Numeration systems for aperiodic Wang tilings

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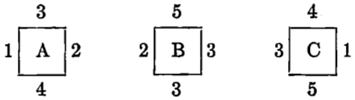
Outline

1. Motivation
2. Kari/Culik, Jeandel–Rao, Ammann sets of Wang tiles
3. A Fibonacci analog of two's complement numeration system
4. Dumont–Thomas complement num. systems for $\mathbb{Z}$ and $\mathbb{Z}^d$
5. Open questions

Outline

Wang tiles



Then we can easily find an infinite solution by the following argument. The following configuration satisfies the constraint on the edges:

A	*B*	*C*
C	*A*	*B*
B	*C*	*A*

Now the colors on the periphery of the above block are seen to be the following:

	3	5	4	
1				1
3				3
2				2
	3	5	4	

Wang's original question : is it true that a set of Wang tiles tile the plane if and only if there exists such a cyclic rectangle ?

H. Wang. Proving theorems by pattern recognition – II. Bell System Technical Journal*, 40(1) :1–41, January 1961.* doi:10.1002/j.1538-7305.1961.tb03975.x

Aperiodic Wang tile sets

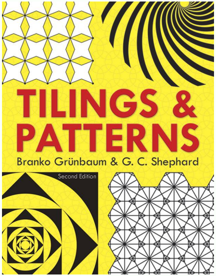
- 1966 (Berger) : 20426 tiles (lowered down later to 104)
- 1968 (Knuth) : 92 tiles
- 1971 (Robinson) : 56 tiles
- **1971 (Ammann) : 16 tiles**
- 1987 (Grunbaum) : 24 tiles
- **1996 (Kari and Culik) : 14 tiles and 13 tiles**
- **2015 (Jeandel, Rao) : 11 tiles**

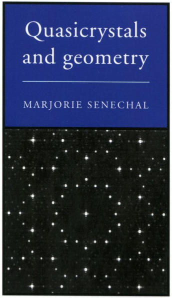
Theorem (Jeandel, Rao, 2015)

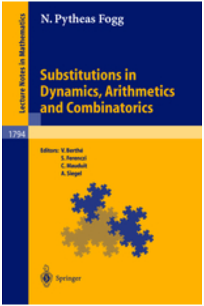
All sets of ≤ 10 Wang tiles are **periodic** or **don't tile** the plane.

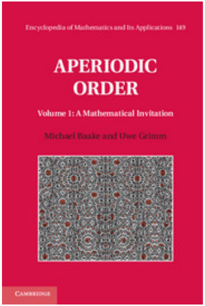
Emmanuel Jeandel and Michaël Rao. An aperiodic set of 11 Wang tiles. Adv. Comb. ***37*** *(2021) Id/No 1.*

Books





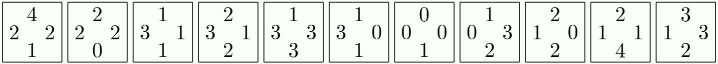




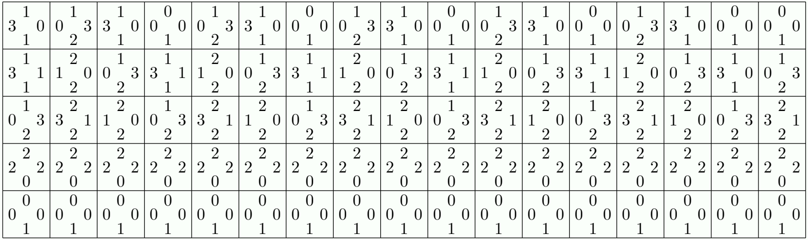
- Tilings and Patterns, by Grünbaum & Shephard, 1987
- Quasicrystals and Geometry, Senechal, 1995
- Pytheas Fogg's book, 2002
- Aperiodic Order, Baake & Grimm, 2013

Small aperiodic sets and Numeration systems

Given : a small aperiodic set $\mathcal{T}$ of Wang tiles



Objective : Construct a valid configuration $\mathbb{Z}^2 \to \mathcal{T}$



Today's question

Can **numeration systems** help us to decide which tile is at position $(n_1, n_2) \in \mathbb{Z}^2$?

Outline

Balanced representation of a real number

The **balanced representation of a real number** $x \in \mathbb{R}$ is the bi-infinite sequence $B(x) \in \{\lfloor x \rfloor, \lceil x \rceil\}^{\mathbb{Z}}$ where

$$B(x)_i = \lfloor ix \rfloor - \lfloor (i-1)x \rfloor.$$

Note : $x \in \mathbb{R} \setminus \mathbb{Q}$ if and only if $B(x)$ is a Sturmian sequence.

Lemma (Kari, 2016)

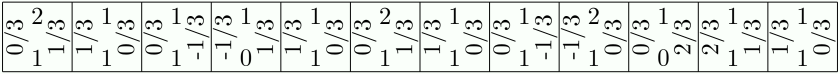
For **every** $q \in \mathbb{Q}_{>0}$ and **every** $x \in \mathbb{R}$, the sequence $\{t_{i,x}\}_{i\in\mathbb{Z}}$

$$B(x)_i$$

$$q\lfloor (i-1)x \rfloor - \lfloor q(i-1)x \rfloor \quad \boxed{t_{i,x}} \quad q\lfloor ix \rfloor - \lfloor qix \rfloor$$

$$B(qx)_i$$

is a **valid tiling of a horizontal strip** by the Wang tiles $t_{i,x}$.



Kari and Culik sets of Wang tiles

Lemma (Kari, 2016)

For every $q \in \mathbb{Q}_{>0}$ and every $a, b \in \mathbb{Z}$,

$$T_{q,a,b} = \left\{ t_{i,x} \mid i \in \mathbb{Z}, x \in [a, a+1], qx \in [b, b+1] \right\}$$

is a **finite set** of Wang tiles.

