Robinson (1971)
- 16 : **Ammann** (1971)
- 11 : **Jeandel-Rao** (2015)

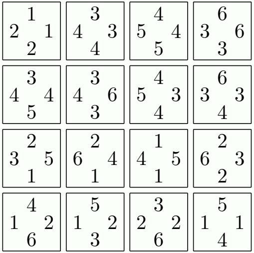
Theorem (Jeandel, Rao, 2015)

All sets of ≤ 10 Wang tiles are **periodic** or **don't tile** the plane.

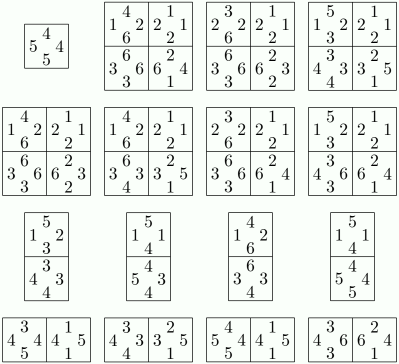
Emmanuel Jeandel and Michaël Rao. An aperiodic set of 11 Wang tiles. Adv. Comb. ***37*** *(2021) Id/No 1.*

Ammann 16 Wang tiles are self-similar

16 tiles :



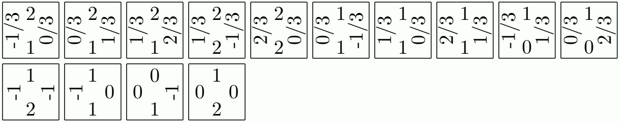
16 equivalent supertiles :

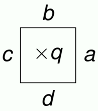


Branko Grünbaum and G. C. Shephard. Tilings and patterns. *W. H. Freeman and Company, New York, 1987.*

M. Senechal. The mysterious Mr. Ammann. Math. Intelligencer*, 26(4) :10–21, 2004.* doi:10.1007/BF02985414

Kari's 14 Wang tiles computing $\times\frac{2}{3}$ and $\times 2$



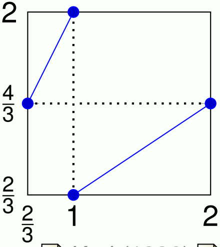


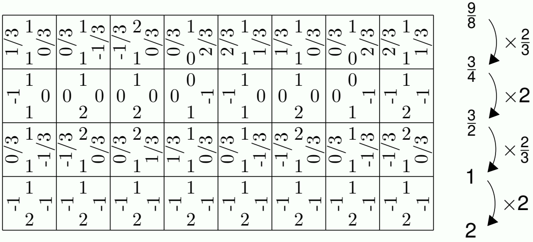
$$\iff qb + c = d + a$$

with $q \in \{\frac{2}{3}, 2\}$

$$g(x) = \begin{cases} 2x & \text{if } x \leq 1, \\ \frac{2}{3}x & \text{if } x > 1. \end{cases}$$

Averages of horizontal labels are orbits of g :

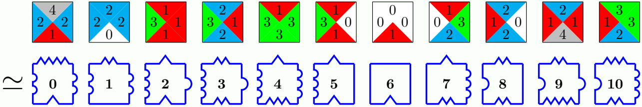




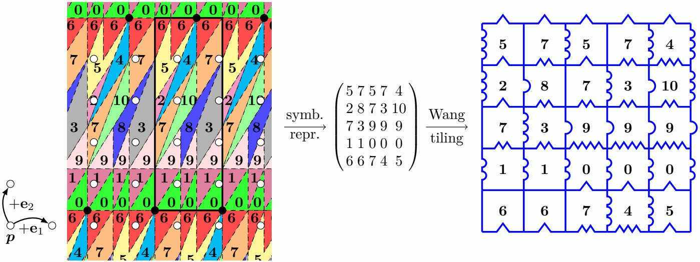
Kari (1996) Durand, Gamard, Grandjean (2007) Kari (2016)

Jeandel–Rao aperiodic set of 11 Wang tiles

The following set of 11 Wang tiles is **aperiodic** (Jeandel, Rao, 2015) :



Tilings are coding the orbits of a $\mathbb{Z}^2$-action on $\mathbb{R}^2/\langle(\varphi, 0),(1,\varphi+3)\rangle$:



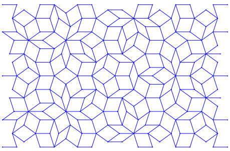
Markov partitions for toral $\mathbb{Z}^2$-rotations featuring Jeandel-Rao Wang shift and model sets. Ann. H. Lebesgue *4 (2021) 283–324.* doi:10.5802/ahl.73

Aperiodic order : from combinatorics to geometry via symbolic dynamics, number theory and algorithms, HDR, Bordeaux, 2025, hal.science/tel-05138330

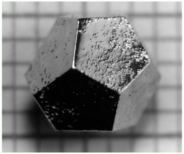
Quasicrystals

1982 (Shechtman) : observed that aluminium-manganese alloys produced a **quasicrystals structure**. 2011 **Nobel Prize** :

"*His discovery of quasicrystals revealed a new principle for packing of atoms and molecules* [that] *led to a* ***paradigm shift*** *within chemistry*."

