

Indistinguishable asymptotic pairs and multidimensional Sturmian configurations

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Contexte : géométrie discrète \iff combinatoire

E.g., "digitally convex = Lyndon + Christoffel"

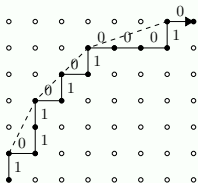
```
sage: w = Word('nicolasbédaride')
sage: w.lyndon_factorization()           # Lyndon 1954
(n, i, col, asbéd, aride)
```

Theorem (Brek, Lachaud, Provençal, Reutenauer, 2009)

A word $w \in \{0, 1\}^*$ is **NW-convex** if and only if its unique Lyndon factorization $l_1^{n_1} l_2^{n_2} \dots l_k^{n_k}$ is such that all l_i are **Christoffel words**.

```
sage: w = Word('1011010100010')
sage: w.lyndon_factorization()
(1, 011, 01, 01, 0001, 0)
```

0, 011, 01, 0001 and 0 are all Christoffel words.



S. Brek, J.-O. Lachaud, X. Provençal, and C. Reutenauer. Lyndon + Christoffel = digitally convex. Pattern Recognition, 42(10) :2239–2246, October 2009.

Outline

- 1 Discrete lines and planes
- 2 Indistinguishable asymptotic pairs of configurations
- 3 Results
- 4 Open questions

Outline

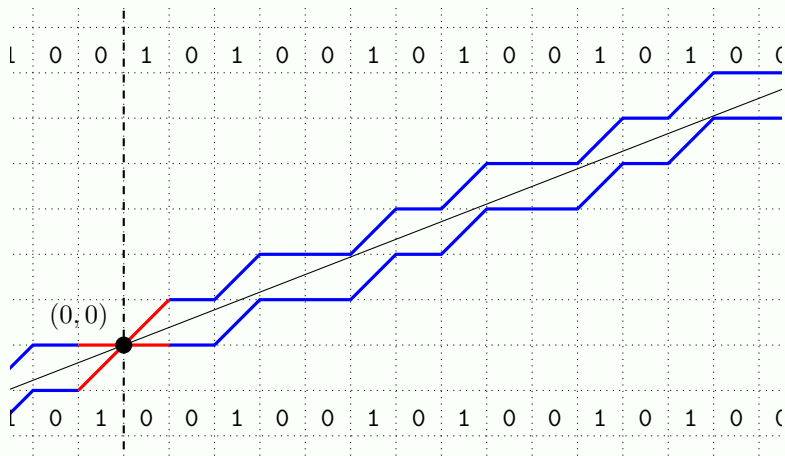
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Sturmian words (Morse, Hedlund, 1940)

Let $\alpha \in [0, 1] \setminus \mathbb{Q}$ and $c_\alpha, c'_\alpha : \mathbb{Z} \rightarrow \{0, 1\}$ be the configurations

$$c_\alpha(n) = \lfloor \alpha(n+1) \rfloor - \lfloor \alpha n \rfloor \quad (\text{lower characteristic Sturmian word})$$

$$c'_\alpha(n) = \lceil \alpha(n+1) \rceil - \lceil \alpha n \rceil \quad (\text{upper characteristic Sturmian word})$$



Pattern complexity

$$c_\alpha = \dots 101001010010 \boxed{1.0} 010010100101 \dots$$

$$c'_\alpha = \dots 101001010010 \boxed{0.1} 010010100101 \dots$$

n	$\mathcal{L}_n(c_\alpha)$	$\#\mathcal{L}_n(c_\alpha)$
0	ε	1
1	0, 1	2
2	00, 01, 10	3
3	001, 010, 100, 101	4
4	0010, 0100, 0101, 1001, 1010	5

Theorem (Morse, Hedlund, 1940 & Coven, Hedlund, 1970)

Let $x \in \{0, 1\}^{\mathbb{Z}}$ be recurrent sequence.

The sequence x has **complexity** $\#\mathcal{L}_n(c_\alpha) = n + 1$

if and only if

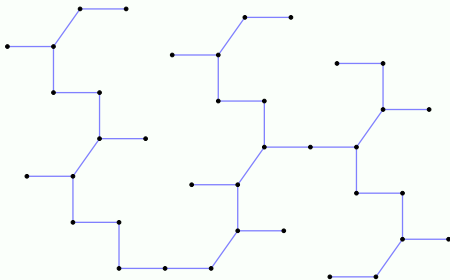
$x = c_\alpha$ is a Sturmian seq. for some **irrational** slope $\alpha \in [0, 1] \setminus \mathbb{Q}$.

Discrete planes

A **discrete plane** of **normal vector** $v \in \mathbb{R}^3$, **intercept** μ and **width** ω is the subset

$$\{p \in \mathbb{Z}^3 \mid 0 \leq p \cdot v + \mu < \omega\} \subset \mathbb{Z}^3.$$

For example, with $\mu = 0$ and $\omega = \|v\|_1/2$, we get :



P. Arnoux, V. Berthé, and S. Ito. *Discrete planes, \mathbb{Z}^2 -actions, Jacobi-Perron algorithm and substitutions*. *Annales de l'Institut Fourier*, 52(2) :305–349, 2002.



