Numeration systems and nonexpansive directions in Aperiodic Wang shifts

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joint work with
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Outline

1. A Fibonacci's complement numeration system for $\mathbb{Z}$ and $\mathbb{Z}^2$
2. Rappels : Combinatorics on words
3. An automatic characterization of an aperiodic Wang shift
4. Nonexpansive direction in the Jeandel-Rao Wang shift

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1. A Fibonacci's complement numeration system for $\mathbb{Z}$ and $\mathbb{Z}^2$
2. Rappels : Combinatorics on words
3. An automatic characterization of an aperiodic Wang shift
4. Nonexpansive direction in the Jeandel-Rao Wang shift

Numeration systems for $\mathbb{N}$ and $\mathbb{Z}$

Let $w = w_N \cdots w_0 \in \{0,1\}^*$ for some $N \in \mathbb{N}$.

Binary NS for $\mathbb{N}$

$\mathrm{val}_2(w) = \sum_{i=0}^{N} w_i 2^i$

Binary numeration system

Addition of integers using the **Binary NS** ($\mathrm{val}_2(w) = \sum_{i=0}^{N} w_i 2^i$) :

$\mathrm{val}_2(w)$	w
11	`01011`
17	`10001`
28	`11100`

Lemma

- (Neutral prefix) For every $u \in \{0,1\}^*$: $\mathrm{val}_2(0u) = \mathrm{val}_2(u)$.
- Let $\Sigma = \{0,1\}$. For **every** $n \in \mathbb{N}$, there exists a **unique word** $w \in \Sigma^* \setminus (0\Sigma^*)$ such that $n = \mathrm{val}_2(w)$.

Definition (Base-2 Numeration system)

For every $n \in \mathbb{N}$, we **denote** this unique word **by** $\mathrm{rep}_2(n)$.

```
sage: (10^9).bits()
[0,0,0,0,0,0,0,0,0,1,0,1,0,0,1,1,0,1,0,1,1,0,0,1,1,1,0,1,1,1]
```

Numeration systems for $\mathbb{N}$ and $\mathbb{Z}$

Let $w = w_N \cdots w_0 \in \{0,1\}^*$ for some $N \in \mathbb{N}$.

Binary NS for $\mathbb{N}$	
$\mathrm{val}_2(w) = \sum_{i=0}^{N} w_i 2^i$	

Numeration systems for $\mathbb{N}$ and $\mathbb{Z}$

Let $w = w_N \cdots w_0 \in \{0,1\}^*$ for some $N \in \mathbb{N}$.

Binary NS for $\mathbb{N}$	**Two's complement notation for $\mathbb{Z}$**
$\mathrm{val}_2(w) = \sum_{i=0}^{N} w_i 2^i$	$\mathrm{val}_{2c}(w) = \sum_{i=0}^{N-1} w_i 2^i - w_N 2^N$

Two's complement numeration system

Two's complement $\mathrm{val}_{2c}(w) = \sum_{i=0}^{N-1} w_i 2^i - w_N 2^N$ advantages are

- representation of all integers $\mathbb{Z}$,
- only one representation of zero,
- addition on $\mathbb{Z}$ **works the same as addition on** $\mathbb{N}$:

$\mathrm{val}_2(w)$	w	$\mathrm{val}_{2c}(w)$
11	`01011`	**+11**
17	`10001`	**-15**
28	`11100`	**-4**

D. E. Knuth. The art of computer programming. Vol. 2. *Addison-Wesley, Reading, MA, 1998. Seminumerical algorithms, Third edition.*

Lemma (Neutral prefix)

For every $u \in \{0,1\}^*$: $\begin{cases} \mathrm{val}_{2c}(00u) = \mathrm{val}_{2c}(0u) \\ \mathrm{val}_{2c}(11u) = \mathrm{val}_{2c}(1u) \end{cases}$

Numeration systems for $\mathbb{N}$ and $\mathbb{Z}$

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Numeration systems for $\mathbb{N}$ and $\mathbb{Z}$

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$\mathrm{val}_2(w) = \sum_{i=0}^{N} w_i 2^i$	$\mathrm{val}_{2c}(w) = \sum_{i=0}^{N-1} w_i 2^i - w_N 2^N$
Fibonacci NS for $\mathbb{N}$	
$\mathrm{val}_Z(w) = \sum_{i=0}^{N} w_i F_i$	

where Fibonacci numbers are indexed as $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$ with $F_0 = 1$, $F_1 = 2$.

Fibonacci numeration system (Zeckendorf)

Proposition (Zeckendorf)

Let $\Sigma = \{0, 1\}$. For **every** $n \in \mathbb{N}$, there exists a **unique word**

$$w \in \Sigma^* \setminus (\Sigma^* 11 \Sigma^* \cup 0\Sigma^*)$$

such that $n = \mathrm{val}_Z(w) = \sum_{i=0}^{N} w_i F_i$, where Fibonacci numbers $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$ are indexed with $F_0 = 1$, $F_1 = 2$.