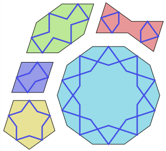


Penrose tiling (1976)



A Ho-Mg-Zn quasicrystal

Smith (2023)



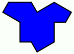
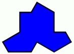
Darb-e Imam, Isfahan, Iran (1453)

en.wikipedia.org/wiki/Quasicrystal, en.wikipedia.org/wiki/Darb-e_Imam

Outline

Golden mean omnipresence

Golden mean (ratio) $\varphi = \frac{1+\sqrt{5}}{2}$ is everywhere !

- **Penrose** tilings : two-tile frequency ratio is $\varphi = \frac{1+\sqrt{5}}{2}$.
- Patch frequencies in **Jeandel-Rao** tilings are in $\mathbb{Q}[\varphi]$.
- Patch frequencies in **Ammann** tilings are in $\mathbb{Q}[\varphi]$.
- In tilings with **David Smith's aperiodic monotile** (the **hat**), the frequency of  versus its mirror image  is φ^4.

In fact, not really :

D. Frettlöh. More inflation tilings. In Aperiodic order. Vol. 2*, volume 166 of* Encyclopedia Math. Appl.*, pages 1–37. Cambridge Univ. Press, Cambridge, 2017.*

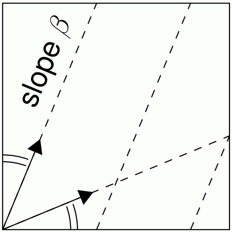
D. Frettlöh, A. Garber, and N. Mañibo. Substitution tilings with transcendental inflation factor, December 2022. `arXiv:2208.01327`

but still, golden mean is very **omnipresent** in aperiodic tilings.

Metallic means

Definition

The ***n*-th metallic mean** is the positive root of $x^2 - nx - 1$.



$$\beta = \frac{n + \sqrt{n^2+4}}{2} = n + \cfrac{1}{n + \cfrac{1}{n + \cfrac{1}{\cdots}}}$$

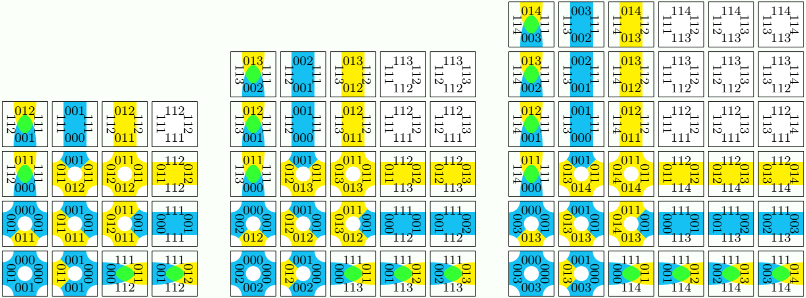
https://oeis.org/wiki/Metallic_means

V. W. de Spinadel. The family of metallic means. Vis. Math., *1(3) :1 HTML document ; approx. 16, 1999.*

Also called ***silver means*** *(Schroeder 1991) or* ***noble means*** *(Baake, Grimm, 2013).*

A family $\{\mathcal{T}_n\}_{n\geq 1}$ of metallic mean Wang tiles

For every integer $n \geq 1$, $\mathcal{T}_n$ is made of n^2 white tiles, $2n$ blue stripe tiles, $2n$ yellow stripe tiles, $2(n+1)$ green tiles and 7 junction tiles. Total : $n^2+6n+9 = (n+3)^2$ tiles.



$\mathcal{T}_1 \equiv$ **Ammann** $\qquad \mathcal{T}_2 \qquad \mathcal{T}_3$

Tile labels are vectors in the finite set

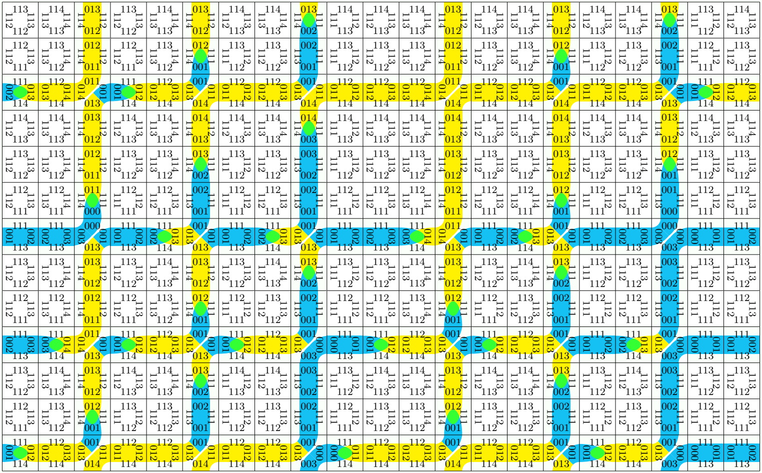
$$V_n = \{(v_0, v_1, v_2) \in \mathbb{N}^3 : 0 \leq v_0 \leq v_1 \leq 1 \text{ and } v_1 \leq v_2 \leq n+1\}$$

that we represent compactly as words, e.g., $113 := (1, 1, 3)$.

Metallic mean Wang shift

The ***n*-th metallic mean Wang shift** is $\mathbb{Z}^2 \overset{\sigma}{\curvearrowright} \Omega_n$ where

$$\Omega_n := \Omega_{\mathcal{T}_n} = \{w : \mathbb{Z}^2 \to \mathcal{T}_n \colon w \text{ is a valid configuration}\}.$$



(a 21×13 valid patch with $\mathcal{T}_3$)

Self-similarity, aperiodicity and minimality

Theorem

For every integer $n \geq 1$, the metallic mean Wang shift Ω_n is

- **self-similar**,
- **aperiodic** and
- **minimal** (if X subshift and $\varnothing \neq X \subset \Omega_n$, then $X = \Omega_n$).