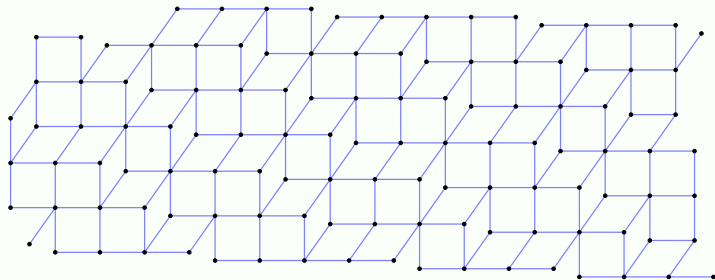
D. Jamet and J.-L. Toutant, *On the connectedness of rational arithmetic discrete hyperplanes*, *LNCS 4245*, 223–234, 2006.

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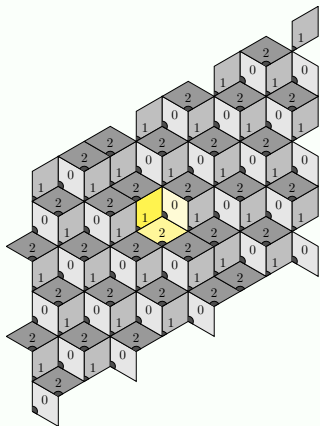
D. Jamet and J.-L. Toutant, *On the connectedness of rational arithmetic discrete hyperplanes*, *LNCS 4245*, 223–234, 2006.

Encoding upper and lower discrete planes

When $\mu = 0$ and $\omega = \|v\|_1$, we get the vertices of the surface of a **standard** discrete plane of normal vector $v \in \mathbb{R}^3$:

$$\{p \in \mathbb{Z}^3 \mid 0 \leq p \cdot v < \|v\|_1\} \subset \mathbb{Z}^3$$

$$\text{Encoding} : \mathbb{Z}^2 \rightarrow \{0, 1, 2\}$$



1	0	2	2	1	0	2	1	0	2	1
0	2	1	1	0	2	1	0	2	1	0
2	1	0	2	2	1	0	2	1	0	2
1	0	2	1	1	0	2	1	0	2	1
0	2	1	0	2	2	1	0	2	1	0
2	1	0	2	1	0	2	2	1	0	2
1	0	2	1	0	2	1	1	0	2	1
0	2	1	0	2	1	0	2	2	1	0

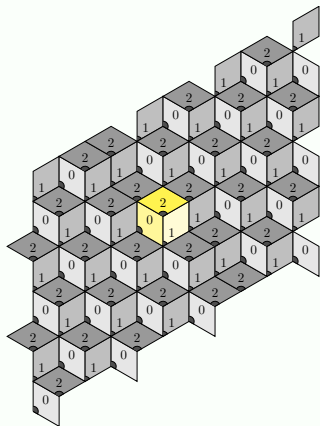


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$$\{p \in \mathbb{Z}^3 \mid 0 < p \cdot v \leq \|v\|_1\} \subset \mathbb{Z}^3$$

$$\text{Encoding} : \mathbb{Z}^2 \rightarrow \{0, 1, 2\}$$



1	0	2	2	1	0	2	1	0	2	1
0	2	1	1	0	2	1	0	2	1	0
2	1	0	2	2	1	0	2	1	0	2
1	0	2	1	0	2	2	1	0	2	1
0	2	1	0	2	1	0	2	1	0	0
2	1	0	2	1	0	2	2	1	0	2
1	0	2	1	0	2	1	1	0	2	1
0	2	1	0	2	1	0	2	2	1	0



d -dimensional Sturmian configurations

Let $\alpha \in [0, 1)^d$ be a **totally irrational vector**, that is,

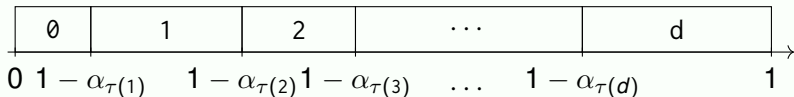
$$\text{for every } \mathbf{n} \in \mathbb{Z}^d, \mathbf{n} \cdot \alpha \in \mathbb{Z} \implies \mathbf{n} = \mathbf{0}.$$

Definition

The **lower** and **upper characteristic d -dimensional Sturmian configurations** with slope α are respectively :

$$\begin{aligned} c_\alpha : \mathbb{Z}^d &\rightarrow \{0, 1, \dots, d\} \\ \mathbf{n} &\mapsto \sum_{i=1}^d ([\alpha_i + \mathbf{n} \cdot \alpha] - [\mathbf{n} \cdot \alpha]) \end{aligned}$$

$$\begin{aligned} c'_\alpha : \mathbb{Z}^d &\rightarrow \{0, 1, \dots, d\} \\ \mathbf{n} &\mapsto \sum_{i=1}^d ([\alpha_i + \mathbf{n} \cdot \alpha] - [\mathbf{n} \cdot \alpha]). \end{aligned}$$



