Structure of Jeandel-Rao aperiodic Wang tilings

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Outline

Outline

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Proving Theorems by Pattern Recognition — II

By HAO WANG

(Manuscript received July 18, 1960)

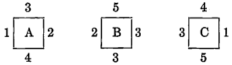
Theoretical questions concerning the possibilities of proving theorems by machines are considered here from the viewpoint that emphasizes the underlying logic. A proof procedure for the predicate calculus is given that contains a few minor peculiar features. A fairly extensive discussion of the decision problem is given, including a partial solution of the (x)(Ey)(z) satisfiability case, an alternative procedure for the (x)(y)(Ez) case, and a rather detailed treatment of Skolem's case. In connection with the (x)(Ey)(z) case, an amusing combinatorial problem is suggested in Section 4.1. Some simple mathematical examples are considered in Section VI.

Editor's Note. This is in form the second and concluding part of this paper' Part I having appeared in another journal.[1] However, an expansion of the author's original plan for Part II has made it a complete paper in its own right.

I. A SURVEY OF THE DECISION PROBLEM

1.1 *The Decision Problem and the Reduction Problem*

With regard to any formula of the predicate calculus, we are interested in knowing whether it is a theorem (the problem of provability), or equivalently, whether its negation has any model at all (the problem of satisfiability). Originally this decision problem was directed to the search for one finite procedure which is applicable to all formulae of the predicate calculus. Since it is known that there can be no such omnipotent



Then we can easily find an infinite solution by the following argument. The following configuration satisfies the constraint on the edges:

A	*B*	*C*
C	*A*	*B*
B	*C*	*A*

Now the colors on the periphery of the above block are seen to be the following:

```
     3 5 4
   1       1
   3       3
   2       2
     3 5 4
```

In other words, the bottom edge repeats the top edge, and the right edge repeats the left edge. Hence, if we repeat the 3 × 3 block in every direction, we obtain a solution of the given set of three plates. In general, we define a "cyclic rectangle."

4.1.1 Given any finite set of plates, a cyclic rectangle of the plates is a rectangle consisting of copies of some or all plates of the set such that: (a) adjoining edges always have the same color; (b) the bottom edge of the rectangle repeats the top edge; (c) the right edge repeats the left edge.

Clearly, a sufficient condition for a set of plates to have a solution is that there exists a cyclic rectangle of the plates.

What appears to be a reasonable conjecture, which has resisted proof or disproof so far, is:

4.1.2 *The fundamental conjecture*: A finite set of plates is solvable (has at least one solution) if and only if there exists a cyclic rectangle of the plates; or, in other words, a finite set of plates is solvable if and only if it has at least one periodic solution.

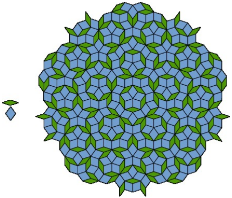
It is easy to prove the following:

4.1.3 If 4.1.2 is true, we can decide effectively whether any given finite set of plates is solvable.

Thus, we proceed to build all possible rectangles from copies of the

Aperiodicity

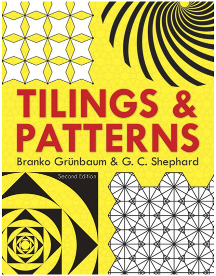
- 1966 (Berger) : The fundamental conjecture is false :
  There **exists an aperiodic** set of Wang tiles (*a tile set is* ***aperiodic*** *if it tiles the plane, but no tiling is periodic*)
- 1976 (Penrose) : discovered an aperiodic set of **two tiles**

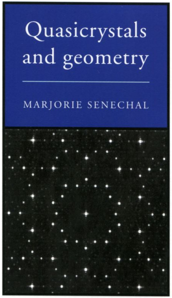


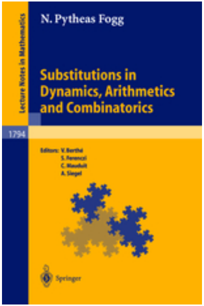
- 1982 (Shechtman) : observed that aluminium-manganese alloys produced a **quasicrystals structure** and receives the 2011 **Nobel Prize** in Chemistry :

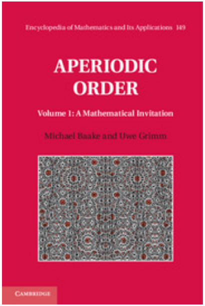
  "*His discovery of quasicrystals revealed a* ***new principle for packing of atoms and molecules***", stated the Nobel Committee that "*led to a* ***paradigm shift*** *within chemistry*".

Books



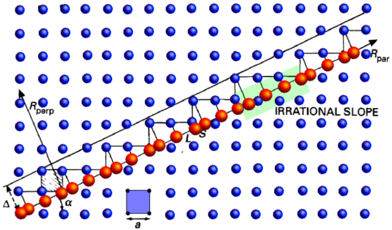
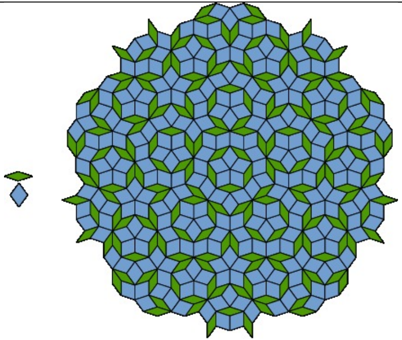






- Tilings and Patterns, by Grünbaum & Shephard, 1987
- Quasicrystals and Geometry, Senechal, 1995
- Pytheas Fogg’s book, 2002
- Aperiodic Order, Baake & Grimm, 2013

Cut and project schemes

$\mathbb{R}^2 \to \mathbb{R}$ (ex : Sturmian sequences)	$\mathbb{R}^5 \to \mathbb{R}^2$ (ex : Penrose tilings)
 Image : `arxiv:2008.05339`	

N. G. de Bruijn. Algebraic theory of Penrose's nonperiodic tilings of the plane. I, II. Nederl. Akad. Wetensch. Indag. Math., *43(1) :39–52, 53–66, 1981.*

Why Penrose Tiles Never Repeat, by minutephysics, Dec 1, 2022, `https://youtu.be/-eqdj63nEr4` *(already 538,583 views after two weeks)*

Aperiodic Wang tile sets

- 1966 (Berger) : 20426 tiles (lowered down later to 104)
- 1968 (Knuth) : 92 tiles
- 1971 (Robinson) : 56 tiles
- 1971 (Ammann) : 16 tiles
- 1987 (Grunbaum) : 24 tiles
- 1996 (Kari) : 14 tiles
- 1996 (Culik) : (same method) 13 tiles

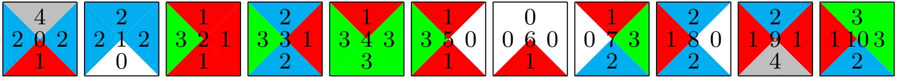
Theorem (Jeandel, Rao, 2015)

- All sets of $\leq$ 10 Wang tiles are **periodic** or don't tile the plane.
- The following set of 11 Wang tiles is **aperiodic** :

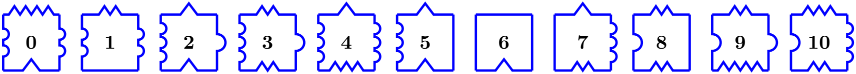
$\mathcal{T}_0 = \{$  $\}$.