Definition (Zeckendorf Numeration system Z)

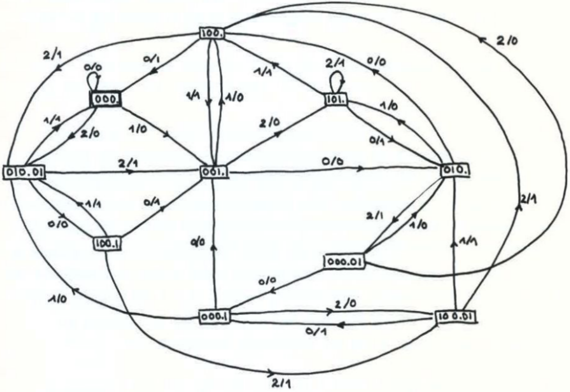
For every $n \in \mathbb{N}$, we **denote** this unique word **by** $\mathrm{rep}_Z(n)$.

$$\begin{aligned} 1000000 &= 832040 + 121393 + 46368 + 144 + 55 \\ &= F_{28} + F_{24} + F_{22} + F_{10} + F_8 \\ \mathrm{rep}_Z(1000000) &= 10001010000000000010100000000 \end{aligned}$$

E. Zeckendorf. Représ. des nombres naturels par une somme de nombres de Fibonacci ou de nombres de Lucas. Bull. Soc. Roy. Sci. Liège, *41 :179–182, 1972.*

The Berstel adder (1986)

Addition is almost like normalization. Given two numbers represented in the Fibonacci number system, the first step for addition is to add the digits at the corresponding positions. This gives a sequence of 0, 1, and 2. This sequence is fed into the adder, which gives as output the corresponding sequence, written only with 0 and 1.



Berstel, Jean. « Fibonacci Words – A Survey ». In The Book of L, ed. G. Rozenberg et A. Salomaa, 13–27. Berlin, Heidelberg : Springer, 1986.

Numeration systems for $\mathbb{N}$ and $\mathbb{Z}$

Let $w = w_N \cdots w_0 \in \{0,1\}^*$ for some $N \in \mathbb{N}$.

Binary NS for $\mathbb{N}$ $\mathrm{val}_2(w) = \sum_{i=0}^{N} w_i 2^i$	**Two's complement notation for $\mathbb{Z}$** $\mathrm{val}_{2c}(w) = \sum_{i=0}^{N-1} w_i 2^i - w_N 2^N$
Fibonacci NS for $\mathbb{N}$ $\mathrm{val}_Z(w) = \sum_{i=0}^{N} w_i F_i$	

where Fibonacci numbers are indexed as $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$ with $F_0 = 1$, $F_1 = 2$.

Numeration systems for $\mathbb{N}$ and $\mathbb{Z}$

Let $w = w_N \cdots w_0 \in \{0,1\}^*$ for some $N \in \mathbb{N}$.

Binary NS for $\mathbb{N}$	**Two's complement notation for $\mathbb{Z}$**
$\mathrm{val}_2(w) = \sum_{i=0}^{N} w_i 2^i$	$\mathrm{val}_{2c}(w) = \sum_{i=0}^{N-1} w_i 2^i - w_N 2^N$
Fibonacci NS for $\mathbb{N}$	**Fib.'s complement NS $\mathcal{F}$ for $\mathbb{Z}$**
$\mathrm{val}_Z(w) = \sum_{i=0}^{N} w_i F_i$	$\mathrm{val}_{\mathcal{F}}(w) = \sum_{i=0}^{2k-1} w_i F_i - w_{2k} F_{2k-1}$

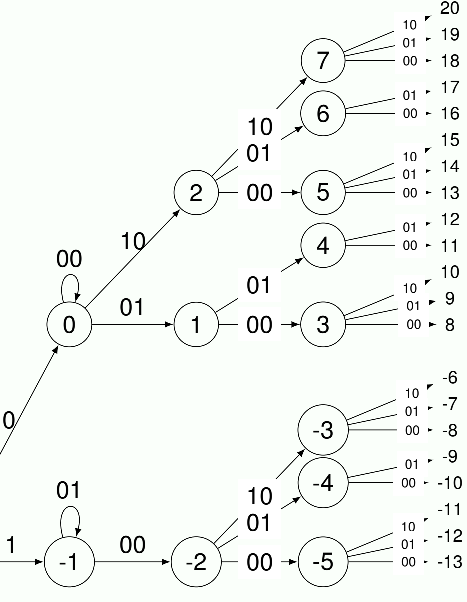
where Fibonacci numbers are indexed as $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$ with $F_0 = 1$, $F_1 = 2$.

Not to be confused with **NegaFibonacci coding**, see :

Knuth, Donald (2009), The Art of Computer Programming, Volume 4, Fascicle 1 : Section 7.1.3, pp. 36–39.

Fibonacci's complement num. system $\mathcal{F}$ for $\mathbb{Z}$

n	$\text{rep}_{\mathcal{F}}(n)$
7	01010
6	01001
5	01000
4	00101
3	00100
2	010
1	001
0	0
-1	1
-2	100
-3	10010
-4	10001
-5	10000



n	$\text{rep}_{\mathcal{F}}(n)$
20	0101010
19	0101001
18	0101000
17	0100101
16	0100100
15	0100010
14	0100001
13	0100000
12	0010101
11	0010100
10	0010010
9	0010001
8	0010000
-6	1001010
-7	1001001
-8	1001000
-9	1000101
-10	1000100
-11	1000010
-12	1000001
-13	1000000

Fibonacci's complement num. system $\mathcal{F}$ for $\mathbb{Z}$

Lemma (Neutral prefix)

Let $\Sigma = \{0,1\}$. For every $u \in (\Sigma\Sigma)^* \setminus \Sigma^* 11 \Sigma^*$:

- $\mathrm{val}_{\mathcal{F}}(000u) = \mathrm{val}_{\mathcal{F}}(0u)$
- $\mathrm{val}_{\mathcal{F}}(101u) = \mathrm{val}_{\mathcal{F}}(1u)$ [since $-F_{2k} + F_{2k-1} = -F_{2k-2}$]

Proposition

For **every** $n \in \mathbb{Z}$, there exists a **unique odd-length word**

$$w \in \Sigma(\Sigma\Sigma)^* \setminus (\Sigma^* 11 \Sigma^* \cup 000\Sigma^* \cup 101\Sigma^*)$$

such that $n = \mathrm{val}_{\mathcal{F}}(w)$.