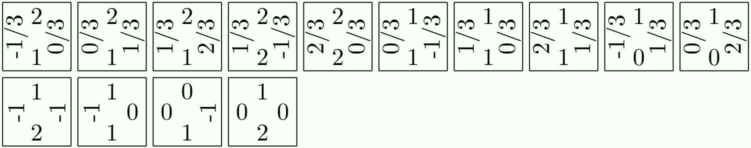
Kari's aperiodic set of 14 tiles is $T_{2,0,1} \cup T_{\frac{2}{3},1,1} \cup T_{\frac{2}{3},1,0}$. The proof of existence of tilings **involves balanced representations** of $x \in [\frac{2}{3}, 2]$ and the invertible map

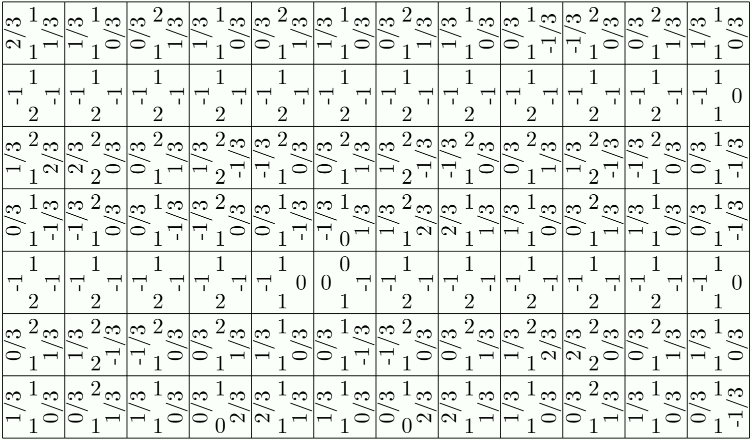
$$x \mapsto \begin{cases} 2x & \text{if } x \leq 1, \\ \frac{2}{3}x & \text{otherwise.} \end{cases}.$$

Culik's aperiodic set of 13 Wang tiles is similar.

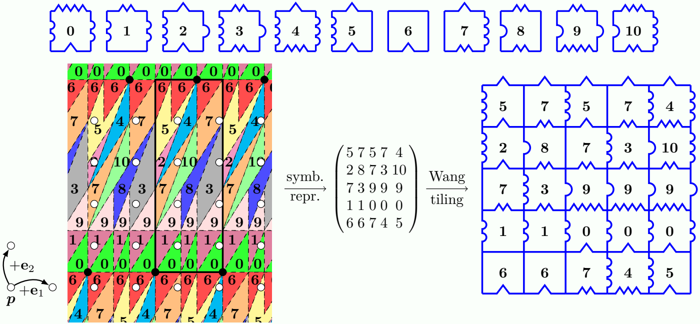
Jarkko Kari. Piecewise affine functions, Sturmian sequences and Wang tiles. Fundam. Inform., *145(3) :257–277, 2016.*

Kari set of 14 Wang tiles & a tiling





Jeandel–Rao aperiodic set of 11 Wang tiles



Markov partitions for toral $\mathbb{Z}^2$*-rotations featuring Jeandel-Rao Wang shift and model sets.* Ann. H. Lebesgue *4 (2021) 283–324.* doi:10.5802/ahl.73

Rauzy induction of polygon partitions and toral $\mathbb{Z}^2$*-rotations.* J. Mod. Dyn. *17 (2021) 481–528.* doi:10.3934/jmd.2021017

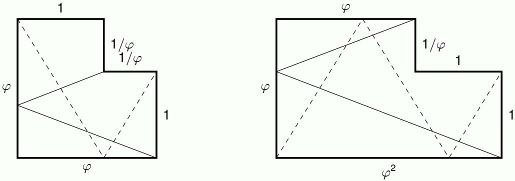
Substitutive structure of Jeandel-Rao aperiodic tilings. Discrete Comput. Geom., *65 (2021) 800–855.* doi:10.1007/s00454-019-00153-3

Open question. Jeandel–Rao is not alone : characterize the family.

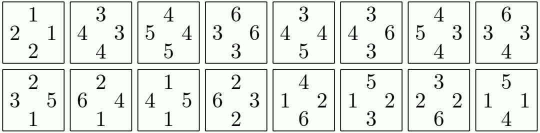
Ammann A2 encoded into 16 Wang tiles

Two shapes belonging to the Ammann A2 family :

(dashed and solid lines in the interior of the tiles must continue straight across the edges of the tiling)



Cutting the solid Ammann bars of a tiling yields parallelograms which can be encoded into 16 Wang tiles :



Branko Grünbaum and G. C. Shephard. Tilings and patterns. *W. H. Freeman and Company, New York, 1987.*

Ammann 16 Wang tiles

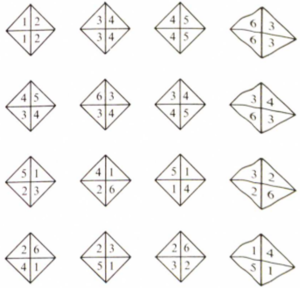


Figure 11.1.13
The 16 Wang tiles that correspond to the tiles of Figure 11.1.12. These form the smallest known aperiodic set.

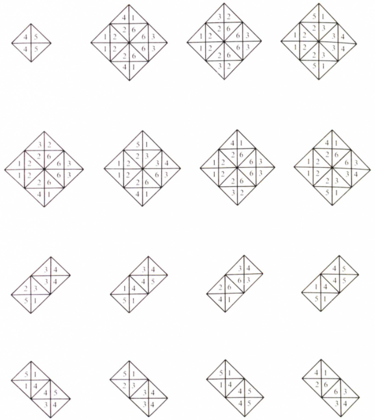


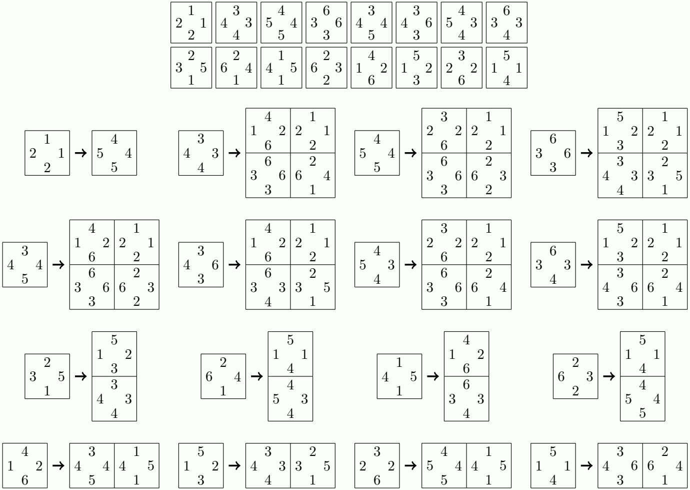
Figure 11.1.16
This diagram shows how the Wang tiles of Figure 11.1.13 can be "decomposed".