Laser cut version of Jeandel-Rao's 11 tiles

We represent the 11 Jeandel-Rao's tiles $\mathcal{T}_0$



geometrically as follows :



Formally, we define the **Jeandel-Rao Wang shift** symbolically as

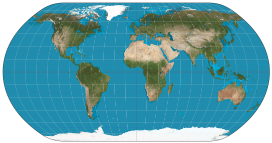
$$\Omega_0 = \left\{ w : \mathbb{Z}^2 \to \{0, 1, \ldots, 10\} \,\middle|\, w \text{ is a valid tiling with } \mathcal{T}_0 \right\}$$

on which the shift $\mathbb{Z}^2 \overset{\sigma}{\curvearrowright} \Omega_0$ acts naturally

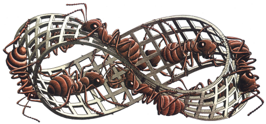
$$\begin{array}{rccc} \sigma : & \mathbb{Z}^2 \times \Omega_0 & \to & \Omega_0 \\ & (\mathbf{k}, w) & \mapsto & \sigma^{\mathbf{k}}(w) := (\mathbf{n} \mapsto w_{\mathbf{n}+\mathbf{k}}). \end{array}$$

A Wang shift Ω is **minimal** if every orbit under the shift is dense in Ω.

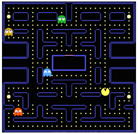
Outline



Source : Natural Earth Projection (wikipedia)



(https://i.imgur.com/R2eAvWi.jpg)

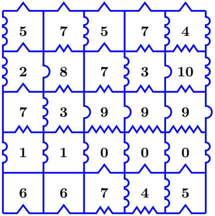


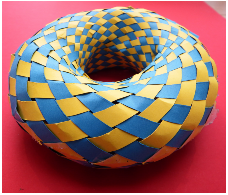
(https://www.todayifoundout.com/)

Henri Poincaré, 1904

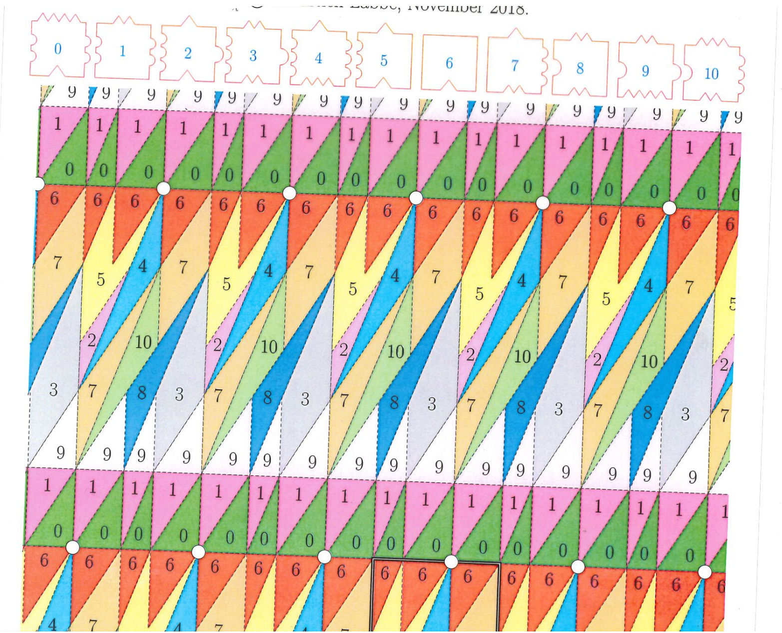
"Imagine an ant walking on a surface. How can this insect know, without rising above it, whether this surface is flat or whether it is moving on a sphere or on any other shape ?"

(La conjecture de Poincaré, George G. Szpiro, 2007)

$\cdots$ 

$\cdots \in \Omega_0 \simeq$ 

An easy way to construct Jeandel-Rao tilings



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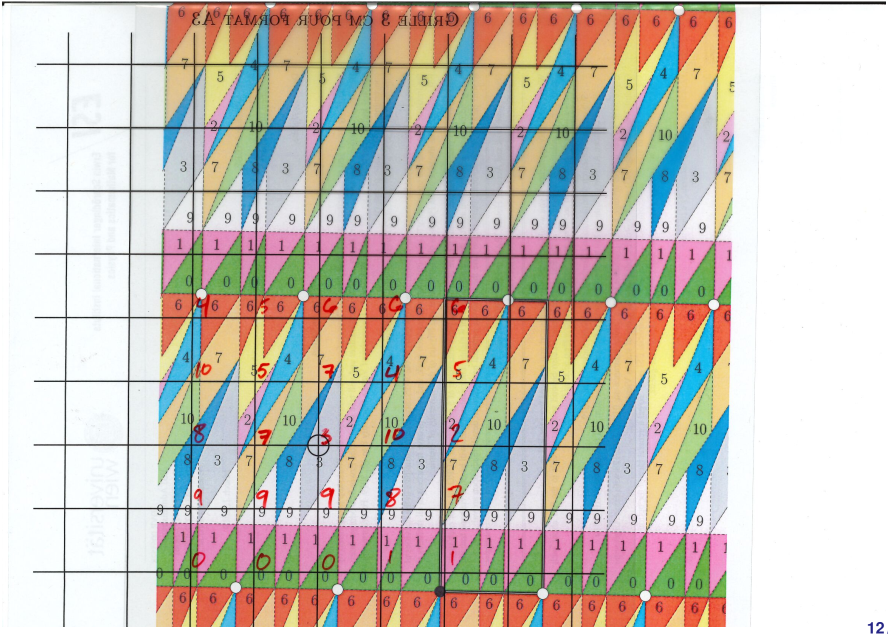