Definition (Numeration system $\mathcal{F}$)

For every $n \in \mathbb{Z}$, we **denote** this unique word **by** $\mathrm{rep}_{\mathcal{F}}(n)$.

Fibonacci's complement num. system $\mathcal{F}$ for $\mathbb{Z}^2$

Definition (Numeration system $\mathcal{F}$ for $\mathbb{Z}^2$)

For $\mathbf{n} = (n_1, n_2) \in \mathbb{Z}^2$, we define

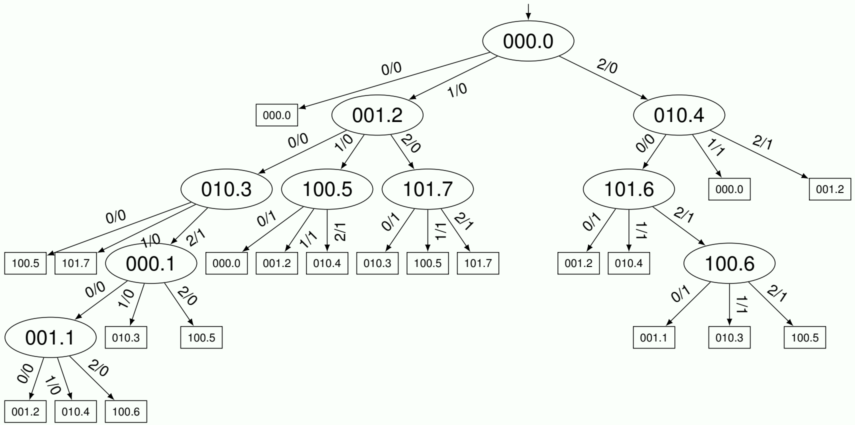
$$\mathrm{rep}_{\mathcal{F}}(\mathbf{n}) = \mathrm{pad}\begin{pmatrix}\mathrm{rep}_{\mathcal{F}}(n_1)\\ \mathrm{rep}_{\mathcal{F}}(n_2)\end{pmatrix},$$

where pad padds the representation shorter in length with the **neutral prefix**.

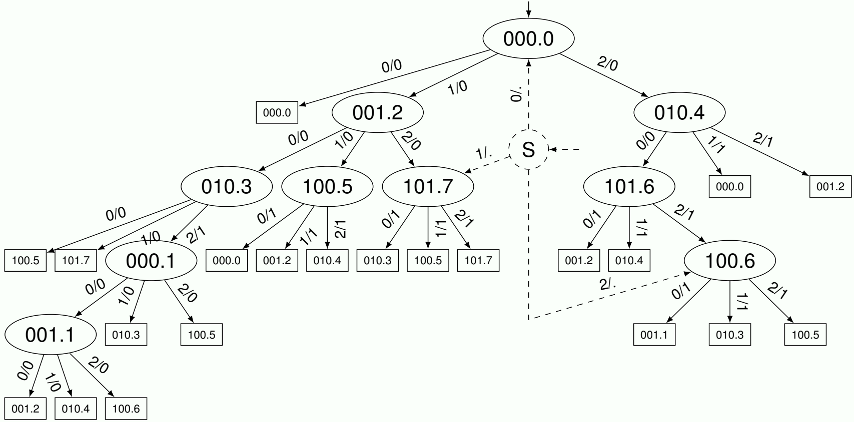
For example :

$$\mathrm{rep}_{\mathcal{F}}((-1,6)) = \mathrm{pad}\begin{pmatrix}\mathrm{rep}_{\mathcal{F}}(-1)\\ \mathrm{rep}_{\mathcal{F}}(6)\end{pmatrix} = \mathrm{pad}\begin{pmatrix}1\\ 01001\end{pmatrix} = \begin{pmatrix}\color{red}{1010}1\\ 01001\end{pmatrix}$$
$$= \begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0\\0\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}.$$

Result #1 : We can modify Berstel adder



Result #1 : We can modify Berstel adder



Addition of $\mathcal{F}$ representations (L., Lepšová, 2022)

The **modified** adder $\mathcal{T}_{\mathcal{F}} : \{0,1,2\}^* \mapsto \{0,1\}^*$ satisfies

$$\mathrm{val}_{\mathcal{F}}(\mathcal{T}_{\mathcal{F}}(\mathrm{rep}_{\mathcal{F}}(m) + \mathrm{rep}_{\mathcal{F}}(n))) = m + n, \text{ for every } \mathbf{m}, \mathbf{n} \in \mathbb{Z}.$$

Outline

Fixed point of a uniform morphism

```
sage: m = WordMorphism('a->ab,b->ba')
sage: for i in range(8): m('a', i)
word: a
word: ab
word: abba
word: abbabaab
word: abbabaabbaababba
word: abbabaabbaababbabaababbaabbabaab
word: abbabaabbaababbabaababbaabbabaabbaababba...
word: abbabaabbaababbabaababbaabbabaabbaababba...
sage: w = m.fixed_point('a'); w
word: abbabaabbaababbabaababbaabbabaabbaababba...
sage: w[10^9]
'b'
sage: (10^9).bits()
[0,0,0,0,0,0,0,0,0,1,0,1,0,0,1,1,0,1,0,1,1,0,0,1,1,1,0,1,1,1]
```

Cobham's theorem

Theorem (Cobham, 1972)

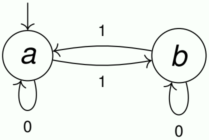
An infinite word $x = x_0 x_1 x_2 \dots$ is a fixed point of a ***k*-uniform morphism**

if and only if

there exists deterministic finite automaton with output $\mathcal{A}$ s.t. $x_n = \mathcal{A}(\mathrm{rep}_k(n))$ for every $n \in \mathbb{N}$.