Example

With $\alpha = (\alpha_1, \alpha_2) = (\sqrt{2}/2, \sqrt{19} - 4)$:

$$c_\alpha : \mathbb{Z}^2 \rightarrow \{0, 1, 2\}$$

1	0	2	2	1	0	2	1	0	2	1
0	2	1	1	0	2	1	0	2	1	0
2	1	0	2	2	1	0	2	1	0	2
1	0	2	1	1	0	2	1	0	2	1
0	2	1	0	2	2	1	0	2	1	0
2	1	0	2	1	0	2	2	1	0	2
1	0	2	1	0	2	1	1	0	2	1
0	2	1	0	2	1	0	2	2	1	0

$$c'_\alpha : \mathbb{Z}^2 \rightarrow \{0, 1, 2\}$$

1	0	2	2	1	0	2	1	0	2	1
0	2	1	1	0	2	1	0	2	1	0
2	1	0	2	2	1	0	2	1	0	2
1	0	2	1	0	2	2	1	0	2	1
0	2	1	0	2	1	1	0	2	1	0
2	1	0	2	1	0	2	2	1	0	2
1	0	2	1	0	2	1	1	0	2	1
0	2	1	0	2	1	0	2	2	1	0

is the encoding of a discrete plane of normal vector

$$(1 - \alpha_1, \alpha_1 - \alpha_2, \alpha_2) \approx (0.293, 0.348, 0.359).$$

Question

Can we **characterize** d -dimensional Sturmian configurations by their **pattern complexity** ?

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Configurations

A map $x : \mathbb{Z}^d \rightarrow \Sigma$ is called a **configuration**.

$$x : \mathbb{Z}^2 \rightarrow \{0, 1, 2\}$$

0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	2	0
0	0	0	0	0	0	0	0	0	1	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0

$$\sigma^{(4,1)}x$$

0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	1	2	0	0	0	0
0	0	0	0	0	0	1	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0

(both restricted to $[-5, 5] \times [-4, 3]$)

The **shift action** $\mathbb{Z}^d \curvearrowright \Sigma^{\mathbb{Z}^d}$ is given by the map $\sigma : \mathbb{Z}^d \times \Sigma^{\mathbb{Z}^d} \rightarrow \Sigma^{\mathbb{Z}^d}$ where

$$\sigma^u(x)_v := \sigma(u, x)_v = x_{u+v} \quad \text{for every } u, v \in \mathbb{Z}^d, x \in \Sigma^{\mathbb{Z}^d}.$$

Asymptotic pair

Two configurations $x, y \in \Sigma^{\mathbb{Z}^d}$ are **asymptotic** if they differ in finitely many sites of \mathbb{Z}^d .

$$x : \mathbb{Z}^2 \rightarrow \{0, 1, 2\}$$

0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	2	0
0	0	0	0	0	0	0	0	0	1	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0

$$\sigma^{(4,1)}x$$

0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0

(both restricted to $[-5, 5] \times [-4, 3]$)

The set $F = \{\mathbf{n} \in \mathbb{Z}^d : x_{\mathbf{n}} \neq y_{\mathbf{n}}\}$ is called the **difference set** of (x, y) .

Patterns

For finite subset $S \subset \mathbb{Z}^d$, a function $p: S \rightarrow \Sigma$ is called a **pattern** and the set S is its **support**. We denote it $p \in \Sigma^S$.

For example,

$$p = \begin{array}{|c|} \hline 0 \\ \hline \end{array} \begin{array}{|c|} \hline 2 \\ \hline \end{array} \begin{array}{|c|} \hline 1 \\ \hline \end{array} \begin{array}{|c|} \hline 0 \\ \hline \end{array}$$

is a pattern of support $S = \{\mathbf{0}, \mathbf{e}_1, 2\mathbf{e}_1, \mathbf{e}_2\}$ and

$$q = \begin{array}{|c|} \hline 0 \\ \hline \end{array} \begin{array}{|c|} \hline 2 \\ \hline \end{array} \begin{array}{|c|} \hline 1 \\ \hline \end{array} \begin{array}{|c|} \hline 0 \\ \hline \end{array} \begin{array}{|c|} \hline 2 \\ \hline \end{array}$$

is a pattern of support $S' = \{\mathbf{0}, \mathbf{e}_1, 2\mathbf{e}_1, \mathbf{e}_2, 3\mathbf{e}_1 + 2\mathbf{e}_2\}$.