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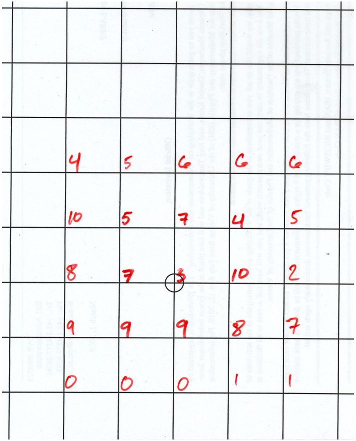
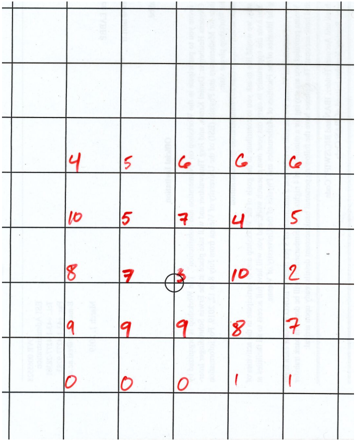


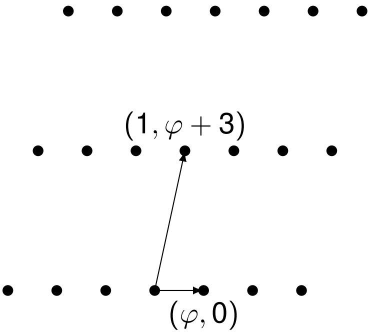
An easy way to construct Jeandel-Rao tilings

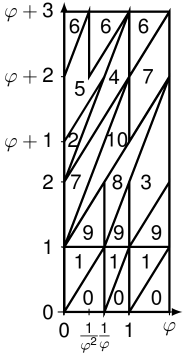


Outline

A lattice Γ_0, a partition $\mathcal{P}_0$ and a $\mathbb{Z}^2$-action R_0

Let $\varphi = \frac{1+\sqrt{5}}{2}$, the **lattice** $\Gamma_0 = \langle(\varphi,0),(1,\varphi+3)\rangle_{\mathbb{Z}}$
and the following topological **partition** $\mathcal{P}_0$ of $\mathbb{R}^2/\Gamma_0$.





We consider the **action** of $\mathbb{Z}^2$ on the **torus** $\mathbb{R}^2/\Gamma_0$:

$$\begin{array}{rccc} R_0 : & \mathbb{Z}^2 \times \mathbb{R}^2/\Gamma_0 & \to & \mathbb{R}^2/\Gamma_0 \\ & (\mathbf{n},\mathbf{x}) & \mapsto & R_0^{\mathbf{n}}(\mathbf{x}) := \mathbf{x}+\mathbf{n} \end{array}$$

for every $\mathbf{n} \in \mathbb{Z}^2$.

Definitions (§6.5 Lind-Marcus for $\mathbb{Z}^2$-actions)

- A **topological partition** of a (compact) metric space M is a finite collection $\mathcal{P} = \{P_a\}_{a\in\mathcal{A}}$ of disjoint open sets whose closures $\overline{P_a}$ together cover M in the sense that $M = \cup_{a\in\mathcal{A}}\overline{P_a}$.
- Let $(M, \mathbb{Z}^2, R)$ be a **dynamical system** with $\mathbb{Z}^2$-action R on M.
- If $S \subset \mathbb{Z}^2$, a **pattern** $w : S \to \mathcal{A}$ is **allowed** for $\mathcal{P}, R$ if

$$\textsc{CodingRegion}(w) = \bigcap_{\mathbf{k}\in S} R^{-\mathbf{k}}(P_{w_\mathbf{k}}) \neq \varnothing.$$

- Let $\mathcal{L}_{\mathcal{P},R}$ be the collection of all allowed patterns for $\mathcal{P}, R$.
- $\mathcal{X}_{\mathcal{P},R}$ is the **symbolic dyn. system corresponding to $\mathcal{P}, R$**.
  It is the unique subshift $\mathcal{X}_{\mathcal{P},R} \subset \mathcal{A}^{\mathbb{Z}^2}$ whose language is $\mathcal{L}_{\mathcal{P},R}$.
- $\mathcal{P}$ gives a **symbolic representation** of $(M, \mathbb{Z}^2, R)$ if for every $w \in \mathcal{X}_{\mathcal{P},R}$ the intersection $\cap_{\mathbf{k}\in\mathbb{Z}^2} R^{-\mathbf{k}}\overline{P_{w_\mathbf{k}}}$ consists of exactly one point $m \in M$.
  This gives rise to a map $f : \mathcal{X}_{\mathcal{P},R} \to M$ such that $f(w) = m$.

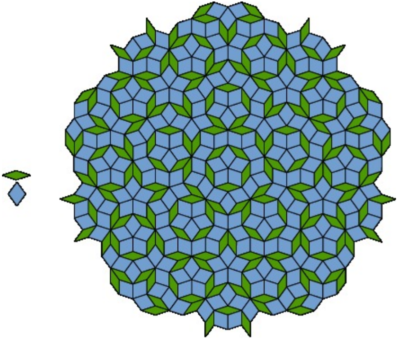
Results on Jeandel-Rao tilings $\mathcal{X}_{\mathcal{P}_0,R_0} \subset \Omega_0$

Theorem

- $\mathcal{P}_0$ gives a **symbolic representation** of $(\mathbb{R}^2/\Gamma_0, \mathbb{Z}^2, R_0)$
- there exists an **almost 1-1 factor** map $f : \mathcal{X}_{\mathcal{P}_0,R_0} \to \mathbb{R}^2/\Gamma_0$
- the set of fiber cardinalities of the factor map is $\{1, 2, 8\}$,
- $\mathcal{X}_{\mathcal{P}_0,R_0}$ is a **minimal, aperiodic and uniquely ergodic** subshift of the Jeandel-Rao Wang shift, i.e., $\mathcal{X}_{\mathcal{P}_0,R_0} \subset \Omega_0$.
- The **measure-preserving** dynamical system $(\mathcal{X}_{\mathcal{P}_0,R_0}, \mathbb{Z}^2, \sigma, \nu)$ is **isomorphic** to $(\mathbb{R}^2/\Gamma_0, \mathbb{Z}^2, R_0, \lambda)$ where
  - ν is the unique shift-invariant probability measure on $\mathcal{X}_{\mathcal{P}_0,R_0}$
  - λ is the Haar measure on $\mathbb{R}^2/\Gamma_0$.
- Occurences of patterns in $\mathcal{X}_{\mathcal{P}_0,R_0}$ is a **4-to-2 C&P set**, more precisely a **regular** (generic or singular) **model set**.