Example (Thue-Morse) :

$$\begin{cases} a \mapsto ab \\ b \mapsto ba \end{cases}$$



$\mathrm{rep}_2(n)$ is the base 2 expansion of $n \in \mathbb{N}$

Cobham, Alan. « Uniform tag sequences ». Mathematical Systems Theory. An International Journal on Mathematical Computing Theory 6 (1972) : 164–92.

Fixed point of a nonuniform morphism

```
sage: m = WordMorphism('a->ab,b->a')
sage: for i in range(9): m('a', i)
word: a
word: ab
word: aba
word: abaab
word: abaababa
word: abaababaabaab
word: abaababaabaababaababa
word: abaababaabaababaababaabaababaabaab
word: abaababaabaababaababaabaababaabaababaaba...
sage: w = m.fixed_point('a'); w
word: abaababaabaababaababaabaababaabaababaaba...
sage: w[10^9]
'b'
sage: (10^9).bits()
[0,0,0,0,0,0,0,0,0,1,0,1,0,0,1,1,0,1,0,1,1,0,0,1,1,1,0,1,1,1]
```

Cobham's theorem for nonuniform morphisms

Theorem (P. Lecomte, M. Rigo, 2001)

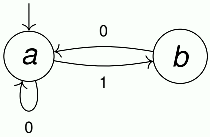
An infinite word $x = x_0 x_1 x_2 \ldots$ is the image under a coding of a fixed point of a **morphism**

if and only if

there exists deterministic finite automaton with output $\mathcal{A}$ and an **abstract numeration system** $\mathcal{S}$ s.t. $x_n = \mathcal{A}(\mathrm{rep}_{\mathcal{S}}(n))$ for all $n \in \mathbb{N}$.

Example (Fibonacci) :

$$\begin{cases} a \mapsto ab \\ b \mapsto a \end{cases}$$



$\mathrm{rep}_Z(n)$ is the Zeckendorf expansion of $n \in \mathbb{N}$

Berthé, Valérie, et Michel Rigo, éd. Combinatorics, automata and number theory. Vol. 135. Encyclopedia of Mathematics and its Applications. Cambridge University

Two-dimensional substitutions (Mozes)

Two-dimensional substitution systems

Definitions. (1) A two-dimensional substitution system is a pair $(\mathscr{A}, \mathscr{P})$ where:

$\mathscr{A}$ — the alphabet, a finite set of symbols called letters.

$\mathscr{P}$ — a finite set of derivation rules of the form

$$a \to \begin{matrix} x_{k1} & \cdots & x_{kl} \\ \vdots & & \vdots \\ x_{11} & \cdots & x_{1l} \end{matrix} \qquad a, x_{ij} \in \mathscr{A}$$

k is called the *height* of the rule and l its *width*. Such a rule is said to *belong to* a.

[...]

derivation rule belonging to it and replacing every x_{ij} by the right-hand side of the chosen derivation rule. This process of choosing must be done so that for all the letters x_{ij}, $1 \leqq j \leqq m$ in one row, the derivation rules chosen all have the same height. For all the letters x_{ij}, $1 \leqq i \leqq n$ in one column, the derivation rules chosen all have the same width. These requirements ensure that for a block α, if it is legally derived, we get a rectangular block β. This process of replacing each letter according to some derivation rule is called *blowing up*. As in the one dimensional

Mozes, Shahar. « Tilings, Substitution Systems and Dynamical Systems Generated by Them ». Journal d'Analyse Mathématique 53 (1989) : 139–86.

Double Fibonacci substitution & a fix. pt over $\mathbb{N}^2$

Let $\mathcal{B} = \{a, b, c, d\}$ and $F : \mathcal{B} \to \mathcal{B}^{*2}$ be a **2-dimensional morphism** :

$$F : a \mapsto \begin{pmatrix} c & d \\ a & b \end{pmatrix}, b \mapsto \begin{pmatrix} c \\ a \end{pmatrix}, c \mapsto \begin{pmatrix} a & b \end{pmatrix}, d \mapsto \begin{pmatrix} a \end{pmatrix}$$

Applying the morphism gives

$$\begin{array}{|l} a \\ \hline \end{array} \mapsto \begin{array}{|ll} c & d \\ a & b \\ \hline \end{array} \mapsto \begin{array}{|lll} a & b & a \\ c & d & c \\ a & b & a \\ \hline \end{array} \mapsto \begin{array}{|lllll} c & \mathbf{d} & c & c & d \\ a & b & a & a & b \\ a & b & a & a & b \\ c & d & c & c & d \\ a & b & a & a & b \\ \hline \end{array} \mapsto \dots$$

Then $x = F(x) = \lim_{k \to +\infty} F^k(a)$ is the **fixed point** of F.

Cobham's theorem generalization in $\mathbb{N}^2$

Theorem (E. Charlier, T. Kärki, M. Rigo, 2010)

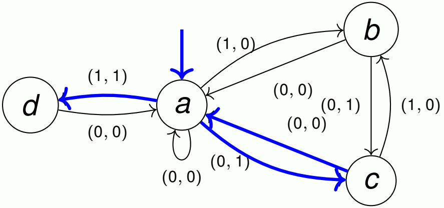
The 2-dimensional infinite word $x \in \mathcal{B}^{\mathbb{N}^2}$ is is the image under a coding of a shape-symmetric pure morphic word if and only if it is $\mathcal{S}$-automatic for some **abstract numeration system** $\mathcal{S} = (L, \Sigma, <)$ with $\varepsilon \in L$.