Language of patterns

The **language** of support $S \subset \mathbb{Z}^d$ in a configuration $x \in \Sigma^{\mathbb{Z}^d}$ is

$$\mathcal{L}_S(x) = \{\sigma^n(x)|_S : \mathbf{n} \in \mathbb{Z}^d\}.$$

For example,

$$x : \mathbb{Z}^2 \rightarrow \{0, 1, 2\}$$

1	0	2	2	1	0	2	1	0	2	1
0	2	1	1	0	2	1	0	2	1	0
2	1	0	2	2	1	0	2	1	0	2
1	0	2	1	1	0	2	1	0	2	1
0	2	1	0	2	2	1	0	2	1	0
2	1	0	2	1	0	2	2	1	0	2
1	0	2	1	0	2	1	1	0	2	1
0	2	1	0	2	1	0	2	2	1	0

$$\mathcal{L}_S(x) = \left\{ \begin{array}{cccc} \begin{array}{|c|} \hline 0 \\ \hline \end{array} & \begin{array}{|c|} \hline 0 \\ \hline \end{array} & \begin{array}{|c|} \hline 1 \\ \hline \end{array} & \begin{array}{|c|} \hline 1 \\ \hline \end{array} \\ \begin{array}{|c|c|c|} \hline 2 & 1 & 0 \\ \hline \end{array} & \begin{array}{|c|c|c|} \hline 2 & 1 & 1 \\ \hline \end{array} & \begin{array}{|c|c|c|} \hline 0 & 2 & 1 \\ \hline \end{array} & \begin{array}{|c|c|c|} \hline 2 & 2 & 1 \\ \hline \end{array} \\ \begin{array}{|c|} \hline 1 \\ \hline \end{array} & \begin{array}{|c|} \hline 2 \\ \hline \end{array} & \begin{array}{|c|} \hline 2 \\ \hline \end{array} & \begin{array}{|c|} \hline 2 \\ \hline \end{array} \\ \begin{array}{|c|c|c|} \hline 0 & 2 & 2 \\ \hline \end{array} & \begin{array}{|c|c|c|} \hline 0 & 2 & 2 \\ \hline \end{array} & \begin{array}{|c|c|c|} \hline 1 & 0 & 2 \\ \hline \end{array} & \begin{array}{|c|c|c|} \hline 1 & 1 & 0 \\ \hline \end{array} \end{array} \right\}$$

Occurrences

The **occurrences** of a pattern $p \in \Sigma^S$ in a configuration $x \in \Sigma^{\mathbb{Z}^d}$ is

$$\text{occ}_p(x) := \{\mathbf{n} \in \mathbb{Z}^d : \sigma^{\mathbf{n}}(x)|_S = p\}.$$

$$x : \mathbb{Z}^2 \rightarrow \{0, 1, 2\}$$

1	0	2	2	1	0	2	1	0	2	1
0	2	1	1	0	2	1	0	2	1	0
2	1	0	2	2	1	0	2	1	0	2
1	0	2	1	1	0	2	1	0	2	1
0	2	1	0	2	2	1	0	2	1	0
2	1	0	2	1	0	2	2	1	0	2
1	0	2	1	0	2	1	1	0	2	1
0	2	1	0	2	1	0	2	2	1	0

$$y : \mathbb{Z}^2 \rightarrow \{0, 1, 2\}$$

1	0	2	2	1	0	2	1	0	2	1
0	2	1	1	0	2	1	0	2	1	0
2	1	0	2	2	1	0	2	1	0	2
1	0	2	1	0	2	2	1	0	2	1
0	2	1	0	2	1	1	0	2	1	0
2	1	0	2	1	0	2	2	1	0	2
1	0	2	1	0	2	1	1	0	2	1
0	2	1	0	2	1	0	2	2	1	0

The occurrences of the pattern $p = \begin{array}{|c|} \hline 1 \\ \hline \end{array} \begin{array}{|c|} \hline 0 \\ \hline \end{array} \begin{array}{|c|} \hline 2 \\ \hline \end{array} \begin{array}{|c|} \hline 1 \\ \hline \end{array}$ in x and y are

$$\text{occ}_p(x) = \{(-5, 2), (1, 1), (3, -3), \dots\},$$

$$\text{occ}_p(y) = \{(-5, 2), (1, 1), (3, -3), \dots\},$$

Occurrences within asymptotic pairs

The **occurrences** of a pattern $p \in \Sigma^S$ in a configuration $x \in \Sigma^{\mathbb{Z}^d}$ is

$$\text{occ}_p(x) := \{\mathbf{n} \in \mathbb{Z}^d : \sigma^{\mathbf{n}}(x)|_S = p\}.$$

$$x : \mathbb{Z}^2 \rightarrow \{0, 1, 2\}$$

1	0	2	2	1	0	2	1	0	2	1
0	2	1	1	0	2	1	0	2	1	0
2	1	0	2	2	1	0	2	1	0	2
1	0	2	1	1	0	2	1	0	2	1
0	2	1	0	2	2	1	0	2	1	0
2	1	0	2	1	0	2	2	1	0	2
1	0	2	1	0	2	1	1	0	2	1
0	2	1	0	2	1	0	2	2	1	0

$$y : \mathbb{Z}^2 \rightarrow \{0, 1, 2\}$$

1	0	2	2	1	0	2	1	0	2	1
0	2	1	1	0	2	1	0	2	1	0
2	1	0	2	2	1	0	2	1	0	2
1	0	2	1	0	2	2	1	0	2	1
0	2	1	0	2	1	1	0	2	1	0
2	1	0	2	1	0	2	2	1	0	2
1	0	2	1	0	2	1	1	0	2	1
0	2	1	0	2	1	0	2	2	1	0