Ammann tiles (p.595)

16 equivalent supertiles (p.597)

Branko Grünbaum and G. C. Shephard. Tilings and patterns. *W. H. Freeman and Company, New York, 1987.*

Ammann 16 Wang tiles & unique decomposition



Branko Grünbaum and G. C. Shephard. Tilings and patterns. *W. H. Freeman and Company, New York, 1987.*

Symbolic 2-dimensional substitution

Encoding each Ammann tile by the indices from the set $A = \{0, 1, \ldots, 15\}$

Tile (center)	Top	Left	Right	Bottom
0	1	2	1	2
1	3	4	3	4
2	4	5	4	5
3	6	3	6	3
4	3	4	4	5
5	3	4	6	3
6	4	5	3	4
7	6	3	3	4
8	2	3	5	1
9	2	6	4	1
10	1	4	5	1
11	2	6	3	2
12	4	1	2	6
13	5	1	2	3
14	3	2	2	6
15	5	1	1	4

,

we obtain a 2-dimensional substitution $\phi : A \to A^{*^2}$:

$$\phi : \begin{cases} 0 \mapsto (2), & 1 \mapsto \begin{pmatrix} 12 & 0 \\ 3 & 9 \end{pmatrix}, 2 \mapsto \begin{pmatrix} 14 & 0 \\ 3 & 11 \end{pmatrix}, 3 \mapsto \begin{pmatrix} 13 & 0 \\ 1 & 8 \end{pmatrix}, \\ 4 \mapsto \begin{pmatrix} 12 & 0 \\ 3 & 11 \end{pmatrix}, & 5 \mapsto \begin{pmatrix} 12 & 0 \\ 7 & 8 \end{pmatrix}, 6 \mapsto \begin{pmatrix} 14 & 0 \\ 3 & 9 \end{pmatrix}, 7 \mapsto \begin{pmatrix} 13 & 0 \\ 5 & 9 \end{pmatrix}, \\ 8 \mapsto \begin{pmatrix} 13 \\ 1 \end{pmatrix}, & 9 \mapsto \begin{pmatrix} 15 \\ 6 \end{pmatrix}, 10 \mapsto \begin{pmatrix} 12 \\ 7 \end{pmatrix}, 11 \mapsto \begin{pmatrix} 15 \\ 2 \end{pmatrix}, \\ 12 \mapsto (4, 10), & 13 \mapsto (1, 8), 14 \mapsto (2, 10), 15 \mapsto (5, 9). \end{cases}$$

Shahar Mozes. Tilings, substitution systems and dynamical systems generated by them. J. Analyse Math., *53 :139–186, 1989.*

A configuration over $\mathbb{N}^2$

Applying the substitution ϕ on seed 1 produces two configurations

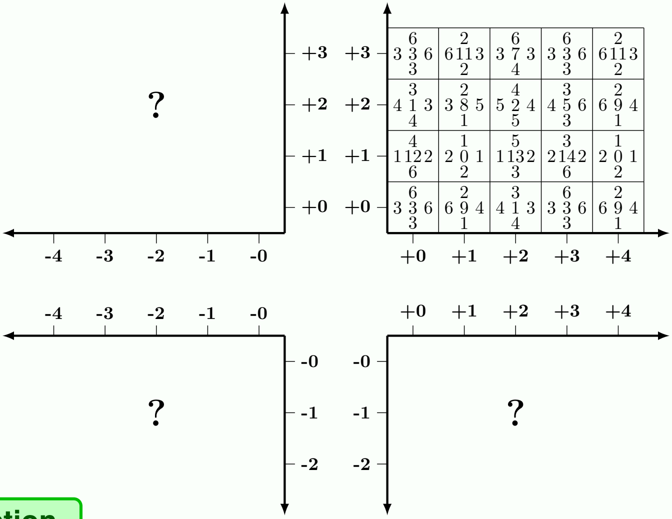
$$\begin{array}{|l}1\\\hline\end{array} \mapsto \begin{array}{|ll}12 & 0\\ 3 & 9\\\hline\end{array} \mapsto \begin{array}{|lll}4 & 10 & 2\\ 13 & 0 & 15\\ 1 & 8 & 6\\\hline\end{array} \mapsto \begin{array}{|lllll}12 & 0 & 12 & 14 & 0\\ 3 & 11 & 7 & 3 & 11\\ 1 & 8 & 2 & 5 & 9\\ 12 & 0 & 13 & 14 & 0\\ 3 & 9 & 1 & 3 & 11\\\hline\end{array} \mapsto \ldots$$

of the first quadrant $\mathbb{N}^2$ which are the **periodic points** $x = \phi^2(x)$ $= \lim_{k\to+\infty} \phi^{2k}(a)$ of period 2 starting from the **seeds** $a = 1$ or $a = 3$.

Theorem (E. Charlier, T. Kärki, M. Rigo, 2010)

The 2-dimensional infinite word $x \in \mathcal{B}^{\mathbb{N}^2}$ is the image under a coding of a shape-symmetric pure morphic word if and only if it is $\mathcal{S}$-automatic for some **abstract numeration system** $\mathcal{S} = (L, \Sigma, <)$ with $\varepsilon \in L$.

What about the other quadrants ?



Question

What is the best representation of -1 ? Is it -0 ?