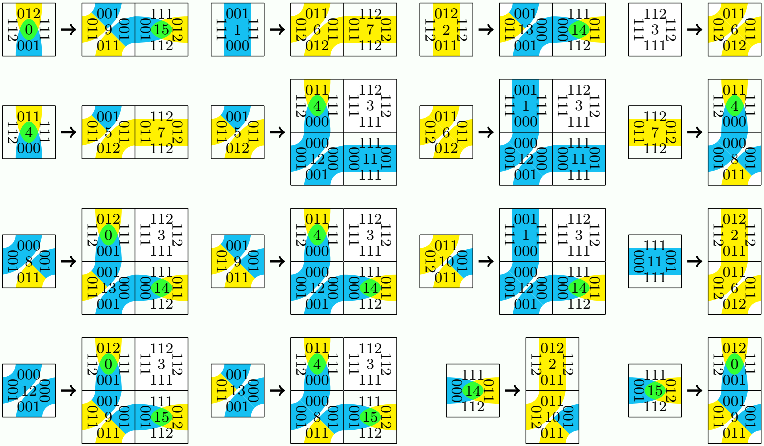
The inflation factor of the self-similarity of Ω_n is the n-th metallic mean, that is, the positive root of $x^2 - nx - 1$.

Self-similarity proof (main idea) : the set of **return blocks** to the junction tiles J_n is in bijection with some **extended set** $\mathcal{T}'_n \supseteq \mathcal{T}_n$.

2D structure associated with 1D **substitution** $a \mapsto ab^n, b \mapsto ab^{n-1}$.

Metallic mean Wang tiles I : self-similarity, aperiodicity and minimality. Forum of Mathematics, Sigma 13 (2025) e133. doi:10.1017/fms.2025.10069

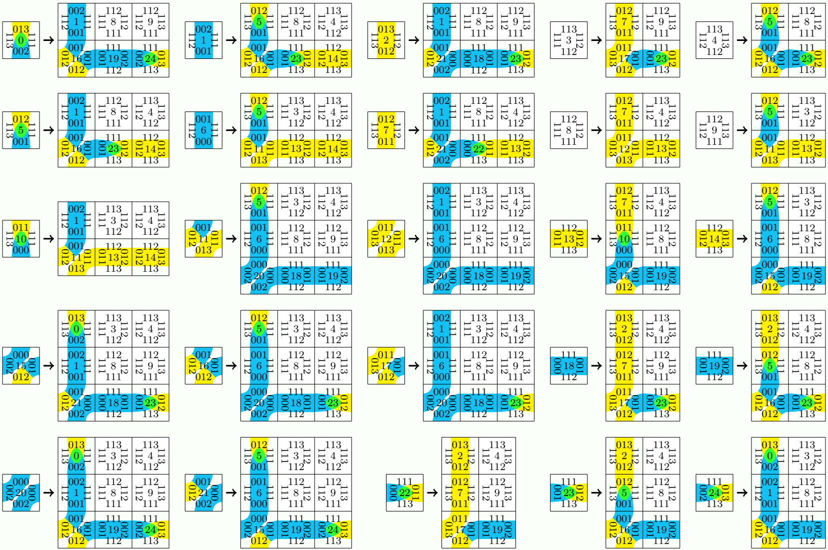
Substitution $\omega_1 : \Omega_1 \to \Omega_1$



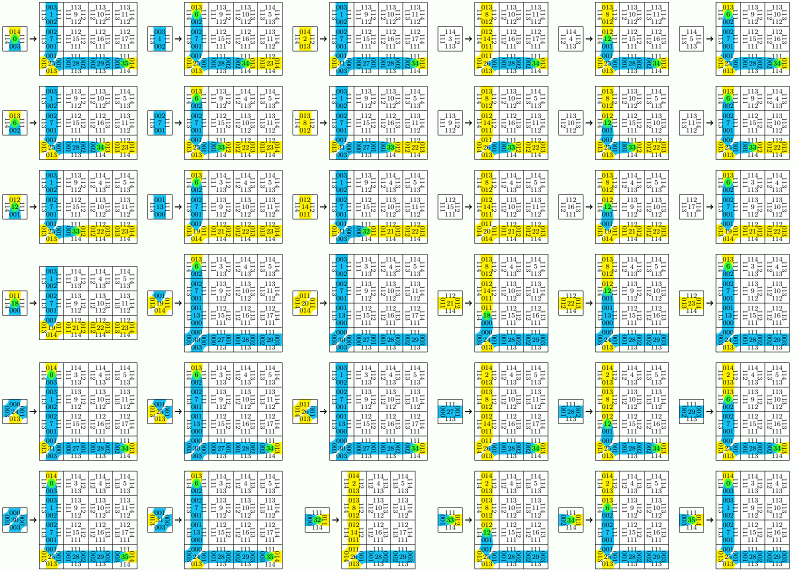
The self-similarity is a non-uniform rectangular 2-dimensional substitution as in Mozes (1989).