The occurrences of the pattern $p = \begin{array}{|c|} \hline 1 \\ \hline \end{array} \begin{array}{|c|} \hline 0 \\ \hline \end{array} \begin{array}{|c|} \hline 2 \\ \hline \end{array} \begin{array}{|c|} \hline 1 \\ \hline \end{array}$ in x and y are

$$\text{occ}_p(x) \setminus \text{occ}_p(y) = \{(0, 0)\},$$

$$\text{occ}_p(y) \setminus \text{occ}_p(x) = \{(-2, -1)\}.$$

Indistinguishable asymptotic pair

Let $p \in \Sigma^S$ is a pattern of finite support $S \subset \mathbb{Z}^d$.

If $x, y \in \Sigma^{\mathbb{Z}^d}$ are asymptotic configurations with difference set F , then

$$\text{occ}_p(x) \setminus \text{occ}_p(y) = \text{occ}_p(x) \cap (F - S)$$

and in particular it **is finite**.

Definition

We say that (x, y) is an **indistinguishable asymptotic pair** if (x, y) is asymptotic and the following equality holds

$$\#(\text{occ}_p(x) \setminus \text{occ}_p(y)) = \#(\text{occ}_p(y) \setminus \text{occ}_p(x))$$

for every pattern p of finite support.

Not all asymptotic pair is indistinguishable

$$x : \mathbb{Z}^2 \rightarrow \{0, 1, 2\}$$

0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	1	0	0	0	0	0	0	0
0	0	0	0	0	2	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0

$$y : \mathbb{Z}^2 \rightarrow \{0, 1, 2\}$$

0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	2	0	0	0	0	0
0	0	0	0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0

The occurrences of the pattern $p = \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array}$ in x and y are

$$\text{occ}_p(x) = \{(-1, -1)\},$$

$$\text{occ}_p(y) = \emptyset,$$

$$\text{occ}_p(x) \setminus \text{occ}_p(y) = \{(-1, -1)\},$$

$$\text{occ}_p(y) \setminus \text{occ}_p(x) = \emptyset.$$

Initial question

In Fall 2019, Sebastian Barbieri asked :

Question

Is there any non trivial pair $x, y \in \Sigma^{\mathbb{Z}^d}$ of indistinguishable asymptotic configurations ?

A trivial pair refers to cases like (x, x) and $(x, \sigma^n(x))$ where $n \in \mathbb{Z}^d$.



S. Barbieri, R. Gómez, B. Marcus, T. Meyerovitch, and S. Taati. Gibbsian representations of continuous specifications : the theorems of Kozlov and Sullivan revisited. Communications in Mathematical Physics, 382(2) :1111–1164, 2021.

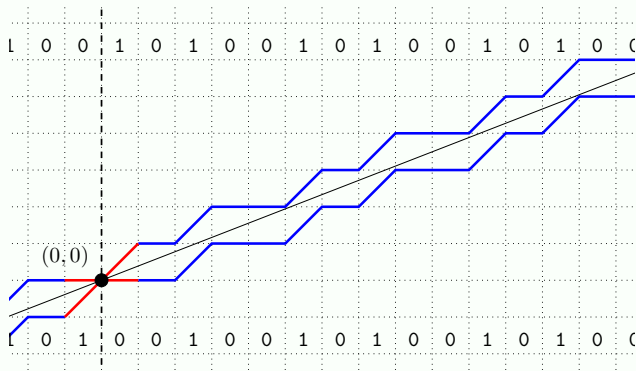
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When $d = 1$

Proposition (Barbieri, L., 2021)

Let $\alpha \in [0, 1] \setminus \mathbb{Q}$ be an irrational slope. The lower and upper characteristic Sturmian sequences (c_α, c'_α) of slope α is an **indistinguishable asymptotic pair** with difference set $F = \{-1, 0\}$.



When $d = 1$

$$c_\alpha = \cdots 101001010010 \boxed{1.0} 010010100101 \cdots$$

$$c'_\alpha = \cdots 101001010010 \boxed{0.1} 010010100101 \cdots$$


Theorem (Barbieri, L, Starosta, 2021)

Let $x, y \in \{0, 1\}^{\mathbb{Z}}$ and assume that x is **recurrent**.

The pair (x, y) is an **indistinguishable asymptotic pair** with difference set $F = \{-1, 0\}$ such that $x_{-1}x_0 = 10$ and $y_{-1}y_0 = 01$

if and only if

there exists $\alpha \in [0, 1] \setminus \mathbb{Q}$ such that $x = c_\alpha$ and $y = c'_\alpha$ are the lower and upper **characteristic Sturmian words** of slope α .