Outline

Negative p-adic integers

On a finite size memory representing unsigned integers with base-10 digits, incrementing by 1 the largest representable number gives

$$\begin{array}{r} \texttt{9999999999999} \\ +1 \\ \hline \texttt{0000000000000} \end{array}$$

In the **ten's complement NS**, value -1 is represented by $\texttt{999}\cdots\texttt{9}$:

$$\cdots \xrightarrow{+1} \texttt{99997} \xrightarrow{+1} \texttt{99998} \xrightarrow{+1} \texttt{99999} \xrightarrow{+1} \texttt{00000} \xrightarrow{+1} \texttt{00001} \xrightarrow{+1} \cdots$$

D. E. Knuth. The art of computer programming. Vol. 2. *Addison-Wesley, Reading, MA, 1998. Seminumerical algorithms, Third edition.*

Likewise

$$\begin{array}{r} \cdots + 2^4 + 2^3 + 2^2 + 2^1 + 2^0 \\ +1 \\ \hline 0 \end{array} \qquad \begin{array}{r} \cdots + 55 + 21 + 8 + 3 + 1 \\ +1 \\ \hline 0 \end{array}$$

Two's complement

Let $\Sigma = \{0, 1\}$. Let $w = w_{k-1} \cdots w_0 \in \Sigma^*$ for some $k \in \mathbb{N}$.

$$\text{val}_2(w) = \sum_{i=0}^{k-1} w_i 2^i \qquad \text{val}_{2c}(w) = \text{val}_2(w) - w_{k-1} 2^k.$$

n	$\text{rep}_2(n)$	$\text{rep}_{2c}(n)$
8	001000	0001000
7	000111	0000111
6	000110	0000110
5	000101	0000101
4	000100	0000100
3	000011	0000011
2	000010	0000010
1	000001	0000001
0	000000	0000000
-1		1111111
-2		1111110
-3		1111101
-4		1111100
-5		1111011

- (Neutral prefix) For every $u \in \Sigma^*$, $\text{val}_{2c}(00u) = \text{val}_{2c}(0u)$ and $\text{val}_{2c}(11u) = \text{val}_{2c}(1u)$.
- For **every** $n \in \mathbb{Z}$, there exists a **unique word** $w \in \Sigma^* \setminus (00\Sigma^* \cup 11\Sigma^*)$ such that $n = \text{val}_{2c}(w)$.
- For every $n \in \mathbb{Z}$, we **denote** this unique word **by** $\text{rep}_{2c}(n)$.

Fibonacci analog of two's complement

Let $\Sigma = \{0,1\}$. Let $w = w_{k-1} \cdots w_0 \in \Sigma^*$ for some $k \in \mathbb{N}$.

$$\mathrm{val}_{\mathcal{F}}(w) = \sum_{i=0}^{k-1} w_i F_i \qquad \mathrm{val}_{\mathcal{F}c}(w) = \mathrm{val}_{\mathcal{F}}(w) - w_{k-1} F_k$$

n	$\mathrm{rep}_{\mathcal{F}}(n)$	$\mathrm{rep}_{\mathcal{F}c}(n)$
8	0010000	000010000
7	0001010	000001010
6	0001001	000001001
5	0001000	000001000
4	0000101	000000101
3	0000100	000000100
2	0000010	000000010
1	0000001	000000001
0	0000000	000000000
-1		101010101
-2		101010100
-3		101010010
-4		101010001
-5		101010000

$F_0 = 1$, $F_1 = 2$ and $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$

- (Neutral prefix) For every $u \in \Sigma^*$, $\mathrm{val}_{\mathcal{F}c}(000u) = \mathrm{val}_{\mathcal{F}c}(0u)$ and $\mathrm{val}_{\mathcal{F}c}(101u) = \mathrm{val}_{\mathcal{F}c}(1u)$.
- For **every** $n \in \mathbb{Z}$, there exists a **unique odd-length word** $w \in \Sigma^* \setminus (\Sigma^* 11 \Sigma^* \cup 000\Sigma^* \cup 101\Sigma^*)$ such that $n = \mathrm{val}_{\mathcal{F}c}(w)$.
- For every $n \in \mathbb{Z}$, we **denote** this unique word **by** $\mathrm{rep}_{\mathcal{F}c}(n)$.

Fibonacci's complement num. system for $\mathbb{Z}^2$

Definition (Numeration system $\mathcal{F}c$ for $\mathbb{Z}^2$)

For $\boldsymbol{n} = (n_1, n_2) \in \mathbb{Z}^2$, we define

$$\mathrm{rep}_{\mathcal{F}c}(\boldsymbol{n}) = \mathrm{pad}\begin{pmatrix}\mathrm{rep}_{\mathcal{F}c}(n_1)\\ \mathrm{rep}_{\mathcal{F}c}(n_2)\end{pmatrix},$$

where pad padds the representation shorter in length with the **neutral prefix**.

For example :

$$\mathrm{rep}_{\mathcal{F}c}\begin{pmatrix}-1\\ 6\end{pmatrix} = \mathrm{pad}\begin{pmatrix}\mathrm{rep}_{\mathcal{F}c}(-1)\\ \mathrm{rep}_{\mathcal{F}c}(6)\end{pmatrix} = \mathrm{pad}\begin{pmatrix}1\\ 01001\end{pmatrix} = \begin{pmatrix}\color{red}{1010}1\\ 01001\end{pmatrix}$$

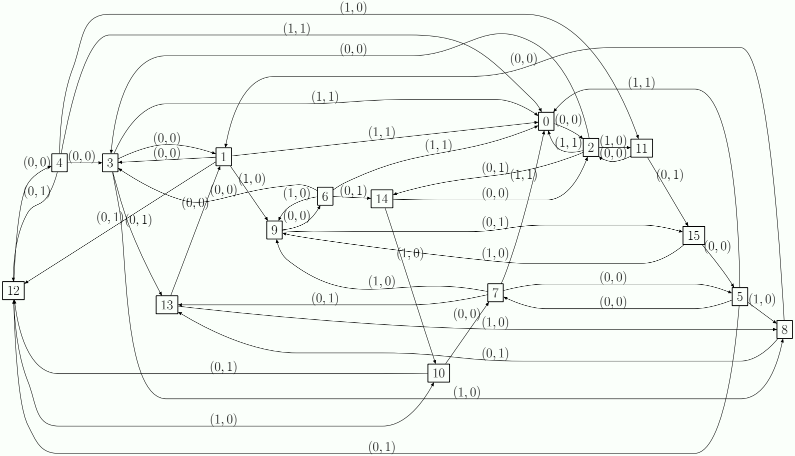
$$= \begin{pmatrix}1\\ 0\end{pmatrix}\begin{pmatrix}0\\ 1\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}\begin{pmatrix}0\\ 0\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix}.$$

L., Lepšová. A Fibonacci analogue of the two's complement numeration system. RAIRO - Theoretical Informatics and Applications 57 (2023) 12.