Markov partitions for toral $\mathbb{Z}^2$-rotations featuring Jeandel-Rao Wang shift and model sets, Annales Henri Lebesgue 4 (2021) 283-324.

5-to-2 vs 4-to-2

Penrose	Jeandel-Rao
	

Outline

$\mathcal{X}_{\mathcal{P}_0,R_0} = \Omega_0$?

Question

$\mathcal{X}_{\mathcal{P}_0,R_0} \subset \Omega_0$, but do we have $\mathcal{X}_{\mathcal{P}_0,R_0} = \Omega_0$?

In other words, if $w \in \Omega_0$ is a valid tiling in the Jeandel-Rao Wang shift, **was it obtained** from the orbit under R_0 of some starting point in $\mathbb{R}^2/\Gamma_0$ coded by the partition $\mathcal{P}_0$?

$\mathcal{X}_{\mathcal{P}_0,R_0} = \Omega_0$?

Question

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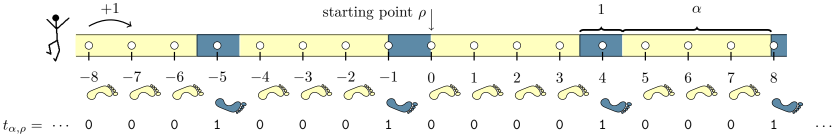
In other words, if $w \in \Omega_0$ is a valid tiling in the Jeandel-Rao Wang shift, **was it obtained** from the orbit under R_0 of some starting point in $\mathbb{R}^2/\Gamma_0$ coded by the partition $\mathcal{P}_0$?

It turns out that such a question was **already answered** years ago for 1-dimensional configurations $\{0, 1\}^{\mathbb{Z}}$.

Sturmian words (Morse, Hedlund, 1940)

A person is walking on a sidewalk made of alternating **dark bricks of size 1** and **light bricks of size $\alpha > 0$**.



Step zero is made at starting position $\rho \in \mathbb{R}$.
They walk from left to right with steps of length 1 and uses

- their **right foot** (1) on dark bricks and
- their **left foot** (0) on light bricks,

thus constructing a bi-infinite binary sequence

$$t_{\alpha,\rho} : \mathbb{Z} \to \{0, 1\}.$$

Pattern complexity

$$x = \cdots 1010010100100101001010010 1\cdots$$

n	$\mathcal{L}_n(x)$	$\#\mathcal{L}_n(x)$
0	ε	1
1	0, 1	2
2	00, 01, 10	3
3	001, 010, 100, 101	4
4	0010, 0100, 0101, 1001, 1010	5

Theorem (Morse, Hedlund, 1940 & Coven, Hedlund, 1970)

Let $x \in \{0,1\}^{\mathbb{Z}}$ be recurrent sequence.
The sequence x has ***pattern complexity*** $\#\mathcal{L}_n(x) = n + 1$
if and only if
$x = t_{\alpha,\rho}$ *for some* ***irrational*** $\alpha > 0$ *and* ***starting point*** $\rho \in \mathbb{R}$.

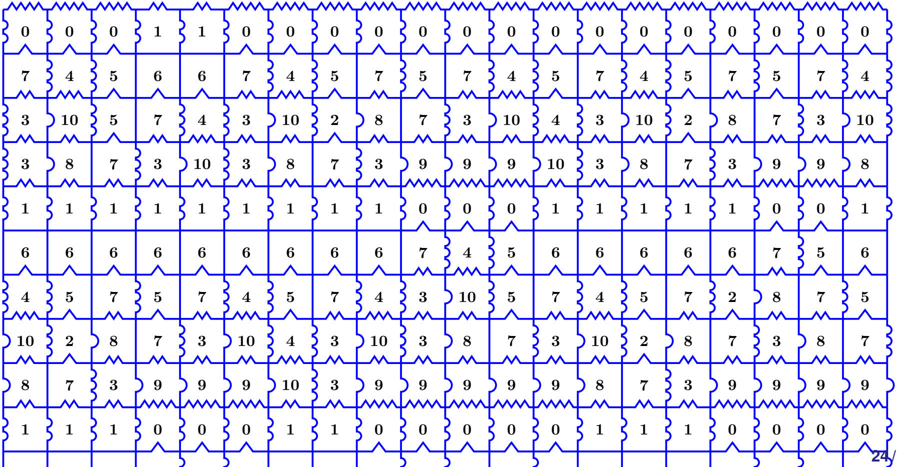
Proof technique : Desubstitute + Rauzy induction + Ostrowki numeration system.
See Pytheas Fogg, chapter 6, written by P. Arnoux.

Outline

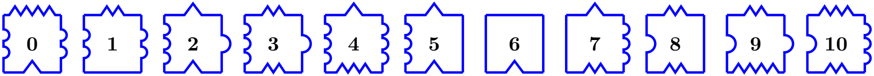
Markers

$\varnothing \neq M \subset \mathcal{T}$ is a set of **markers in the direction $\mathbf{e}_2$** if for all valid configurations $w : \mathbb{Z}^2 \to \mathcal{T} \in \Omega_{\mathcal{T}}$ there exists $P \subset \mathbb{Z}$ s.t.

$$w^{-1}(M) = \mathbb{Z} \times P \quad \text{with} \quad 1 \notin P - P.$$



Markers allows to desubstitute tilings



Markers and recognizable 2-dim. morphisms

Computing $\mathcal{T}_1$

In [10]:
```
T0.tikz(font=r'\small', ncolumns=11, size=1.2, color=color)
```

Out[10]:

In [11]:
```
%time T0.find_markers(i=2, radius=1, solver=solver) # 229ms with dancing_links, 32s with Glucose, 1.4s with Gurobi
```

```
CPU times: user 260 ms, sys: 0 ns, total: 260 ms
Wall time: 258 ms
```

Out[11]:
```
[[0, 1]]
```

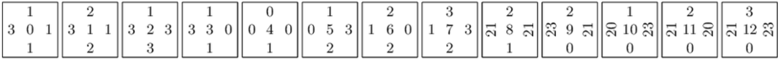
In [12]:
```
M0 = [0,1]
T1,omega0 = T0.find_substitution(M=M0, i=2, side='left', solver=solver)
```