 Barbieri, L., Starosta, A characterization of Sturmian sequences by indistinguishable asymptotic pairs, *European Journal of Combinatorics* **95** (2021) 103318, doi:10.1016/j.ejc.2021.103318

When $d = 1$: without recurrence hypothesis on x

Theorem (Barbieri, L, Starosta, 2021)


Let $x, y \in \{0, 1\}^{\mathbb{Z}}$.

The pair (x, y) is an **indistinguishable asymptotic pair** with difference set $F = \{-1, 0\}$ such that $x_{-1}x_0 = 10$ and $y_{-1}y_0 = 01$

if and only if

there exists a monotone sequence $(\alpha_n)_{n \in \mathbb{N}}$ with $\alpha_n \in [0, 1] \setminus \mathbb{Q}$ s.t. $x = \lim_{n \rightarrow \infty} c_{\alpha_n}$ and $y = \lim_{n \rightarrow \infty} c'_{\alpha_n}$ are the limits of **characteristic Sturmian words** of slope α_n .

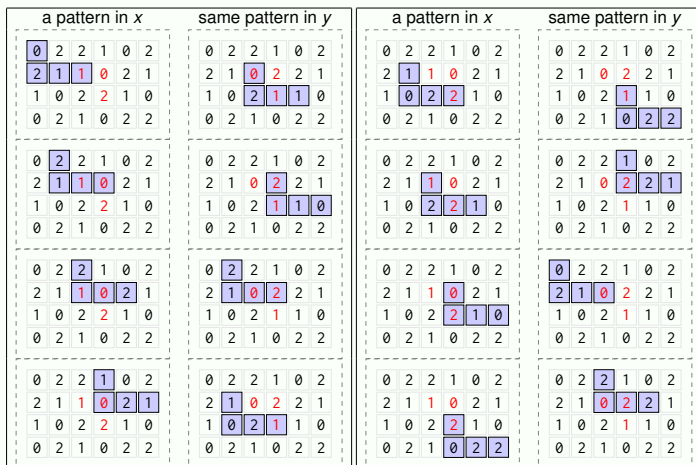
Moreover, indistinguishable asymptotic pairs over \mathbb{Z} for any finite difference set F are described in terms of derived sequences.

 Barbieri, L., Starosta, A characterization of Sturmian sequences by indistinguishable asymptotic pairs, *European Journal of Combinatorics* **95** (2021) 103318, doi:10.1016/j.ejc.2021.103318

When $d \geq 1$

Proposition (Barbieri, L., 2022)

Let $d \geq 1$. Let $\alpha \in [0, 1)^d$ be a totally irrational vector. The lower and upper **characteristic d -dimensional Sturmian configurations** (c_α, c'_α) with slope α is an **indistinguishable asymptotic pair**.



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Let $d \geq 1$. Let $\alpha \in [0, 1)^d$ be a totally irrational vector. The lower and upper **characteristic d -dimensional Sturmian configurations** (c_α, c'_α) with slope α is an **indistinguishable asymptotic pair**.

Question

What about the reciprocal ?

Flip condition

Definition

An asymptotic pair $x, y \in \{0, 1, \dots, d\}^{\mathbb{Z}^d}$ satisfies the **flip condition** if

- 1 the difference set of x and y is $F = \{\mathbf{0}, -\mathbf{e}_1, \dots, -\mathbf{e}_d\}$,
- 2 the restriction $x|_F$ is a **bijection** $F \rightarrow \{0, 1, \dots, d\}$,
- 3 the map defined by $x_n \mapsto y_n$ for every $\mathbf{n} \in F$ is a **cyclic permutation** on the alphabet $\{0, 1, \dots, d\}$.

Without loss of generality, we assume that $x_{\mathbf{0}} = 0$ and $y_{\mathbf{n}} = x_{\mathbf{n}} - 1 \pmod{d+1}$ for every $\mathbf{n} \in F$.

$$x : \mathbb{Z}^2 \rightarrow \{0, 1, 2\}$$

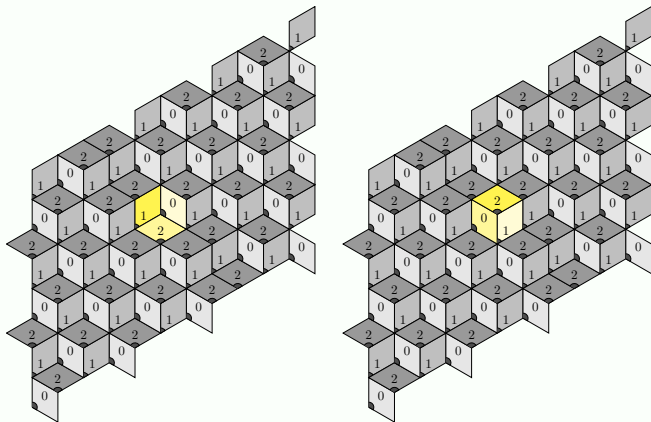
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	1	0	0	0	0	0	0
0	0	0	0	2	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0

$$y : \mathbb{Z}^2 \rightarrow \{0, 1, 2\}$$

0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	2	0	0	0	0	0
0	0	0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0