If $x = F(x)$, we have

$$x_{\mathbf{n}} = \mathcal{A}_F(\mathrm{rep}_Z(\mathbf{n})), \quad \forall \mathbf{n} \in \mathbb{N}^2$$

For example,

$$\mathrm{rep}_Z(1,4) = \mathrm{pad}\begin{pmatrix} 1 \\ 101 \end{pmatrix} = \begin{pmatrix} 001 \\ 101 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$



n	$\mathrm{rep}_Z(n)$
0	ε
1	1
2	10
3	100
4	101
5	1000
6	1001
7	1010
8	10000

Question

Question

Can we generalize Cobham's theorem to $\mathbb{Z}^2$?

$$F : \begin{array}{c|c} b & a \\ \hline d & c \end{array} \mapsto \begin{array}{c|cc} c & c & d \\ a & a & b \\ \hline a & a & b \end{array} \mapsto \begin{array}{cc|ccc} a & b & a & b & a \\ c & d & c & d & c \\ a & b & a & b & a \\ \hline c & d & c & d & c \\ a & b & a & b & a \end{array} \mapsto \dots$$

Motivation : understanding aperiodic Wang tilings of the plane

Outline

Wang tiles and configurations

A **Wang tile** is a square tile with 4 edge labels (and an index) :

$$\mathcal{Z} = \left\{ \begin{array}{cccccccc}
\boxed{\begin{matrix} & \mathrm{O} & \\ \mathrm{J} & 0 & \mathrm{D} \\ & \mathrm{O} & \end{matrix}} &
\boxed{\begin{matrix} & \mathrm{O} & \\ \mathrm{H} & 1 & \mathrm{D} \\ & \mathrm{L} & \end{matrix}} &
\boxed{\begin{matrix} & \mathrm{M} & \\ \mathrm{D} & 2 & \mathrm{J} \\ & \mathrm{P} & \end{matrix}} &
\boxed{\begin{matrix} & \mathrm{M} & \\ \mathrm{D} & 3 & \mathrm{D} \\ & \mathrm{K} & \end{matrix}} &
\boxed{\begin{matrix} & \mathrm{P} & \\ \mathrm{J} & 4 & \mathrm{H} \\ & \mathrm{P} & \end{matrix}} &
\boxed{\begin{matrix} & \mathrm{P} & \\ \mathrm{H} & 5 & \mathrm{H} \\ & \mathrm{N} & \end{matrix}} &
\boxed{\begin{matrix} & \mathrm{K} & \\ \mathrm{D} & 6 & \mathrm{H} \\ & \mathrm{P} & \end{matrix}} &
\boxed{\begin{matrix} & \mathrm{O} & \\ \mathrm{I} & 7 & \mathrm{B} \\ & \mathrm{O} & \end{matrix}} \\
\boxed{\begin{matrix} & \mathrm{L} & \\ \mathrm{E} & 8 & \mathrm{I} \\ & \mathrm{O} & \end{matrix}} &
\boxed{\begin{matrix} & \mathrm{L} & \\ \mathrm{C} & 9 & \mathrm{I} \\ & \mathrm{L} & \end{matrix}} &
\boxed{\begin{matrix} & \mathrm{L} & \\ \mathrm{I} & 10 & \mathrm{A} \\ & \mathrm{O} & \end{matrix}} &
\boxed{\begin{matrix} & \mathrm{P} & \\ \mathrm{I} & 11 & \mathrm{E} \\ & \mathrm{P} & \end{matrix}} &
\boxed{\begin{matrix} & \mathrm{P} & \\ \mathrm{I} & 12 & \mathrm{I} \\ & \mathrm{K} & \end{matrix}} &
\boxed{\begin{matrix} & \mathrm{K} & \\ \mathrm{B} & 13 & \mathrm{I} \\ & \mathrm{M} & \end{matrix}} &
\boxed{\begin{matrix} & \mathrm{K} & \\ \mathrm{A} & 14 & \mathrm{I} \\ & \mathrm{K} & \end{matrix}} &
\boxed{\begin{matrix} & \mathrm{N} & \\ \mathrm{I} & 15 & \mathrm{C} \\ & \mathrm{P} & \end{matrix}}
\end{array} \right\}$$

Adjacent tiles with distinct label on common edge are **forbidden** :

$$\boxed{\begin{matrix} & \mathrm{O} & \\ \mathrm{H} & 1 & \mathrm{D} \\ & \mathrm{L} & \end{matrix}}\boxed{\begin{matrix} & \mathrm{M} & \\ \mathrm{D} & 3 & \mathrm{D} \\ & \mathrm{K} & \end{matrix}} \qquad\qquad \boxed{\begin{matrix} & \mathrm{M} & \\ \mathrm{D} & 2 & \mathrm{J} \\ & \mathrm{P} & \end{matrix}}\boxed{\begin{matrix} & \mathrm{M} & \\ \mathrm{D} & 3 & \mathrm{D} \\ & \mathrm{K} & \end{matrix}}$$

Allowed. Forbidden.

A **configuration** $x : \mathbb{Z}^2 \to \mathcal{Z}$ without forbidden pattern is said **valid**. The set

$$\Omega_{\mathcal{Z}} = \left\{ x : \mathbb{Z}^2 \to \{0, \dots, 15\} \mid x \text{ is a valid configuration} \right\}$$

is called the **Wang shift $\Omega_{\mathcal{Z}}$** associated to the set $\mathcal{Z}$ of Wang tiles.