Automaton associated with a 2-dim. substitution

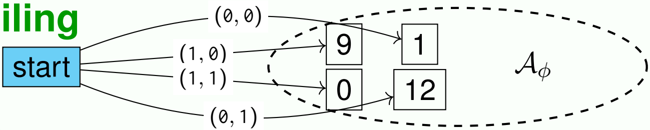
If $\phi : A \to A^{*^2}$ is a 2-dim. substitution, we define the **directed graph**

$$\mathcal{A}_\phi = \left\{ a \xrightarrow{(i,j)} b \;\middle|\; a \in A \text{ and } b \text{ is at position } (i,j) \text{ in } \phi(a) \right\}.$$



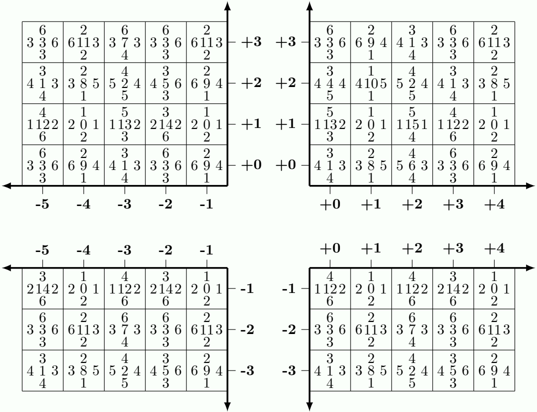
An Ammann tiling

Let $\mathcal{A}_{\phi,\begin{pmatrix}9&1\\0&12\end{pmatrix}} =$

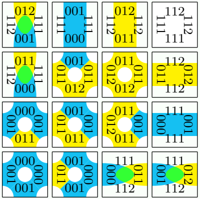


A valid tiling

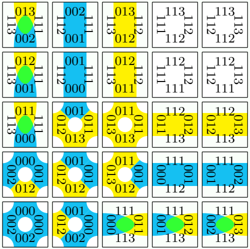
$\begin{pmatrix}n_1\\n_2\end{pmatrix} \mapsto \mathcal{A}_{\phi,\begin{pmatrix}9&1\\0&12\end{pmatrix}}\left(\mathrm{rep}_{\mathcal{F}c}\begin{pmatrix}n_1\\n_2\end{pmatrix}\right)$ is a valid Ammann tiling.



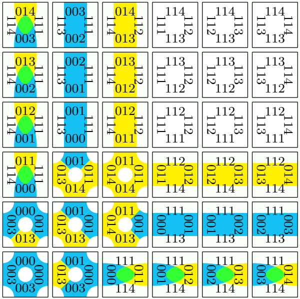
A family $\{\mathcal{T}_n\}_{n\geq 1}$ of metallic mean Wang tiles



$\mathcal{T}_1 \equiv$ **Ammann**



$\mathcal{T}_2$



$\mathcal{T}_3$

- $\Omega_{\mathcal{T}_n}$ related to the ***n*-th metallic mean** (pos. root of $x^2 - nx - 1$),
- structure associated with **substitution** $a \mapsto ab^n, b \mapsto ab^{n-1}$,
- related to the **balanced representation** of real numbers,

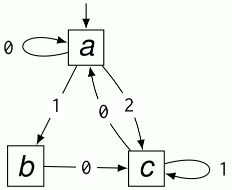
Metallic mean Wang tiles I : self-similarity, aperiodicity and minimality.
arXiv:2312.03652

Metallic mean Wang tiles II : the dynamics of an aperiodic computer chip.
arXiv:2403.03197

Outline

Paths in a graph (automaton)

The graph $\mathcal{A}_{\eta,a}$ associated with $\eta : a \mapsto abc, b \mapsto c, c \mapsto ac$ is



The **number of paths of length *n*** starting from vertex *a* is

$$\begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

In particular, there are

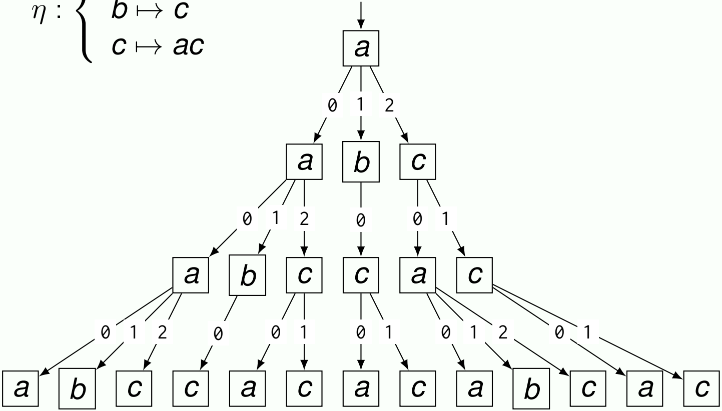
$$\begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}^2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 2 \\ 1 & 0 & 1 \\ 3 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 6$$

paths of length 2 starting in vertex *a*.

Set of paths

The set of paths starting in state *a* in the directed graph :

$$\eta : \begin{cases} a \mapsto abc \\ b \mapsto c \\ c \mapsto ac \end{cases}$$



paths
000
001
002
010
020
021
100
101
200
201
202
210
211

Dumont–Thomas (1989)

Definition (admissible sequence)

Let $\eta : A^* \to A^*$ be a substitution. Let $a \in A$ be a letter, k an integer and, for each integer i, $0 \leq i \leq k$, (m_i, a_i) be an element of $A^* \times A$. We say that the finite sequence $(m_i, a_i)_{i=0,\ldots,k}$ is **admissible with respect to η**

if and only if,

$m_{i-1}a_{i-1}$ is a prefix of $\eta(a_i)$ for all i with $1 \leq i \leq k$.

We say that this sequence is **a-admissible with respect to η** if it is admissible with respect to η and, moreover, $m_k a_k$ is a prefix of $\eta(a)$.