In [13]:
```
show(omega0)
```

$$0 \mapsto (2), \quad 1 \mapsto (3), \quad 2 \mapsto (4), \quad 3 \mapsto (5), \quad 4 \mapsto (6), \quad 5 \mapsto (7), \quad 6 \mapsto (8), \quad 7 \mapsto (10),$$
$$8 \mapsto \begin{pmatrix} 9 \\ 0 \end{pmatrix}, \quad 9 \mapsto \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad 10 \mapsto \begin{pmatrix} 7 \\ 1 \end{pmatrix}, \quad 11 \mapsto \begin{pmatrix} 8 \\ 1 \end{pmatrix}, \quad 12 \mapsto \begin{pmatrix} 10 \\ 1 \end{pmatrix}.$$

In [14]:
```
T1.tikz(font=r'\small', size=1.2, ncolumns=13)
```

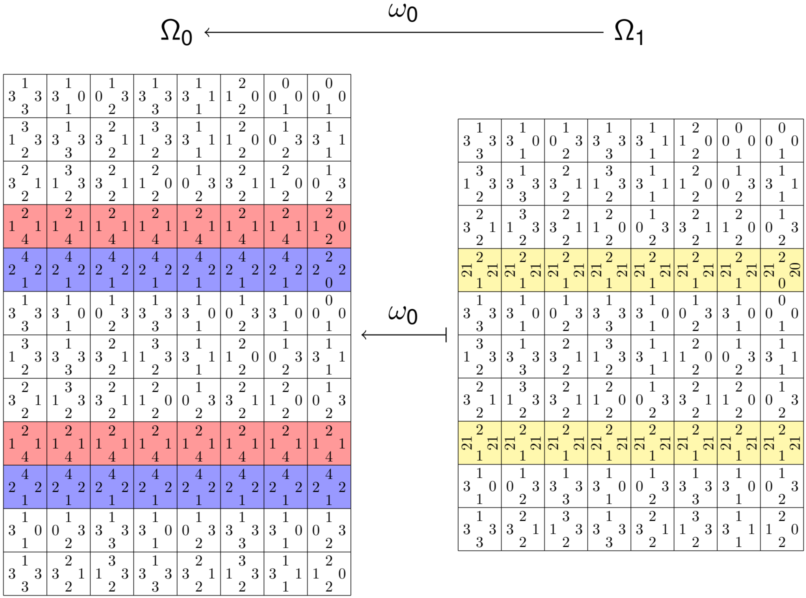
Out[14]:



https://nbviewer.org/url/www.slabbe.org/Publications/arxiv_1808_07768_v4.ipynb

Substitutive structure of Jeandel-Rao aperiodic tilings, Discrete & Computational Geometry 65 (2021) 800–855. doi:10.1007/s00454-019-00153-3

Markers and recognizable 2-dim. morphisms



Substitutive structure of Jeandel-Rao tilings Ω_0

$$\begin{array}{ccccccccccccccccc}
\Omega_0 & \xleftarrow{\omega_0} & \Omega_1 & \xleftarrow{\omega_1} & \Omega_2 & \xleftarrow{\omega_2} & \Omega_3 & \xleftarrow{\omega_3} & \Omega_4 & & & & & & & & \circlearrowleft \omega_{\mathcal{U}} \\
\cup & & \cup & & \cup & & \cup & & \cup & & & & & & & & \\
X_0 & \xleftarrow{\omega_0} & X_1 & \xleftarrow{\omega_1} & X_2 & \xleftarrow{\omega_2} & X_3 & \xleftarrow{\omega_3} & X_4 & \xleftarrow{\pi} & \Omega_5 & \xleftarrow{\eta} & \Omega_6 & \xleftarrow{\omega_6} & \Omega_7 & \xleftarrow{\omega_7\omega_8\omega_9\omega_{10}\omega_{11}} \Omega_{12} \xleftarrow{\rho} & \Omega_{\mathcal{U}}
\end{array}$$

Proposition

Let Ω_0 be the Jeandel-Rao Wang shift. There exist $(\Omega_i)_{0\leq i\leq 12}$ Wang shifts, and there exist

- (i) morphisms $\omega_i : \Omega_{i+1} \to \Omega_i$ that are **recognizable** and **onto up to a shift** for each $i \in \{0, ..., 3\} \cup \{6, ..., 12\}$;
- (ii) $\pi : \Omega_5 \to \Omega_4$ an **embbeding** ;
- (iii) $\eta : \Omega_6 \to \Omega_5$ a **sheering topological conjugacy** ;
- (iv) $\omega_{\mathcal{U}} : \Omega_{\mathcal{U}} \to \Omega_{\mathcal{U}}$ is **primitive** and **expansive**.

Substitutive structure of Jeandel-Rao tilings Ω_0

$$\begin{array}{ccccccccccccccc}
\Omega_0 & \xleftarrow{\omega_0} & \Omega_1 & \xleftarrow{\omega_1} & \Omega_2 & \xleftarrow{\omega_2} & \Omega_3 & \xleftarrow{\omega_3} & \Omega_4 & & & & & & \\
\cup & & \cup & & \cup & & \cup & & \cup & & & & & & \\
X_0 & \xleftarrow{\omega_0} & X_1 & \xleftarrow{\omega_1} & X_2 & \xleftarrow{\omega_2} & X_3 & \xleftarrow{\omega_3} & X_4 & \xleftarrow{\pi} & \Omega_5 & \xleftarrow{\eta} & \Omega_6 & \xleftarrow{\omega_6} & \Omega_7 \xleftarrow{\omega_7\omega_8\omega_9\omega_{10}\omega_{11}} \Omega_{12} \xleftarrow{\rho} \Omega_{\mathcal{U}} \circlearrowleft \omega_{\mathcal{U}}
\end{array}$$

Theorem

Let $X_{12} = \Omega_{12}$ and $X_i = \overline{\omega_i(X_{i+1})}^{\sigma}$ for every i with $0 \leq i \leq 11$.