Shahar Mozes. Tilings, substitution systems and dynamical systems generated by them. J. Analyse Math., *53 :139–186, 1989.*

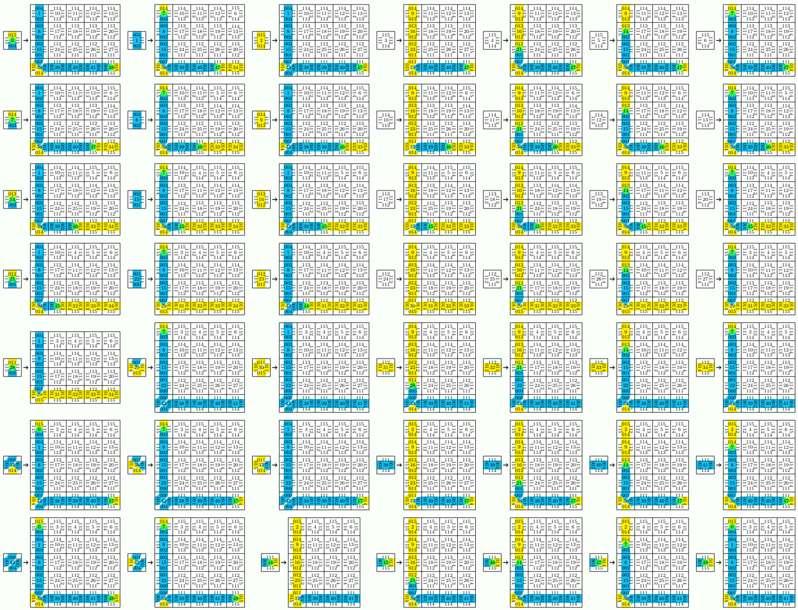
Substitution $\omega_2 : \Omega_2 \to \Omega_2$



Substitution $\omega_3 : \Omega_3 \to \Omega_3$



Substitution $\omega_4 : \Omega_4 \to \Omega_4$



Outline

The θ_n-computer chip

For every integer $n \geq 1$, we define the finite set of vectors

$$V_n = \{(v_0, v_1, v_2) \in \mathbb{N}^3 \colon 0 \leq v_0 \leq v_1 \leq 1 \text{ and } v_1 \leq v_2 \leq n+1\}$$

with **nondecreasing** entries. Let

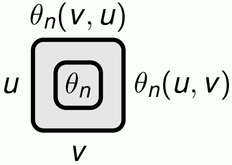
$$\begin{array}{rccc} \boldsymbol{\theta_n} : & V_n \times V_n & \to & \mathbb{Z}^3 \\ & (u_0, u_1, u_2), (v_0, v_1, v_2) & \mapsto & (r_0, r_1, r_2), \end{array}$$

be the **map** defined by the rule

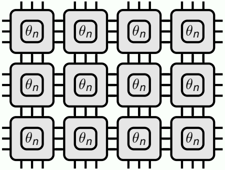
$$r_0 = u_0, \quad r_1 = \begin{cases} v_2 - n & \text{if } u_0 = 0, \\ 1 & \text{if } u_0 = 1, \end{cases}, \quad r_2 = \begin{cases} v_1 + u_0 & \text{if } v_0 = 0, \\ u_2 + 1 & \text{if } v_0 = 1. \end{cases}$$

Let

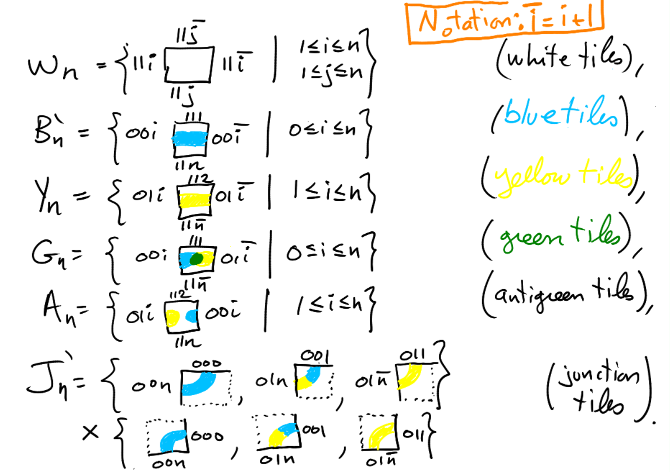
$\mathcal{C}_n = \Bigg\{$



$\Bigg| \; u, v \in V_n$ and $\theta_n(u, v)$, $\theta_n(v, u) \in V_n \Bigg\}$



be the set of instances of the θ_n-chip with outputs **restricted** to V_n.



An **extended set** $\mathcal{T}'_n$ of metallic mean Wang tiles :

$$\mathcal{T}'_n = W_n \cup B'_n \cup G_n \cup Y_n \cup A_n \cup \widehat{B'_n} \cup \widehat{G_n} \cup \widehat{Y_n} \cup \widehat{A_n} \cup J'_n.$$

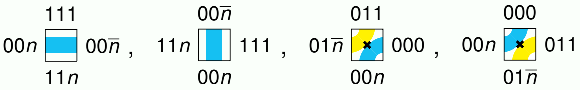
Theorem

For every integer $n \geq 1$, $\mathcal{C}_n = \mathcal{T}'_n$.