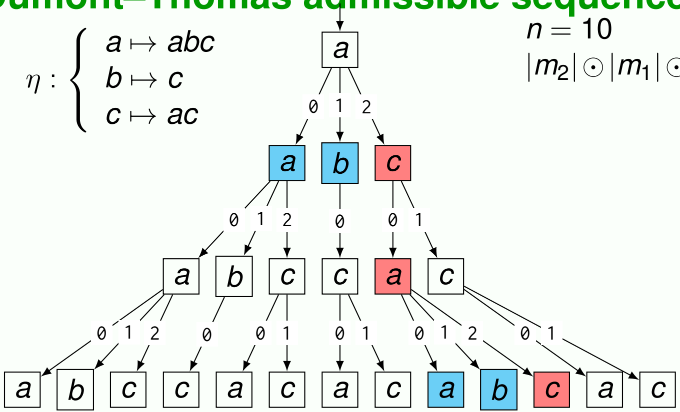
J.-M. Dumont and A. Thomas. Systemes de numeration et fonctions fractales relatifs aux substitutions. Theoret. Comput. Sci.*, 65(2) :153–169, 1989.*

Dumont–Thomas admissible sequences

$$\eta : \begin{cases} a \mapsto abc \\ b \mapsto c \\ c \mapsto ac \end{cases}$$

$n = 10$

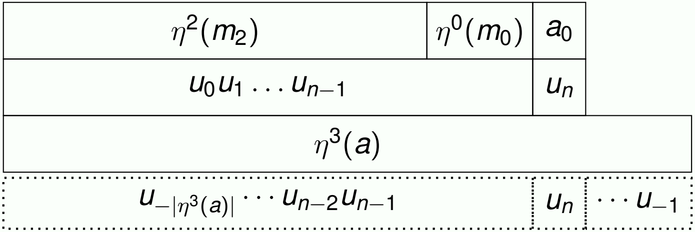
$|m_2| \odot |m_1| \odot |m_0| = 202$



$(m_2, a_2) = (ab, c)$

$(m_1, a_1) = (\varepsilon, a)$

$(m_0, a_0) = (ab, c)$



(as in Python!)

Dumont–Thomas (1989)

Theorem (Dumont, Thomas, 1989)

Let $a \in A$ and let $\eta : A^* \to A^*$ be a substitution. Let $u = \eta(u)$ be a **right-infinite fixed point of η with growing seed** $u_0 = a$. For every integer $n \geq 1$, there exists **a unique integer** $k = k(n)$ and **a unique sequence** $(m_i, a_i)_{i=0,\ldots,k}$ such that

- this sequence is a-admissible and $m_k \neq \varepsilon$,
- $u_0 u_1 \cdots u_{n-1} = \eta^k(m_k)\eta^{k-1}(m_{k-1}) \cdots \eta^0(m_0)$.

This leads to a representation of $n \in \mathbb{N}$:

$$\mathrm{rep}(n) = |m_k| \odot |m_{k-1}| \odot \cdots \odot |m_0|$$

J.-M. Dumont and A. Thomas. Systemes de numeration et fonctions fractales relatifs aux substitutions. Theoret. Comput. Sci., *65(2) :153–169, 1989.*

Dumont–Thomas for right-infinite periodic points

Theorem

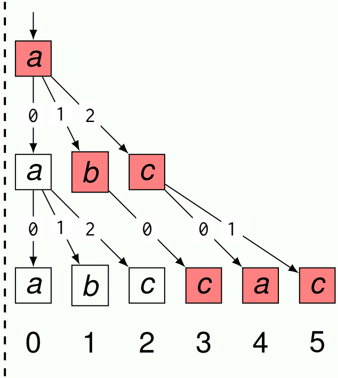
Let $a \in A$ and $\eta : A^* \to A^*$ be a substitution. Let $u \in \mathrm{Per}_{\mathbb{Z}_{\geq 0}}(\eta)$ be **a right-infinite periodic point** with growing seed $u_0 = a$.
Let $p \geq 1$ be a period of u.
For every integer $n \geq 1$, there exists **a unique integer** $k = k(n)$ such that p divides k and **a unique sequence** $(m_i, a_i)_{i=0,\ldots,k-1}$ such that

- this sequence is a-admissible and $m_{k-1} m_{k-2} \cdots m_{k-p} \neq \varepsilon$,
- $u_0 u_1 \cdots u_{n-1} = \eta^{k-1}(m_{k-1}) \eta^{k-2}(m_{k-2}) \cdots \eta^0(m_0)$.

L., Lepšová. Dumont-Thomas complement numeration systems for $\mathbb{Z}$.
arXiv:2302.14481

Example $\mathrm{rep} : \mathbb{Z}_{\geq 0} \to \{0, 1, 2\}^*$

The substitution $\eta : a \mapsto abc, b \mapsto c, c \mapsto ac$ has a **right-infinite periodic point** of period $p = 1$ (a fixed point) with **growing seed** a.



n	path	rep(n)
0	00	ε
1	01	1
2	02	2
3	10	10
4	20	20
5	21	21

The **neutral prefix** is the minimal path of length p :

$$\mathtt{W}_{\min} = 0^p = 0.$$

Dumont–Thomas for left-infinite periodic points

Theorem

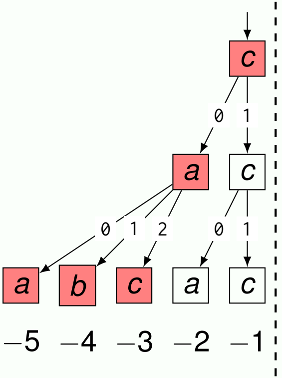
Let $b \in A$ and $\eta : A^* \to A^*$ be a substitution. Let $u \in \mathrm{Per}_{\mathbb{Z}_{<0}}(\eta)$ be **a left-infinite periodic point** with growing seed $u_{-1} = b$.
Let $p \geq 1$ be a period of u.
For every integer $n \leq -2$, there exists **a unique integer** $k = k(n)$ such that p divides k and **a unique sequence** $(m_i, a_i)_{i=0,\ldots,k-1}$ such that

- this sequence is b-admissible and
$$\eta^{p-1}(m_{k-1})\eta^{p-2}(m_{k-2})\cdots\eta^{0}(m_{k-p})a_{k-p} \neq \eta^{p}(b),$$
- $u_{-|\eta^k(b)|}\cdots u_{n-2}u_{n-1} = \eta^{k-1}(m_{k-1})\eta^{k-2}(m_{k-2})\cdots\eta^{0}(m_0)$.