Self-similarity of $\Omega_{\mathcal{Z}}$

Let $\mathcal{A} = \{0, 1, ..., 15\}$ be an alphabet and ϕ be the two-dimensional morphism defined by

$$\phi: \quad \mathcal{A} \to \mathcal{A}^{*^2}$$

$$\begin{cases} 0 \mapsto (14)\,, & 1 \mapsto (13)\,, & 2 \mapsto (12,\ 10)\,, & 3 \mapsto (11,\ 8)\,, \\ 4 \mapsto (14,\ 7)\,, & 5 \mapsto (13,\ 7)\,, & 6 \mapsto (12,\ 7)\,, & 7 \mapsto \begin{pmatrix} 6 \\ 12 \end{pmatrix}, \\ 8 \mapsto \begin{pmatrix} 3 \\ 14 \end{pmatrix}, & 9 \mapsto \begin{pmatrix} 3 \\ 13 \end{pmatrix}, & 10 \mapsto \begin{pmatrix} 2 \\ 12 \end{pmatrix}, & 11 \mapsto \begin{pmatrix} 6 & 1 \\ 12 & 10 \end{pmatrix}, \\ 12 \mapsto \begin{pmatrix} 6 & 1 \\ 11 & 8 \end{pmatrix}, & 13 \mapsto \begin{pmatrix} 5 & 1 \\ 15 & 9 \end{pmatrix}, & 14 \mapsto \begin{pmatrix} 4 & 1 \\ 11 & 8 \end{pmatrix}, & 15 \mapsto \begin{pmatrix} 2 & 0 \\ 12 & 7 \end{pmatrix}. \end{cases}$$

The substitutive subshift :

$$\mathcal{X}_\phi = \{w \in \mathcal{A}^{\mathbb{Z}^2} \mid \mathcal{L}(w) \subset \mathcal{L}(\phi)\}$$

Lemma (L., Lepšová, 2021)

$\Omega_{\mathcal{Z}} = \mathcal{X}_\phi$

Three characterizations of a self-similar aperiodic 2-dimensional subshift. (2020) arXiv:2012.03892

A valid configuration in the Wang shift $\Omega_{\mathcal{Z}}$

7	P I 12 I K	O I 7 B O	K B 13 I M	P I 11 E P	L E 8 I O	P I 12 I K	O I 7 B O	K B 13 I M	P I 12 I K	O I 7 B O	K B 13 I M	O I 7 B O	K B 13 I M
6	K D 6 H P	O H 1 D L	M D 2 J P	P J 4 H P	O H 1 D L	K D 6 H P	O H 1 D L	M D 3 D K	K D 6 H P	O H 1 D L	M D 2 J P	O J 0 D O	M D 3 D K
5	P I 11 E P	L E 8 I O	P I 12 I K	P I 11 E P	L E 8 I O	P I 12 I K	L I 10 A O	K A 14 I K	P I 11 E P	L E 8 I O	P I 12 I K	O I 7 B O	K B 13 I M
4	P J 4 H P	O H 1 D L	K D 6 H P	P H 5 H N	O H 1 D L	K D 6 H P	O H 1 D L	K D 6 H P	P H 5 H N	O H 1 D L	K D 6 H P	O H 1 D L	M D 3 D K
3	P I 11 E P	L E 8 I O	P I 12 I K	N I 15 C P	L C 9 I L	P I 11 E P	L E 8 I O	P I 12 I K	N I 15 C P	L C 9 I L	P I 12 I K	L I 10 A O	K A 14 I K
2	P I 12 I K	O I 7 B O	K B 13 I M	P I 11 E P	L E 8 I O	P I 12 I K	O I 7 B O	K B 13 I M	P I 12 I K	L I 10 A O	K A 14 I K	O I 7 B O	K B 13 I M
1	K D 6 H P	O H 1 D L	M D 2 J P	P J 4 H P	O H 1 D L	K D 6 H P	O H 1 D L	M D 3 D K	K D 6 H P	O H 1 D L	K D 6 H P	O H 1 D L	M D 3 D K
0	P I 11 E P	L E 8 I O	P I 12 I K	P I 11 E P	L E 8 I O	P I 12 I K	L I 10 A O	K A 14 I K	P I 11 E P	L E 8 I O	P I 12 I K	L I 10 A O	K A 14 I K
-1	P H 5 H N	O H 1 D L	K D 6 H P	P H 5 H N	O H 1 D L	K D 6 H P	O H 1 D L	K D 6 H P	P H 5 H N	O H 1 D L	K D 6 H P	O H 1 D L	K D 6 H P
	-5	**-4**	**-3**	**-2**	**-1**	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**

A finite part of a particular configuration $x \in \Omega_{\mathcal{Z}}$.