A family $\{\mathcal{T}_n\}_{n\geq 1}$ of metallic mean Wang tiles

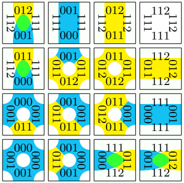
The following tiles are **non-extendible** (proof is not trivial) :

$\mathcal{D} = A_n \cup \widehat{A_n} \cup \{$

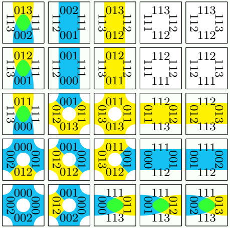


$\}$.

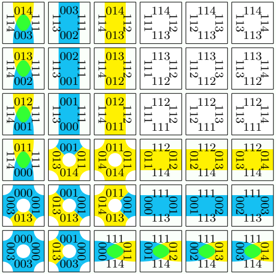
The **subset $\mathcal{T}_n = \mathcal{T}'_n \setminus \mathcal{D}$ of metallic mean Wang tiles** contains $(n+3)^2$ tiles :



$\mathcal{T}_1$



$\mathcal{T}_2$

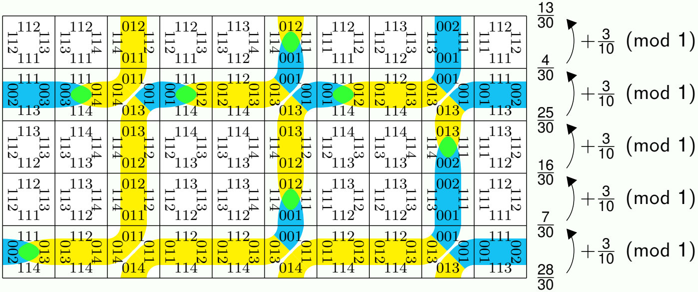


$\mathcal{T}_3$

An explicit factor map (example)

A 10×5 valid rectangular tiling with the set $\mathcal{T}_n$ with $n = 3$.

The numbers indicated in the right margin are the average of the inner products $\langle \frac{1}{n} d, v \rangle$ over the vectors v appearing as top (or bottom) labels of a horizontal row of tiles and where $d = (0, -1, 1)$.



We observe that these numbers increase by $\frac{3}{10} \pmod 1$ from row to row. The number $\frac{3}{10}$ is equal to the frequency of columns containing junction tiles (a junction tile is a tile whose labels all start with 0).

An explicit factor map

Theorem

Let $d = (0, -1, 1)$, $n \geq 1$ be an integer and Ω_n be the n^{th} metallic mean Wang shift. The map

$$\begin{array}{rccl} \Phi_n : & \Omega_n & \to & \mathbb{T}^2 \\ & w & \mapsto & \lim\limits_{k\to\infty} \dfrac{1}{2k+1} \displaystyle\sum_{i=-k}^{k} \begin{pmatrix} \langle \frac{1}{n} d, \textsc{Right}(w_{0,i}) \rangle \\ \langle \frac{1}{n} d, \textsc{Top}(w_{i,0}) \rangle \end{pmatrix} \end{array}$$

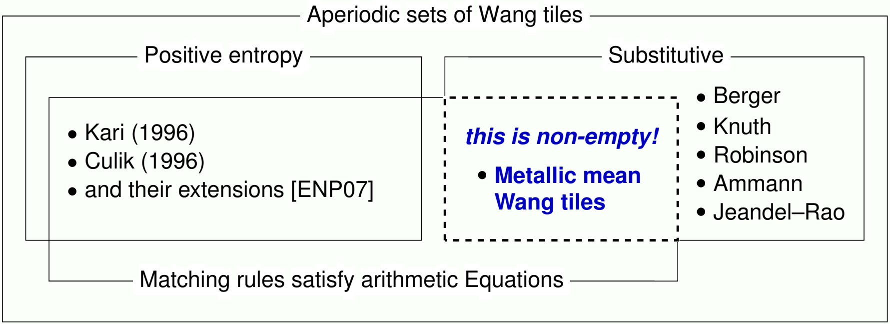
is a factor map commuting the shift $\mathbb{Z}^2 \overset{\sigma}{\curvearrowright} \Omega_n$ with $\mathbb{Z}^2 \overset{R_n}{\curvearrowright} \mathbb{T}^2$ by the equation $\Phi_n \circ \sigma^k = R_n^k \circ \Phi_n$ for every $k \in \mathbb{Z}^2$ where

$$\begin{array}{rccl} R_n & : \mathbb{Z}^2 \times \mathbb{T}^2 & \to & \mathbb{T}^2 \\ & (k, x) & \mapsto & R_n^k(x) := x + \beta k \end{array}$$

and $\beta = \frac{n+\sqrt{n^2+4}}{2}$ is the n^{th} metallic mean, that is, the positive root of the polynomial $x^2 - nx - 1$.

Metallic mean Wang tiles II : the dynamics of an aperiodic computer chip. Forum of Mathematics, Sigma 13 (2025) e155. doi:10.1017/fms.2025.10098

Venn Diagram again



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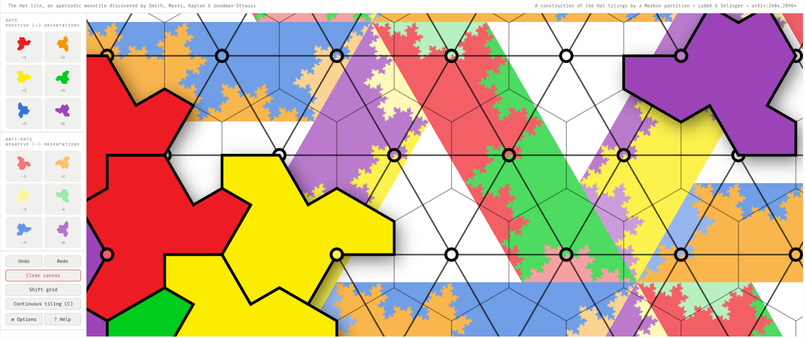
Metallic mean Wang tiles II : the dynamics of an aperiodic computer chip.
Forum of Mathematics, Sigma 13 (2025) e155. doi:10.1017/fms.2025.10098

A constr. of the Hat tilings by a Markov partition



L., Selinger, A construction of the hat tilings by a Markov partition, arXiv:2604.20964 .