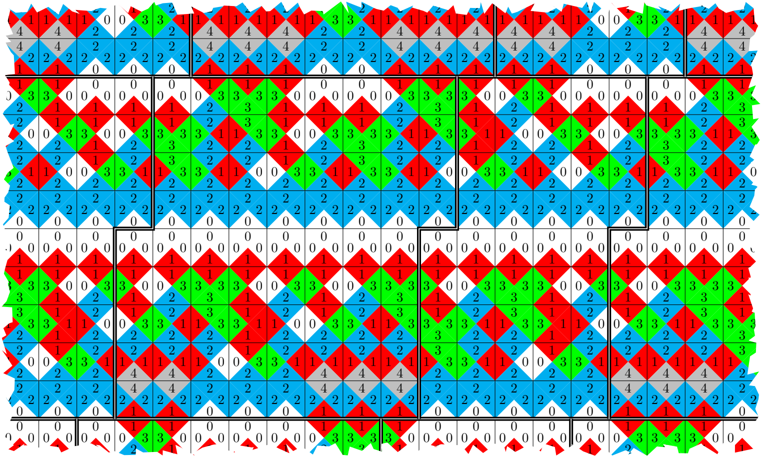
- (i) Ω_{12} and $\Omega_{\mathcal{U}}$ are **self-similar**, **aperiodic** and **minimal**,
- (ii) $X_i = \Omega_i$ and Ω_i is **aperiodic** and **minimal**, $5 \leq i \leq 11$,
- (iii) $X_i \subsetneq \Omega_i$ is an **aperiodic** and **minimal proper** subshift of Ω_i for every i with $0 \leq i \leq 4$.

Conjecture

$\Omega_0 \setminus X_0$ is of **measure 0** for any shift-invariant probability measure on Ω_0 (it consists of sliding half-plane along a fault line).

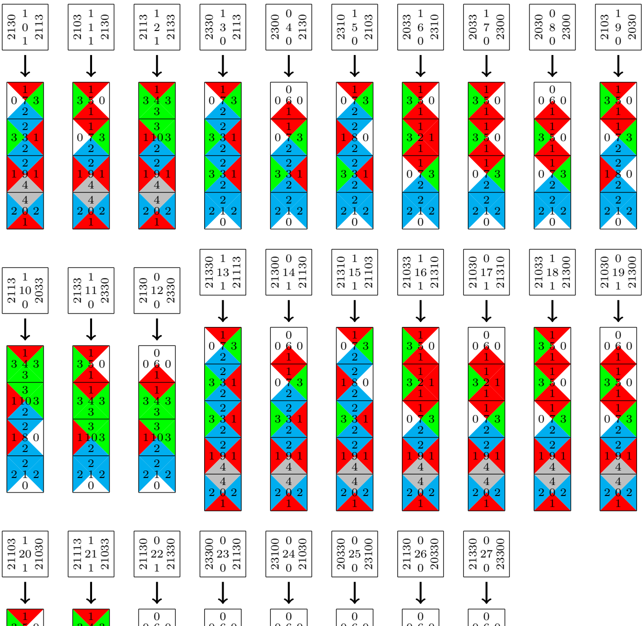
A decomposition into 19 self-similar supertiles

Any tiling in the minimal subshift X_0 of Ω_0 can be decomposed uniquely into **19 supertiles**.

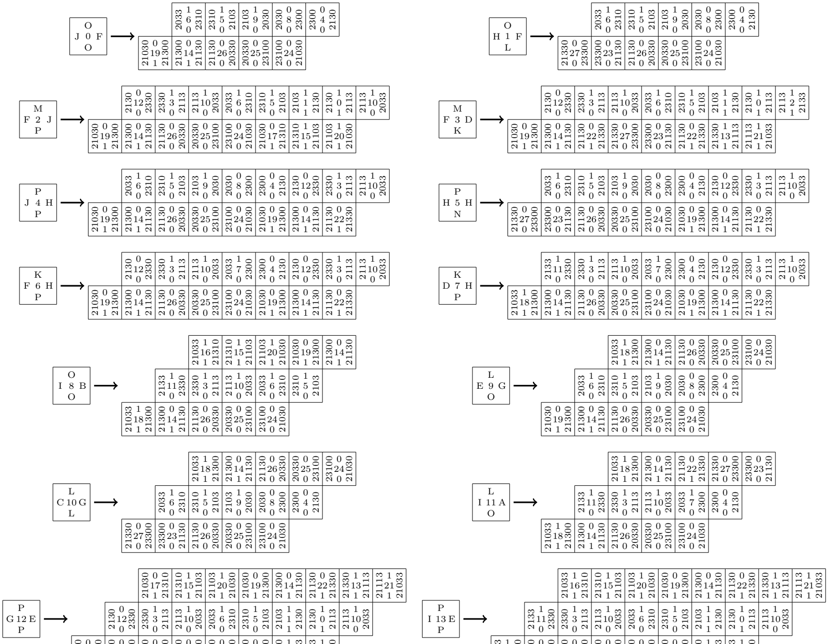


(two of size 45, six of size 72, four of size 70 and seven of size 112).

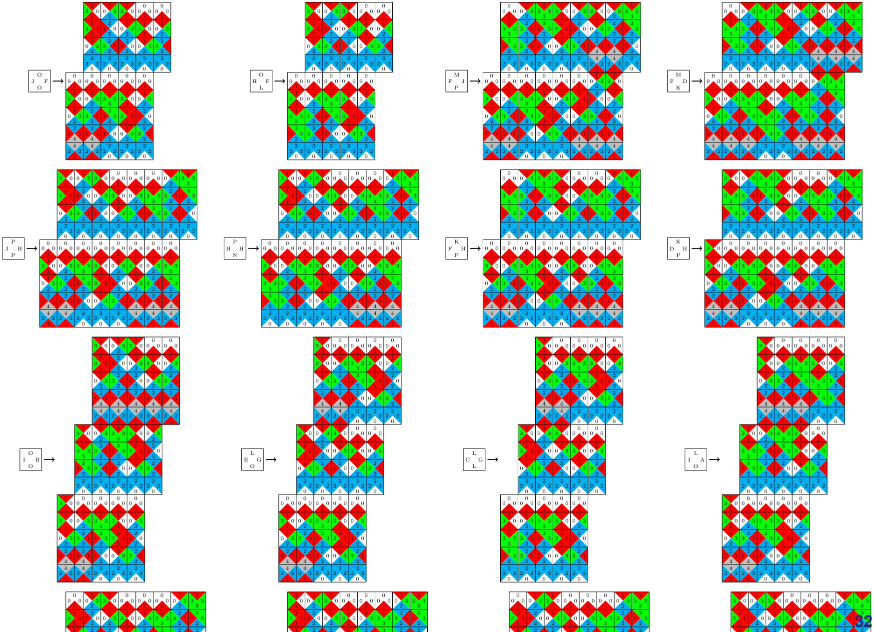
The morphism $\omega_0\,\omega_1\,\omega_2\,\omega_3 : \Omega_4 \to \Omega_0$