Flip condition

The flip condition may be interpreted as the **geometrical flip** of the faces of a hypercube at the origin of a **discrete hyperplane** :



T. Jolivet. Combinatorics of Pisot Substitutions. PhD Thesis, 2013.



Damien Jamet, Coding Stepped Planes and Surfaces by Two-Dimensional Sequences over a Three-Letter Alphabet 05047, 2005, pp.21

Main result : Theorem B

Theorem B (Barbieri, L., 2022)

Let $d \geq 1$ and $x, y \in \{0, 1, \dots, d\}^{\mathbb{Z}^d}$ s.t. x is **uniformly recurrent**. The pair (x, y) is an **indistinguishable asymptotic pair** satisfying the flip condition

if and only if

there exists a **totally irrational vector** $\alpha \in [0, 1)^d$ such that $x = c_\alpha$ and $y = c'_\alpha$ are the lower and upper **characteristic d -dimensional Sturmian configurations** with slope α .

Theorem A

(Theorem B depends on Theorem A)

Theorem A (Barbieri, L., 2022)

Let $d \geq 1$ and $x, y \in \{0, 1, \dots, d\}^{\mathbb{Z}^d}$ be an asymptotic pair satisfying the **flip condition** with difference set $F = \{\mathbf{0}, -\mathbf{e}_1, \dots, -\mathbf{e}_d\}$. The following are equivalent :

- (i) For every nonempty finite **connected** subset $S \subset \mathbb{Z}^d$ and $p \in \mathcal{L}_S(x) \cup \mathcal{L}_S(y)$, we have

$$\#(\text{occ}_p(x) \setminus \text{occ}_p(y)) = 1 = \#(\text{occ}_p(y) \setminus \text{occ}_p(x)).$$

- (ii) The asymptotic pair (x, y) is **indistinguishable**.
- (iii) For every nonempty finite **connected** subset $S \subset \mathbb{Z}^d$, the **pattern complexity** of x and y is

$$\#\mathcal{L}_S(x) = \#\mathcal{L}_S(y) = \#(F - S).$$

Complexity $\#(F - S)$

Complexity $\#(F - S)$ matches what is known :

- When $d = 1$ and $S = \{0, 1, \dots, n - 1\}$:

$$\#(F - S) = \#(\{0, -1\} - \{0, 1, \dots, n - 1\}) = n + 1$$

is the factor complexity of **Sturmian words**.

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is the factor complexity of **Sturmian words**.

- When $d = 2$ and $S = \{0, 1, \dots, n - 1\} \times \{0, 1, \dots, m - 1\}$:,

$$\begin{aligned}\#(F - S) &= \#(\{\mathbf{0}, -\mathbf{e}_1, -\mathbf{e}_2\} - \{(i, j) : 0 \leq i < n, 0 \leq j < m\}) \\ &= mn + m + n\end{aligned}$$

is the rectangular pattern complexity of a **discrete plane** with totally irrational slope.

 V. Berthé, L. Vuillon. *Tilings and rotations on the torus : a two-dimensional generalization of Sturmian sequences*. Discrete Mathematics, 223(1-3) :27–53, 2000.

Language of a discrete plane

The $\#(F - S)$ distinct patterns of connected support S appearing in

$$c_\alpha : \mathbb{Z}^2 \rightarrow \{0, 1, 2\}$$

1	0	2	2	1	0	2	1	0	2	1
0	2	1	1	0	2	1	0	2	1	0
2	1	0	2	2	1	0	2	1	0	2
1	0	2	1	1	0	2	1	0	2	1
0	2	1	0	2	2	1	0	2	1	0
2	1	0	2	1	0	2	2	1	0	2
1	0	2	1	0	2	1	1	0	2	1
0	2	1	0	2	1	0	2	2	1	0

are obtained by sliding the support S on top of the difference set F :

$$\mathcal{L}_{\{0, \mathbf{e}_1, 2\mathbf{e}_1, \mathbf{e}_2\}}(c_\alpha) = \left\{ \begin{array}{cccc} \begin{array}{|c|c|c|} \hline 0 \\ \hline 2 & 1 & 0 \\ \hline \end{array}, & \begin{array}{|c|c|c|} \hline 0 \\ \hline 2 & 1 & 1 \\ \hline \end{array}, & \begin{array}{|c|c|c|} \hline 1 \\ \hline 0 & 2 & 1 \\ \hline \end{array}, & \begin{array}{|c|c|c|} \hline 1 \\ \hline 2 & 2 & 1 \\ \hline \end{array}, \\ \begin{array}{|c|c|c|} \hline 1 \\ \hline 0 & 2 & 2 \\ \hline \end{array}, & \begin{array}{|c|c|c|} \hline 2 \\ \hline 0 & 2 & 2 \\ \hline \end{array}, & \begin{array}{|c|c|c|} \hline 2 \\ \hline 1 & 0 & 2 \\ \hline \end{array}, & \begin{array}{|c|c|c|} \hline 2 \\ \hline 1 & 1 & 0 \\ \hline \end{array} \end{array} \right\}$$