A JavaScript application for placing tiles on the Markov partition :

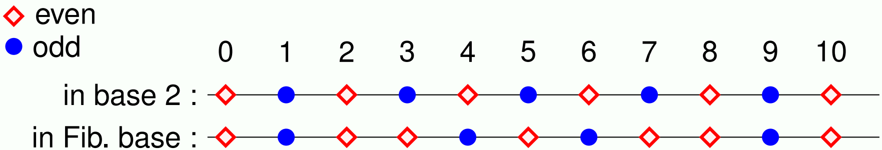
https://www.mathstat.dal.ca/~selinger/hat-partition/

Additional DIY documents :

http://www.slabbe.org/blogue/2026/03/a-construction-of-the-hat-tilings-by-a-markov-partition/

Outline

One-dimensional crystallography

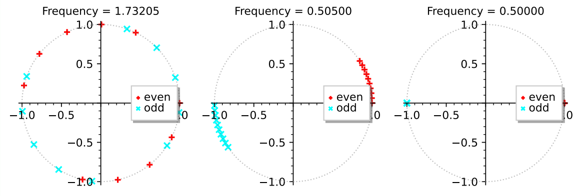


n	$\mathrm{rep}_2(n)$	parity
0=0	0	even
1=1	1	odd
2=2	10	even
3=2+1	11	odd
4=4	100	even
5=4+1	101	odd
6=4+2	110	even
7=4+2+1	111	odd
8=8	1000	even
9=8+1	1001	odd
10=8+2	1010	even
11=8+2+1	1011	odd
12=8+4	1100	even
13=8+4+1	1101	odd

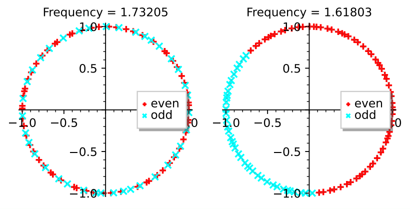
n	$\mathrm{rep}_F(n)$	parity
0=0	0	even
1=1	1	odd
2=2	10	even
3=3	100	even
4=3+1	101	odd
5=5	1000	even
6=5+1	1001	odd
7=5+2	1010	even
8=8	10000	even
9=8+1	10001	odd
10=8+2	10010	even
11=8+3	10100	even
12=8+3+1	10101	odd
13=13	100000	even

Guessing a frequency (rotation angle)

The odd/even in base 2 has frequency $\frac{1}{2}$:



The odd/even in Fibonacci base has frequency $\frac{1}{2}(1+\sqrt{5}) \approx 1.618$:

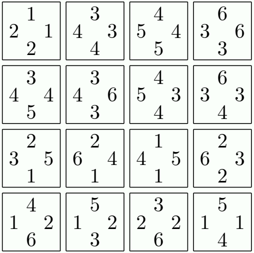


This is an experimental proof of a theorem on Sturmian sequences.

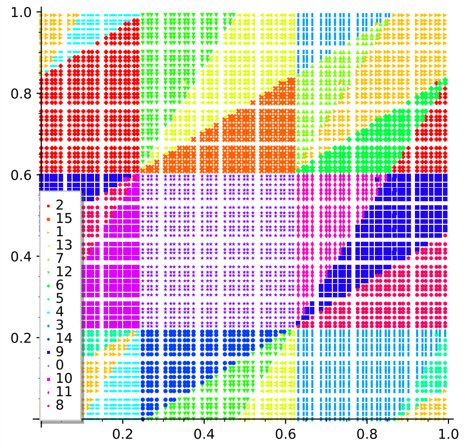
Morse, Hedlund, 1940 *Coven, Hedlund, 1970* *Fogg, 2002* *Lothaire 2002*

Wrapping Ammann tilings

Wrapping Ammann tilings on the 2-torus with frequency $\gamma = \begin{pmatrix} \varphi & 0 \\ 0 & \varphi \end{pmatrix}^{-1}$:



gives



Let $w : \mathbb{Z}^2 \to \mathcal{T}$ be a valid configuration. For every tile $t \in \mathcal{T}$, let