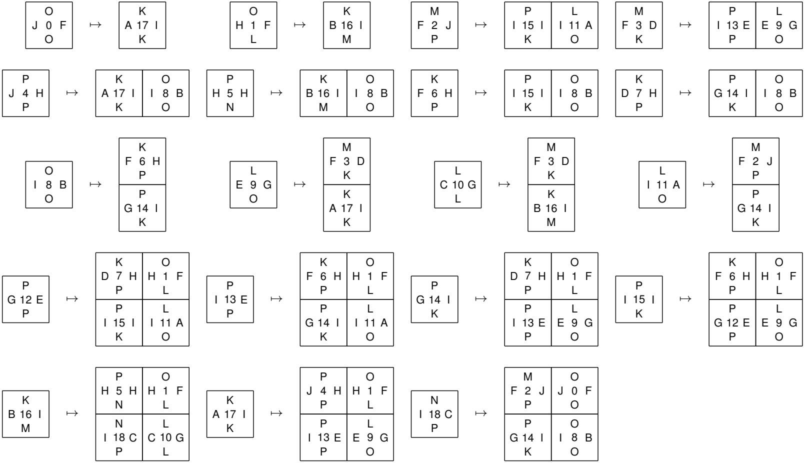
The morphism $\pi\,\eta\,\omega_6\,\omega_7\,\omega_8\,\omega_9\,\omega_{10}\,\omega_{11}\,\rho : \Omega_{\mathcal{U}} \to \Omega_4$



$\omega_0\, \omega_1\, \omega_2\, \omega_3\, \pi\, \eta\, \omega_6\, \omega_7\, \omega_8\, \omega_9\, \omega_{10}\, \omega_{11}\, \rho : \Omega_{\mathcal{U}} \to \Omega_0$



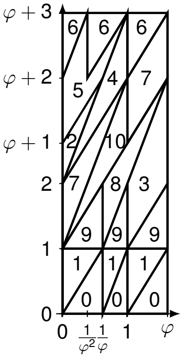
The morphism $\omega_{\mathcal{U}} : \Omega_{\mathcal{U}} \to \Omega_{\mathcal{U}}$



A self-similar aperiodic set of 19 Wang tiles,
Geometriae Dedicata 201 (2019) 81-109 doi:10.1007/s10711-018-0384-8

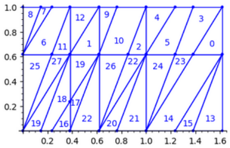
Outline

Rauzy induction of toral $\mathbb{Z}^2$-actions




```
Entrée [13]: y_le_1 = [1, 0, -1]    # syntax for the inequality y <= 1
             P1,beta0 = R0e2.induced_partition(y_le_1, P0, substitution_type='column')
             R1e1,_ = R0e1.induced_transformation(y_le_1)
             R1e2,_ = R0e2.induced_transformation(y_le_1)
```

```
Entrée [37]: P1.plot().show(figsize=4)
```




```
Entrée [35]: beta0._latex_(ncolumns=10)
```

Out[35]:

$0 \mapsto \begin{pmatrix}7\\3\\9\\0\end{pmatrix}, 1 \mapsto \begin{pmatrix}5\\7\\9\\0\end{pmatrix}, 2 \mapsto \begin{pmatrix}4\\10\\9\\0\end{pmatrix}, 3 \mapsto \begin{pmatrix}7\\3\\3\\1\end{pmatrix}, 4 \mapsto \begin{pmatrix}6\\7\\3\\1\end{pmatrix}, 5 \mapsto \begin{pmatrix}7\\8\\3\\1\end{pmatrix}, 6 \mapsto \begin{pmatrix}5\\2\\7\\1\end{pmatrix}, 7 \mapsto \begin{pmatrix}5\\5\\7\\1\end{pmatrix}, 8 \mapsto \begin{pmatrix}6\\5\\7\\1\end{pmatrix}, 9 \mapsto \begin{pmatrix}5\\7\\8\\1\end{pmatrix}$

$10 \mapsto \begin{pmatrix}4\\10\\8\\1\end{pmatrix}, 11 \mapsto \begin{pmatrix}5\\4\\10\\1\end{pmatrix}, 12 \mapsto \begin{pmatrix}6\\4\\10\\1\end{pmatrix}, 13 \mapsto \begin{pmatrix}7\\3\\3\\9\\0\end{pmatrix}, 14 \mapsto \begin{pmatrix}6\\7\\3\\9\\0\end{pmatrix}, 15 \mapsto \begin{pmatrix}7\\8\\3\\9\\0\end{pmatrix}, 16 \mapsto \begin{pmatrix}5\\2\\7\\9\\0\end{pmatrix}, 17 \mapsto \begin{pmatrix}6\\2\\7\\9\\0\end{pmatrix}, 18 \mapsto \begin{pmatrix}5\\5\\7\\9\\0\end{pmatrix}, 19 \mapsto \begin{pmatrix}6\\5\\7\\9\\0\end{pmatrix}$

$20 \mapsto \begin{pmatrix}5\\7\\8\\9\\0\end{pmatrix}, 21 \mapsto \begin{pmatrix}4\\10\\8\\9\\0\end{pmatrix}, 22 \mapsto \begin{pmatrix}6\\4\\10\\9\\0\end{pmatrix}, 23 \mapsto \begin{pmatrix}6\\7\\3\\3\\1\end{pmatrix}, 24 \mapsto \begin{pmatrix}6\\7\\8\\3\\1\end{pmatrix}, 25 \mapsto \begin{pmatrix}6\\5\\2\\7\\1\end{pmatrix}, 26 \mapsto \begin{pmatrix}6\\4\\10\\8\\1\end{pmatrix}, 27 \mapsto \begin{pmatrix}6\\5\\4\\10\\1\end{pmatrix}.$

https://nbviewer.org/url/www.slabbe.org/Publications/arXiv_1906_01104.ipynb

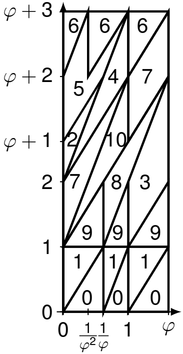
Rauzy induction of polygon partitions and toral $\mathbb{Z}^2$-rotations

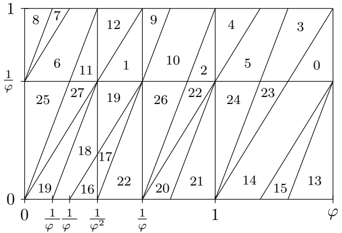
Journal of Modern Dynamics 17 (2021) 481–528. doi:10.3934/jmd.2021017