Outline

- 1 Discrete lines and planes
- 2 Indistinguishable asymptotic pairs of configurations
- 3 Results
- 4 Open questions**

Open question 1

Conjecture

Let $d \geq 1$ and $x, y \in \{0, 1, \dots, d\}^{\mathbb{Z}^d}$. The pair (x, y) is an **indistinguishable asymptotic pair** satisfying the flip condition

if and only if

there exists a sequence $(\alpha_n)_{n \in \mathbb{N}}$ of totally irrational vectors $\alpha_n \in [0, 1)^d$ such that $x = \lim_{n \rightarrow \infty} c_{\alpha_n}$ and $y = \lim_{n \rightarrow \infty} c'_{\alpha_n}$ **are the limits** of lower and upper characteristic d -dimensional Sturmian configurations.

- Proved when $d = 1$. When $d = 1$, there are two kinds of sequences $(\alpha_n)_{n \in \mathbb{N}}$ such that the limits exist : decreasing ones and increasing ones.
- Open when $d > 1$. More difficult, because there are many kinds of sequences $(\alpha_n)_{n \in \mathbb{N}}$ such that the limits exist.

Open question 2

Question

Let $d \geq 1$ and $x \in \{0, 1, \dots, d\}^{\mathbb{Z}^d}$ be a unif. rec. configuration. Let $F = \{0, -\mathbf{e}_1, \dots, -\mathbf{e}_d\}$. Are the following equivalent?

- 1 **pattern complexity** is $\#\mathcal{L}_S(x) = \#(F - S)$ for every nonempty finite connected support $S \subset \mathbb{Z}^d$
- 2 there exists a totally irrational vector $\alpha \in [0, 1)^d$ and intercept $\rho \in [0, 1)$ s.t. $x = s_{\alpha, \rho}$ or $x = s'_{\alpha, \rho}$ is a lower or upper **d -dimensional Sturmian configuration** :

$$\begin{aligned} s_{\alpha, \rho} : \mathbb{Z}^d &\rightarrow \{0, 1, \dots, d\} \\ \mathbf{n} &\mapsto \sum_{i=1}^d ([\alpha_i + \mathbf{n} \cdot \alpha + \rho] - \lfloor \mathbf{n} \cdot \alpha + \rho \rfloor), \\ s'_{\alpha, \rho} : \mathbb{Z}^d &\rightarrow \{0, 1, \dots, d\} \\ \mathbf{n} &\mapsto \sum_{i=1}^d ([\alpha_i + \mathbf{n} \cdot \alpha + \rho] - \lceil \mathbf{n} \cdot \alpha + \rho \rceil). \end{aligned}$$

(We know that (2) implies (1).)

Open question 3

A sequence $w \in \Sigma^{\mathbb{Z}}$ with $\#\mathcal{L}_n(w) \leq n$ is eventually periodic.

Nivat's conjecture

A configuration $x \in \Sigma^{\mathbb{Z}^2}$ for which there are $n, m \in \mathbb{N}$ with $\#\mathcal{L}_{(n,m)}(x) \leq nm$ is periodic.

Equivalently, a sequence $w \in \Sigma^{\mathbb{Z}}$ with totally irrational vector of symbol frequencies has complexity $\#\mathcal{L}_n(w) \geq n + 1$.

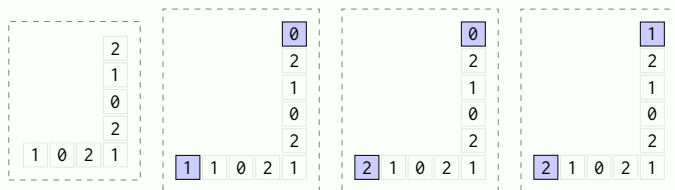
Dual Nivat Conjecture

Let $d \geq 1$ and $F = \{\mathbf{0}, -\mathbf{e}_1, \dots, -\mathbf{e}_d\}$. Let $x \in \{0, 1, \dots, d\}^{\mathbb{Z}^d}$ be a configuration with trivial stabilizer, i.e., $\sigma^n(x) = x$ only holds for $n = \mathbf{0}$. If the **frequencies** of symbols in x **exist and are rationally independent**, then $\#\mathcal{L}_S(x) \geq \#(F - S)$ for every nonempty connected finite support $S \subset \mathbb{Z}^d$.



Open question 4

The pattern below is bispecial within the language of c_α and c'_α :



Bispecial factors within the language of a Sturmian sequence of slope $\alpha \in [0, 1)$ are related to the convergents of the continued fraction expansion of α (de Luca, 1997).

Question

Let $d \geq 1$ and $\alpha \in [0, 1)^d$ be a totally irrational vector. What is the relation between the set

$$V_\alpha = \left\{ b - a : \exists w \in \mathcal{L}(c_\alpha) \text{ bispecial at positions } a, b \in \mathbb{Z}^d \right\}$$

and **simultaneous Diophantine approximations** of α ?