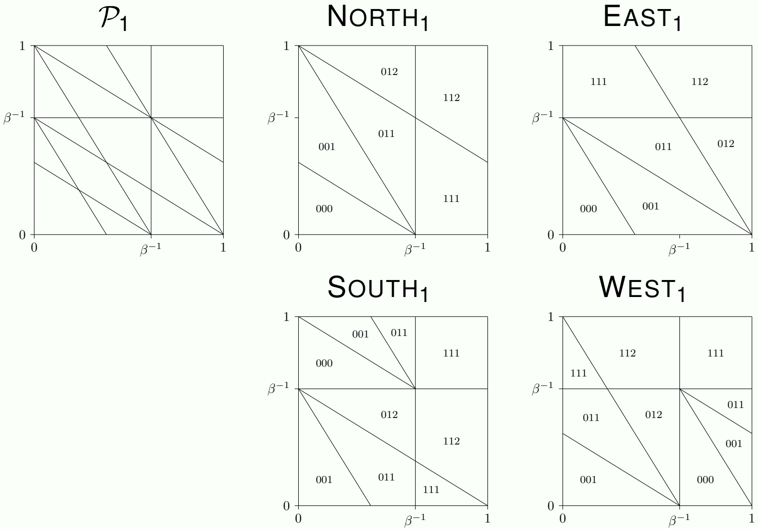
$$P_t(\gamma) = e^{2\pi i \, \gamma \, w^{-1}(t)} \subset \mathbb{R}^2/\mathbb{Z}^2.$$

For Ammann tiles, $\{\overline{P_t(\gamma)} \mid t \in \mathcal{T}\}$ is a polygonal partition of $\mathbb{R}^2/\mathbb{Z}^2$.

Appendix

Understanding the Ammann partition

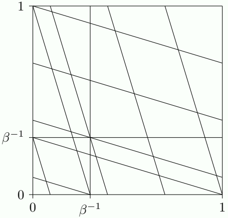
Unfold the partition into 4 partitions, one for each tile label :



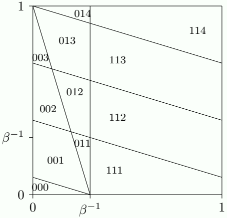
$$(1 \mapsto 112, \quad 2 \mapsto 111, \quad 3 \mapsto 001, \quad 4 \mapsto 011, \quad 5 \mapsto 012, \quad 6 \mapsto 000).$$

Replace golden ratio by other metallic mean

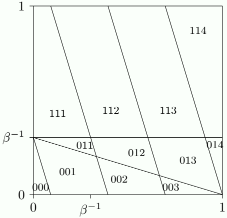
$\mathcal{P}_3$



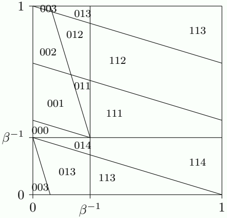
$\textsc{North}_3$



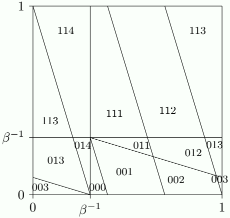
$\textsc{East}_3$



$\textsc{South}_3$



$\textsc{West}_3$

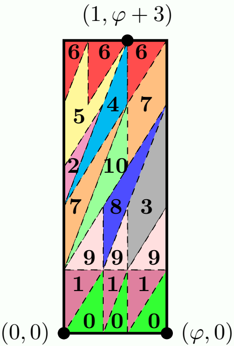


Works for all metallic means : Golden, Silver, Bronze, Copper, Nickel, etc.

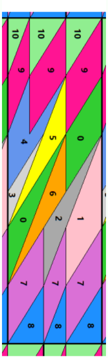
Open questions on Jeandel-Rao tilings

- Describe all of the 33 candidates of 11 Wang tiles listed by JR
  recent progress was made by Thompson (2022) and Mann (2024)

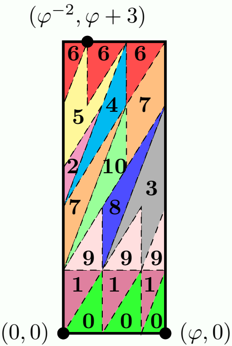
partition for JR



Thompson (2022)



Mann et al. (2024)

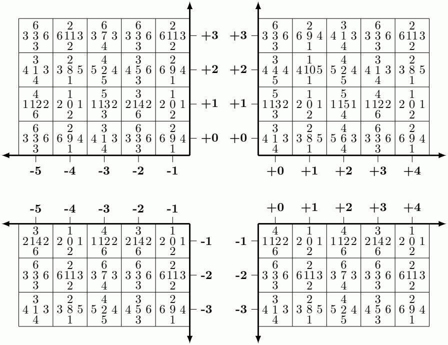


R. D. Thompson. "The Jeandel-Rao Aperiodic Wang Tilings of the Plane". MSc in Mathematics. The Open University, Milton Keynes, UK, May 2022

Hults, Jitsukawa, Mann, Zhang, Experimental Results on Potential Markov Partitions for Wang Shifts arXiv:2302.13516

Jana Lepšová's Ph. D. thesis

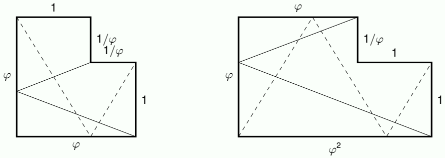
Let $\mathrm{rep}_{\mathcal{F}c}$ be the (padded) **Fibonacci comp. num. syst.** for $\mathbb{Z}^2$.
There exists a **deterministic finite automaton with output** (DFAO) $\mathcal{A}$ such that $\binom{n_1}{n_2} \mapsto \mathcal{A}\left(\mathrm{rep}_{\mathcal{F}c}\binom{n_1}{n_2}\right)$ is a **valid tiling** with Ammann tiles



L., Lepšová. *A Fibonacci analogue of the two's complement numeration system. RAIRO - Theoretical Informatics and Applications 57 (2023) 12.*

L., Lepšová. *Dumont-Thomas complement numeration systems for $\mathbb{Z}$. Integers 24 (2024) Paper No. A112, 27 pages.* doi:10.5281/zenodo.14340125

Ammann A2 encoded into 16 Wang tiles



Tilings in the Ammann A2 family can be encoded into 16 Wang tiles :

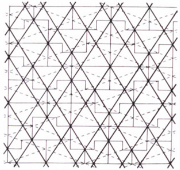


Figure 11.1.10
A tiling by the set A2 of Ammann prototiles with the four families of Ammann bars indicated, two by solid and two by dashed lines.