2-dim. Rauzy induction of $\mathbb{Z}^2$-PETs

$$\mathcal{X}_{\mathcal{P}_0,R_0} \xleftarrow{\quad\beta_0\quad} \mathcal{X}_{\mathcal{P}_1,R_1}$$

$\Gamma_0 = \langle(\varphi, 0), (1, \varphi+3)\rangle_{\mathbb{Z}}$ $\qquad$ $\Gamma_1 = \varphi\mathbb{Z} \times \mathbb{Z}$

$\mathcal{P}_0 =$ 

$\mathcal{P}_1 =$ 

$R_0^{(n_1,n_2)}(\mathbf{x}) := \mathbf{x} + n_1\mathbf{e}_1 + n_2\mathbf{e}_2 \mod \Gamma_0$ $\qquad$ $R_1^{(n_1,n_2)}(\mathbf{x}) = \mathbf{x} + n_1\mathbf{e}_1 + n_2(\varphi^{-1}, \varphi^{-2}) \mod \Gamma_1$

Proof idea : a common substitutive structure

The symbolic dynamical system $\mathcal{X}_{\mathcal{P}_0,R_0}$ and the subshift $X_0 \subsetneq \Omega_0$ of the Jeandel-Rao Wang shift have a **common** substitutive structure :

$$
\begin{array}{ccccccccccccccccccc}
\Omega_0 & \xleftarrow{\omega_0} & \Omega_1 & \xleftarrow{\omega_1} & \Omega_2 & \xleftarrow{\omega_2} & \Omega_3 & \xleftarrow{\omega_3} & \Omega_4 & & & & & & & & & & \\
\cup & & \cup & & \cup & & \cup & & \cup & & & & & & & & & & \\
X_0 & \xleftarrow{\omega_0} & X_1 & \xleftarrow{\omega_1} & X_2 & \xleftarrow{\omega_2} & X_3 & \xleftarrow{\omega_3} & X_4 & \xleftarrow{\pi} & \Omega_5 & \xleftarrow{\eta} & \Omega_6 & \xleftarrow{\omega_6} & \Omega_7 & \xleftarrow{\omega_7\omega_8\omega_9\omega_{10}\omega_{11}} & \Omega_{12} & \xleftarrow{\rho} & \Omega_{\mathcal{U}} \circlearrowleft \omega_{\mathcal{U}} \\
\| & & & & & & & & \| & & & & & & \| & & \| & & \| \\
\mathcal{X}_{\mathcal{P}_0,R_0} & & & \xleftarrow{\beta_0} & & & & & \mathcal{X}_{\mathcal{P}_1,R_1} & \xleftarrow{\beta_1} & \mathcal{X}_{\mathcal{P}_2,R_2} & \xleftarrow{\beta_2} & \mathcal{X}_{\mathcal{P}_3,R_3} & & & \xleftarrow{\beta_3\beta_4\beta_5\beta_6\beta_7} & \mathcal{X}_{\mathcal{P}_8,R_8} & \xleftarrow{\rho} & \mathcal{X}_{\mathcal{P}_{\mathcal{U}},R_{\mathcal{U}}} \circlearrowleft \omega_{\mathcal{U}}
\end{array}
$$

since

$$\omega_0\omega_1\omega_2\omega_3 = \beta_0$$

$$\pi\eta\omega_6 = \beta_1\beta_2$$

$$\omega_7 = \beta_3,\ \omega_8 = \beta_4,\ \omega_9 = \beta_5,\ \omega_{10} = \beta_6 \text{ and } \omega_{11} = \beta_7.$$

Outline

Ongoing work

Chapter written for a book edited by N. Aubrun and M. Rao :

Three characterizations of a self-similar aperiodic 2-dimensional subshift, `arxiv:2012.03892` *(Dec 2020)*

With Casey Mann and Jennifer McLoud-Mann :

Nonexpansive directions in the Jeandel-Rao Wang shift, `arxiv:2206.02414` *(June 2022)*

With Jana Lepšová :

A Numeration System for Fibonacci-Like Wang Shifts. In : Lecroq T., Puzynina S. (eds) Combinatorics on Words. WORDS 2021. Lecture Notes in Computer Science, vol 12847. Springer, Cham. `doi:10.1007/978-3-030-85088-3_9`

A Fibonacci's complement numeration system, `arxiv:2205.02574` *(May 2022)*

The Ellis semigroup

"An action α of a group G on the compact space X is nothing else than a representation of the group in terms of transformations of X; i.e., for each $t \in G$, α_t is a function from X to X. The set of all functions from X to X is the product set X^X and becomes a compact space when equipped with the Tychonoff topology."

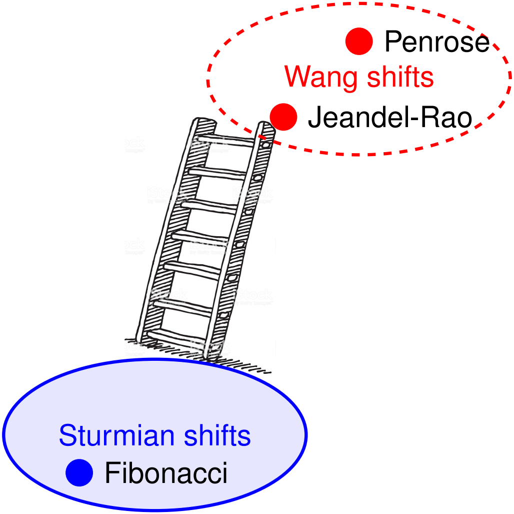
Definition

The Ellis semigroup $E(X, G)$ is the closure of $\{\alpha_t | t \in G\}$ in X^X.

J.-B. Aujogue, M. Barge, J. Kellendonk, and D. Lenz. Equicontinuous factors, proximality and Ellis semigroup for Delone sets. In Mathematics of Aperiodic Order, *volume 309 of* Progr. Math., *pages 137–194. Birkhäuser/Springer, Basel, 2015.*
doi:10.1007/978-3-0348-0903-0_5

Preliminary observation : The Ellis semigroup $E(\Omega_0, \mathbb{Z}^2)$ of the Jeandel-Rao Wang shift contains 8 idempotents each being associated to a way to approach the origin in the partition $\mathcal{P}_0$.

Next steps : continue the exploration



- Can we do with Penrose and others what holds with JR ?
- Can we do with JR what holds with Penrose and others ?