L., Lepšová. Dumont-Thomas complement numeration systems for $\mathbb{Z}$.
arXiv:2302.14481

Example rep : $\mathbb{Z}_{<0} \to \{0, 1, 2\}^*$

The substitution $\eta : a \mapsto abc, b \mapsto c, c \mapsto ac$ has a **left-infinite periodic point** of period $p = 1$ (a fixed point) with **growing seed** c.



n	path	rep(n)
-5	00	00
-4	01	01
-3	02	02
-2	10	0
-1	11	ε

The **neutral prefix** is the **maximal** path starting from c of length p :

$$\mathtt{W}_{\mathrm{max}} = 1.$$

Dumont–Thomas complement NS for $\mathbb{Z}$

Definition

Let $\eta : A^* \to A^*$ be a substitution and $u \in \mathrm{Per}(\eta)$ **be a two-sided periodic point with growing seed** $u_{-1}|u_0$. Let $p \geq 1$ be the period of u. Let $\mathcal{D} = \{0, ..., \max_{c \in A} |\eta(c)| - 1\}$. We define

$$\begin{array}{rccl} \mathrm{rep}_u : & \mathbb{Z} & \to & \{0,1\} \odot \mathcal{D}^* \\ & n & \mapsto & \begin{cases} 0 \odot |m_{k-1}| \odot |m_{k-2}| \odot \ldots \odot |m_0| & \text{if } n \geq 1; \\ 0 & \text{if } n = 0; \\ 1 & \text{if } n = -1; \\ 1 \odot |m_{k-1}| \odot |m_{k-2}| \odot \ldots \odot |m_0| & \text{if } n \leq -2, \end{cases} \end{array}$$

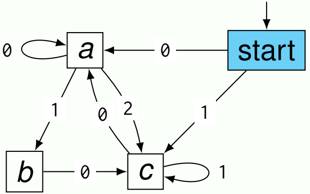
where $k = k(n) \geq 0$ **is the unique integer** and $(m_i, a_i)_{i=0,\ldots,k-1}$ **is the unique sequence** obtained from the previous theorems applied on the right-infinite periodic point $u|_{\mathbb{Z}_{\geq 0}}$ if $n \geq 1$ or on the left-infinite periodic point $u|_{\mathbb{Z}_{<0}}$ if $n \leq -2$, both with period p.

L., Lepšová. Dumont-Thomas complement numeration systems for $\mathbb{Z}$.

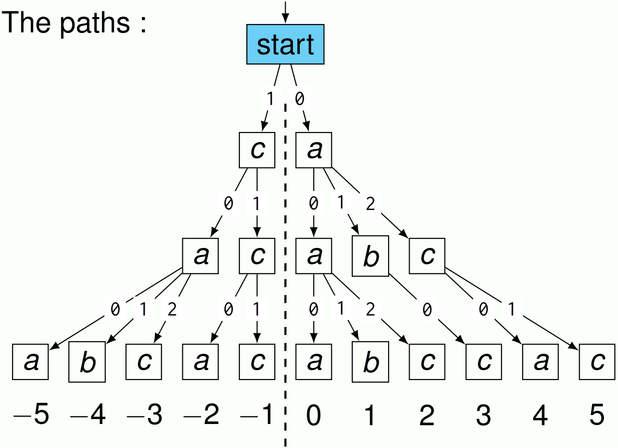
Example $\mathrm{rep}_u : \mathbb{Z} \to \{0,1\} \odot \mathcal{D}^*$

$u = \ldots c|a \cdots \in \mathrm{Per}_{\mathbb{Z}}(\eta)$

$\mathcal{A}_{\eta,c|a} =$



The paths :



n	path	$\mathrm{rep}_u(n)$
-5	100	100
-4	101	101
-3	102	102
-2	110	10
-1	111	1
0	000	0
1	001	01
2	002	02
3	010	010
4	020	020
5	021	021

Automatic characterization

Theorem

Let $\eta : A^* \to A^*$ be a **substitution** and
$u \in \mathrm{Per}(\eta)$ be a **two-sided periodic point** with growing seed $s = u_{-1}|u_0$.
Then for **every** $n \in \mathbb{Z}$

$$u_n = \mathcal{A}_{\eta,s}(\mathrm{rep}_u(n)).$$

L., Lepšová. Dumont-Thomas complement numeration systems for $\mathbb{Z}$.
`arXiv:2302.14481`

A total order $\prec$ on $\{0,1\} \odot \mathcal{D}^*$

The **radix order** $(L, <_{rad})$ and **reversed-radix order** $(L, <_{rev})$ on a language $L \subset \mathcal{D}^*$ are total orders such that

$u <_{rad} v$ if and only if $|u| < |v|$ or $|u| = |v|$ and $u <_{lex} v$,

$u <_{rev} v$ if and only if $|u| > |v|$ or $|u| = |v|$ and $u <_{lex} v$.

Definition (total order $\prec$)

For every $u, v \in \{0,1\}\mathcal{D}^*$, we define $u \prec v$ if and only if

- $u \in 1\mathcal{D}^*$ and $v \in 0\mathcal{D}^*$, or
- $u, v \in 0\mathcal{D}^*$ and $u <_{rad} v$, or
- $u, v \in 1\mathcal{D}^*$ and $u <_{rev} v$.

Thus, if $\mathcal{D} = \{0,1\}$, we get

$$\cdots \prec 110 \prec 111 \prec 10 \prec 11 \prec 1 \prec 0 \prec 00 \prec 01 \prec 000 \prec 001 \prec \cdots .$$

Characterization of rep_u by the total order $\prec$

Theorem (L., Lepšová)

Let $\eta : A^* \to A^*$ be a substitution and $u \in \text{Per}(\eta)$ be a **two-sided periodic point** with growing seed $s = u_{-1}|u_0$. Let $p \geq 1$ be the period of u. Let $f : \mathbb{Z} \to \{\mathtt{0},\mathtt{1}\}\mathcal{D}^*$ be some map. Then

- $f = \text{rep}_u$ **if and only if**
- f is **increasing** with respect to $\prec$, its **image** is $f(\mathbb{Z}) = \bigcup_{\ell\in\mathbb{N}} \mathcal{L}_{\ell p+1}(\mathcal{A}_{\eta,s}) \setminus \{\mathtt{0W}_{\min}, \mathtt{1W}_{\max}\}\mathcal{D}^*$ and $f(0) = \mathtt{0}$.