Figure 11.1.10

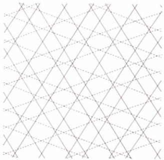


Figure 11.1.11
The Ammann bars of Figure 11.1.10 after the tiles have been detected. The solid bars are to be regarded as the edges of a new tiling by rhombs and parallelograms, the dashed bars are to be regarded as markings on the tiles specifying the matching condition.

Figure 11.1.11

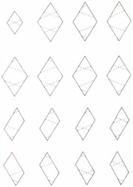


Figure 11.1.12
The 16 tiles that arise as indicated in Figure 11.1.11.

Figure 11.1.12

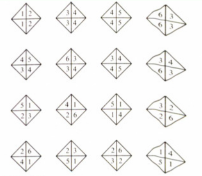


Figure 11.1.13
The 16 Wang tiles that correspond to the tiles of Figure 11.1.12. These form the smallest known aperiodic set.

Figure 11.1.13

- Find **geometric shapes** with Ammann bars on them associated with metallic-mean Wang tiles for $n \geq 2$.

Branko Grünbaum and G. C. Shephard. 1987

Existence of valid tilings

For every $(x, y) \in [0,1)^2$, let $\Lambda_n(x,y) = \begin{pmatrix} \lfloor y - \beta^{-1} + 1 \rfloor \\ \lfloor \beta^{-1} x + y - \beta^{-1} + 1 \rfloor \\ \lfloor \beta x + y - \beta^{-1} + 1 \rfloor \end{pmatrix} \in \mathbb{N}^3$

where β is the positive root of the polynomial $x^2 - nx - 1$ and

$$\Lambda_n(y, x)$$

$$t_n(x,y) = \quad \Lambda_n(\{x - \beta^{-1}\}, y) \ \square \ \Lambda_n(x, y)$$ be a Wang tile

$$\Lambda_n(\{y - \beta^{-1}\}, x)$$

where $\{x\} = x - \lfloor x \rfloor$ is the fractional part of a number $x \in \mathbb{R}$.

Theorem

For every integer $n \geq 1$ and every $(x, y) \in [0,1)^2$,

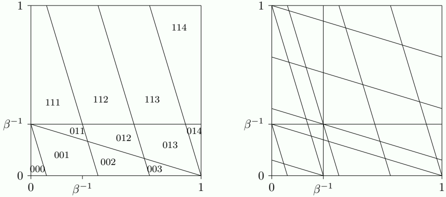
$$\begin{array}{rccc} c_{(x,y)}: & \mathbb{Z}^2 & \to & \mathcal{T}_n \\ & (i,j) & \mapsto & t_n\left(\{x + i\beta^{-1}\}, \{y + j\beta^{-1}\}\right) \end{array}$$

is a **valid tiling** with the **metallic mean** Wang tiles $\mathcal{T}_n$.

A Markov partition

$\mathcal{P}_n = \{\Phi_n([t])\}_{t\in\mathcal{T}_n}$ partitions the unit square into $(n+3)^2$ polygons.

$\{\Lambda_3^{-1}(v)\}_{v\in V_3}$ $\qquad$ $\mathcal{P}_3$



Theorem

For every integer $n \geq 1$, the symbolic dynamical system $\mathcal{X}_{\mathcal{P}_n,R_n}$ corresponding to $\mathcal{P}_n$, R_n **is the metallic mean Wang shift** Ω_n :

$$\Omega_n = \mathcal{X}_{\mathcal{P}_n,R_n}.$$

In particular, $\mathcal{P}_n$ is a **Markov partition** for $\mathbb{Z}^2 \overset{R_n}{\curvearrowright} \mathbb{T}^2$.

(this is the partition of the window in the internal space of a 4-to-2 CAP)

An isomorphism (mod 0)

Theorem

The Wang shift Ω_n and the $\mathbb{Z}^2$-action R_n have the following additional properties :

- $\mathbb{Z}^2 \overset{R_n}{\curvearrowright} \mathbb{T}^2$ is the **maximal equicontinuous factor** of $\mathbb{Z}^2 \overset{\sigma}{\curvearrowright} \Omega_n$,
- the factor map $\Phi_n : \Omega_n \to \mathbb{T}^2$ is **almost one-to-one** and its **set of fiber cardinalities** is $\{1, 2, 8\}$,
- the shift-action $\mathbb{Z}^2 \overset{\sigma}{\curvearrowright} \Omega_n$ on the metallic mean Wang shift is **uniquely ergodic**,
- the measure-preserving dynamical system $(\Omega_n, \mathbb{Z}^2, \sigma, \nu)$ is **isomorphic** to $(\mathbb{T}^2, \mathbb{Z}^2, R_n, \lambda)$ where ν is the unique shift-invariant probability measure on Ω_n and λ is the Haar measure on $\mathbb{T}^2$.