Result #2

$$\mathcal{Z} = \left\{ \begin{array}{cccccccc}
\boxed{\begin{smallmatrix} & \mathrm{O} & \\ \mathrm{J} & 0 & \mathrm{D} \\ & \mathrm{O} & \end{smallmatrix}} &
\boxed{\begin{smallmatrix} & \mathrm{O} & \\ \mathrm{H} & 1 & \mathrm{D} \\ & \mathrm{L} & \end{smallmatrix}} &
\boxed{\begin{smallmatrix} & \mathrm{M} & \\ \mathrm{D} & 2 & \mathrm{J} \\ & \mathrm{P} & \end{smallmatrix}} &
\boxed{\begin{smallmatrix} & \mathrm{M} & \\ \mathrm{D} & 3 & \mathrm{D} \\ & \mathrm{K} & \end{smallmatrix}} &
\boxed{\begin{smallmatrix} & \mathrm{P} & \\ \mathrm{J} & 4 & \mathrm{H} \\ & \mathrm{P} & \end{smallmatrix}} &
\boxed{\begin{smallmatrix} & \mathrm{P} & \\ \mathrm{H} & 5 & \mathrm{H} \\ & \mathrm{N} & \end{smallmatrix}} &
\boxed{\begin{smallmatrix} & \mathrm{K} & \\ \mathrm{D} & 6 & \mathrm{H} \\ & \mathrm{P} & \end{smallmatrix}} &
\boxed{\begin{smallmatrix} & \mathrm{O} & \\ \mathrm{I} & 7 & \mathrm{B} \\ & \mathrm{O} & \end{smallmatrix}} \\
\boxed{\begin{smallmatrix} & \mathrm{L} & \\ \mathrm{E} & 8 & \mathrm{I} \\ & \mathrm{O} & \end{smallmatrix}} &
\boxed{\begin{smallmatrix} & \mathrm{L} & \\ \mathrm{C} & 9 & \mathrm{I} \\ & \mathrm{L} & \end{smallmatrix}} &
\boxed{\begin{smallmatrix} & \mathrm{L} & \\ \mathrm{I} & 10 & \mathrm{A} \\ & \mathrm{O} & \end{smallmatrix}} &
\boxed{\begin{smallmatrix} & \mathrm{P} & \\ \mathrm{I} & 11 & \mathrm{E} \\ & \mathrm{P} & \end{smallmatrix}} &
\boxed{\begin{smallmatrix} & \mathrm{P} & \\ \mathrm{I} & 12 & \mathrm{I} \\ & \mathrm{K} & \end{smallmatrix}} &
\boxed{\begin{smallmatrix} & \mathrm{K} & \\ \mathrm{B} & 13 & \mathrm{I} \\ & \mathrm{M} & \end{smallmatrix}} &
\boxed{\begin{smallmatrix} & \mathrm{K} & \\ \mathrm{A} & 14 & \mathrm{I} \\ & \mathrm{K} & \end{smallmatrix}} &
\boxed{\begin{smallmatrix} & \mathrm{N} & \\ \mathrm{I} & 15 & \mathrm{C} \\ & \mathrm{P} & \end{smallmatrix}}
\end{array} \right\}$$

Theorem (L., Lepšová, 2021)

There exist

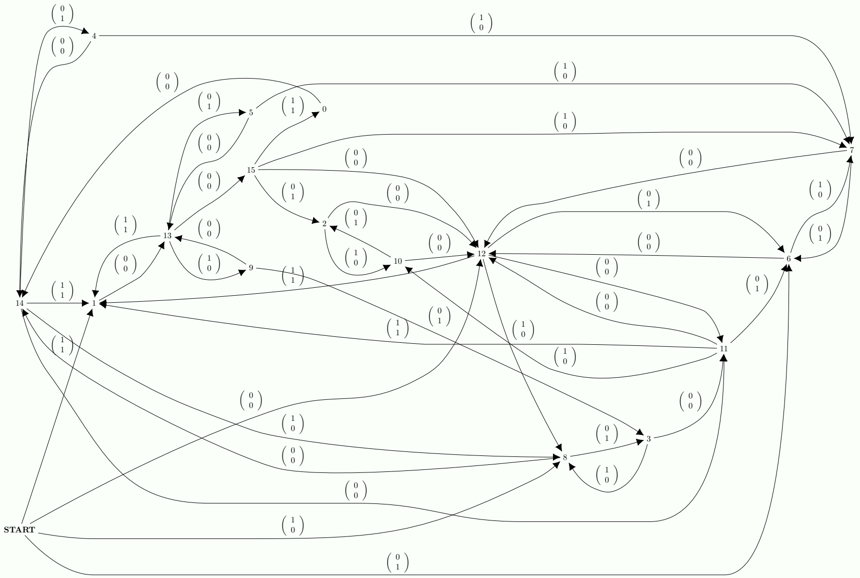
- a **numeration system $\mathcal{F}$ for $\mathbb{Z}^2$** with a representation function $\mathrm{rep}_{\mathcal{F}} : \mathbb{Z}^2 \to \left\{ \binom{0}{0}, \binom{0}{1}, \binom{1}{0}, \binom{1}{1} \right\}^*$ and
- a deterministic finite automaton $\mathcal{A}$ with output

such that the configuration $x : \mathbb{Z}^2 \to \{0, 1, \dots, 15\}$ given by

$$x_{\mathbf{n}} = \mathcal{A}(\mathrm{rep}_{\mathcal{F}}(\mathbf{n})) \qquad \text{for every } \mathbf{n} \in \mathbb{Z}^2$$

is a valid Wang configuration $x \in \Omega_{\mathcal{Z}}$.

The automaton $\mathcal{A}$



Outline

Conway worms

The notion of **Conway worms** was considered in the context of tilings by Penrose kites and darts. It was then defined as

"*a sequence of bow ties placed end to end*"

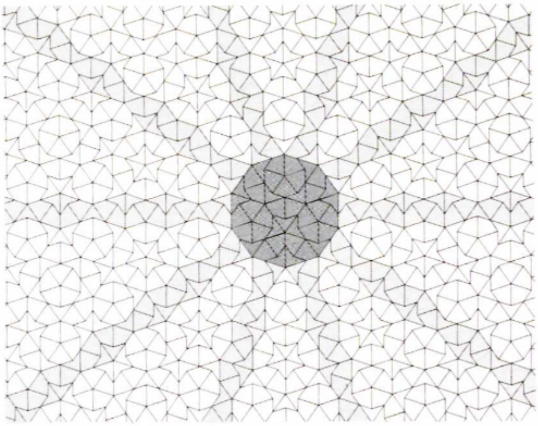
(a)

Lemma

- Every tiling by Penrose kites and darts contains arbitrarily long finite Conway worms.
- There are 5 different possible slopes for these Conway worms and the difference between any two of them is a multiple of $\frac{\pi}{5}$.