We recover the **two's complement** and its **Fibonacci analog** :

Corollary

- Let $\psi : A \to A^*$ be 2-uniform and let $u \in \text{Fix}(\psi)$ be some two-sided fixed-point. Then $\text{rep}_u = \text{rep}_{2c}$.
- Let $\varphi : a \mapsto ab, b \mapsto a$ and let $v \in \text{Per}(\varphi)$ be the periodic point of period 2 with seed $s = b|a$. Then $\text{rep}_v = \text{rep}_{\mathcal{F}c}$.

Dumont–Thomas complement NS for $\mathbb{Z}^d$

Definition

Let $\eta : A^* \to A^*$ be a substitution and $u_1, u_2, \ldots, u_d \in \mathrm{Per}(\eta)$ be **periodic points with growing seeds** and of the **same period**.
For every $\boldsymbol{n} = (n_1, n_2, \ldots, n_d) \in \mathbb{Z}^d$, we define

$$\mathrm{rep}_u(\boldsymbol{n}) = \begin{pmatrix} \mathrm{pad}_t(\mathrm{rep}_{u_1}(n_1)) \\ \mathrm{pad}_t(\mathrm{rep}_{u_2}(n_2)) \\ \ldots \\ \mathrm{pad}_t(\mathrm{rep}_{u_d}(n_d)) \end{pmatrix} \in \{0,1\}^d(\mathcal{D}^d)^*,$$

where $t = \max\{|\mathrm{rep}_{u_i}(n_i)| : 1 \le i \le d\}$.

L., Lepšová. Dumont-Thomas complement numeration systems for $\mathbb{Z}$.
arXiv:2302.14481

Example : Tribonacci

Substitution
$\psi_T : a \mapsto ab, b \mapsto ac, c \mapsto a$.

Let $\tau \in \mathrm{Per}(\psi_T)$ be the periodic point with period $p = 3$ and seed $c|a$.

$$W_{\min} = 000$$
$$W_{\max} = 011$$

n	$\mathrm{rep}_\tau(n)$	$\mathrm{pad}_7(\mathrm{rep}_\tau(n))$
8	0001001	0001001
7	0001000	0001000
6	0110	0000110
5	0101	0000101
4	0100	0000100
3	0011	0000011
2	0010	0000010
1	0001	0000001
0	0	0000000
-1	1	1011011
-2	1010	1011010
-3	1001	1011001
-4	1000	1011000
-5	1010110	1010110

Thus, the coordinate $(-1, 8) \in \mathbb{Z}^2$ can thus be written as a word

$$\mathrm{rep}_\tau \begin{pmatrix} -1 \\ 8 \end{pmatrix} = \begin{pmatrix} \mathrm{pad}_7(\mathrm{rep}_\tau(-1)) \\ \mathrm{pad}_7(\mathrm{rep}_\tau(8)) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

whose alphabet is the Cartesian product of $\{0, 1\}$ with itself.

Outline

1. Motivation
2. Kari/Culik, Jeandel–Rao, Ammann sets of Wang tiles
3. A Fibonacci analog of two's complement numeration system
4. Dumont–Thomas complement num. systems for $\mathbb{Z}$ and $\mathbb{Z}^d$
5. **Open questions**

Open questions / future work

Question 1

Which Dumont–Thomas comp. num. syst. are **positional** ?

Progress made in : *Jana Lepšová, Ph.D. thesis, 2024.*

Question 2 (Abstract complement numeration system)

Can we **extend Abstract numeration system** for $\mathbb{N}$ and $\mathbb{N}^d$ to $\mathbb{Z}$ and $\mathbb{Z}^d$ using the total order $\prec$ on a language $\{\mathtt{0}, \mathtt{1}\} \odot \mathcal{D}^*$?

Charlier, Kärki, Rigo. Multidimensional generalized automatic sequences and shape-symmetric morphic words. Discrete Mathematics 310 (2010) 1238-52.

Question 3 (Complement Bratteli–Vershik NS)

For **every Bratteli–Vershik diagram**, can we define a complement numeration system by representing -1 by some maximal path, and 0 by some minimal path in the diagram ?

Open questions / future work

Bertrand-Mathis (1986)

The β-shift S_β is **sofic** if and only if β is a **Parry number**.

Question 4 : sofic toral $\mathbb{Z}^2$-rotations

For **which matrices** $M \in \mathrm{GL}_2(\mathbb{R})$ does the $\mathbb{Z}^2$-action

$$R_M : \begin{array}{ccc} \mathbb{Z}^2 \times \mathbb{T}^2 & \to & \mathbb{T}^2 \\ (\mathbf{k}, \mathbf{x}) & \mapsto & \mathbf{x} + M\,\mathbf{k} \end{array}$$

on the 2-dimensional torus $\mathbb{T}^2 = (\mathbb{R}/\mathbb{Z})^2$ **have a symbolic extension of finite type / sofic ?**

YES : if $M = \beta$ where β is the n-th metallic mean number.

Metallic mean Wang tiles II : the dynamics of an aperiodic computer chip. arXiv:2403.03197

Question : find an example involving non-quadratic alg. numbers.

Open questions / future work

What about

$$\begin{array}{r} \ldots+89+34+13+5+2 \\ +1 \\ \hline 0 \end{array} ?$$

odd/even Fibonacci integers

Is -1 an **odd or even** Fibonacci integer ?

Solution : define the Dumont–Thomas complement numeration system by starting with seed $b|a$ for describing the periodic point of seed $a|a$ (we thus get even lengths words).

substitution images	Fibonacci (ab, a)	Fibonacci (ab, a)
periodic point	γ	δ
seed	$b\|a$	$a\|a$
period	2	2
n	$\mathrm{rep}_\gamma(n)$	$\mathrm{rep}_\delta(n)$
8	0010000	0010000
7	01010	01010
6	01001	01001
5	01000	01000
4	00101	00101
3	00100	00100
2	010	010
1	001	001
0	0	0
-1	1	1
-2	100	101
-3	10010	100
-4	10001	10101
-5	10000	10100
-6	1001010	10010
-7	1001001	10001
-8	1001000	10000