Grunbaum, Branko, G. C. Shephard. Tilings and Patterns : Second Edition. Mineola, New York : Dover Publications, 2016. (see §10.5 and Fig. 10.5.5)

Penrose tiling : 10 semi-infinite Conway worms



(c)

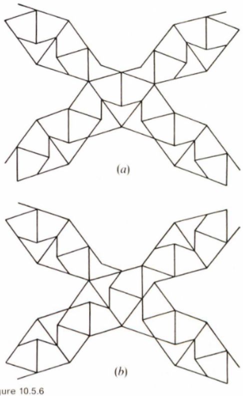


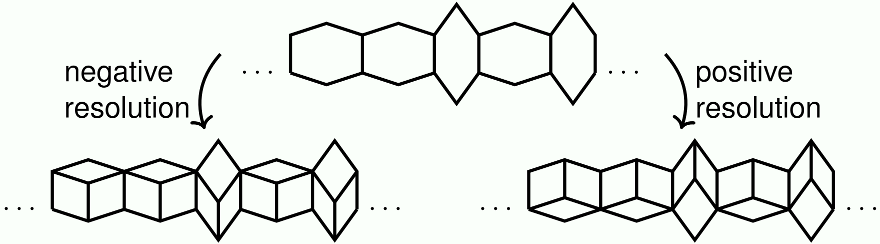
Figure 10.5.6
When two worms meet at 72° they either cross as in (a) and have an ace in common, or one interrupts the other as in (b).

"In (c), we have marked by shading the first-order cartwheeel and the ten semi-infinite Conway worms that radiate from its sides."

Grunbaum, Branko, G. C. Shephard. Tilings and Patterns : Second Edition. Mineola, New York : Dover Publications, 2016. (Figures 10.5.1 and 10.5.6)

Positive/negative resolution of a Conway worm

An unresolved Conway worm made of two kinds of hexagons :



together with its **positive** and **negative resolutions** within a Penrose tiling (according to some orientation).

Nicolaas Govert de Bruijn. Algebraic theory of Penrose's nonperiodic tilings of the plane. I, II. Nederl. Akad. Wetensch. Indag. Math., *43(1) :39–52, 53–66, 1981.*

E. Arthur Robinson, Jr. The dynamical properties of Penrose tilings. Trans. Amer. Math. Soc., *348(11) :4447–4464, 1996.*

Nonexpansive subspaces in subshifts

Let F be a subspace of $\mathbb{R}^d$. Given $t > 0$, the ***t*-neighbourhood** of F is defined by $F^t := \{g \in \mathbb{Z}^d : \mathrm{dist}(g, F) \leq t\}$.

Definition (expansive subspace)

A subspace $F \subset \mathbb{R}^d$ is **expansive** on a subshift $X \subset \mathcal{A}^{\mathbb{Z}^d}$ if there exists $t > 0$ such that for any $x, y \in X$,

$$x|_{F^t} = y|_{F^t} \text{ implies that } x = y.$$

Definition (nonexpansive subspace)

A subspace F is **nonexpansive** if for all $t > 0$, there exist $x, y \in X$ such that $x|_{F^t} = y|_{F^t}$ but $x \neq y$.

Mike Boyle and Douglas Lind. Expansive subdynamics. Trans. Amer. Math. Soc., *349(1) :55–102, 1997.*

Nonexpansive subspaces in subshifts

Theorem (Boyle, Lind, 1997)

If $X \subset \mathcal{A}^{\mathbb{Z}^d}$ is an infinite subshift, then, for each $0 \leq n < d$, there exists a n-dim. subspace of $\mathbb{R}^d$ that is nonexpansive on X.

Mike Boyle and Douglas Lind. Expansive subdynamics. Trans. Amer. Math. Soc., *349(1) :55–102, 1997.*

Theorem (Hochman, 2011)

Any one-dimensional subspace in the plane $\mathbb{R}^2$ occurs as the unique nonexpansive one-dimensional subspace of a $\mathbb{Z}^2$-action.

As a consequence, a set of one-dimensional subspaces occurs as the set of nonexpansive directions for a subshift $X \subset \mathcal{A}^{\mathbb{Z}^2}$ if and only if it is closed and non-empty.

Michael Hochman. Non-expansive directions for $\mathbb{Z}^2$ actions. Ergodic Theory Dynam. Systems, *31(1) :91–112, 2011.*

Nonexpansive directions

If F is expansive, then every translate of F is expansive.
Thus, in the 2-dimensional case, we refer to **nonexpansive directions**.

The set of nonexpansive directions is difficult to compute in general and brings a deeper understanding of a subshift.

Lemma

Let $(X, \mathbb{Z}^d, f)$ and $(Y, \mathbb{Z}^d, g)$ be two topologically conjugate subshifts and $F \subset \mathbb{R}^d$ be a codimension 1 subspace. If F is a nonexpansive in X, then F is nonexpansive in Y.

The notions of expansive and nonexpansive directions was used to obtain partial results toward solving Nivat's conjecture.

Van Cyr and Bryna Kra. Nonexpansive $\mathbb{Z}^2$-subdynamics and Nivat's conjecture. Trans. Amer. Math. Soc., *367(9) :6487–6537, 2015.*

Cleber Fernando Colle. Nivat's Conjecture, Nonexpansiveness and Periodic Decomposition. 2019. arXiv :1909.08195.

Jeandel-Rao Wang tiles

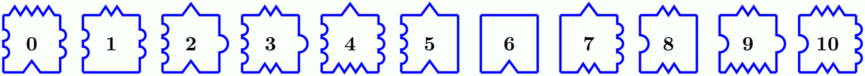
Theorem (Jeandel, Rao, 2021)

No set of less than 11 Wang tiles is aperiodic and

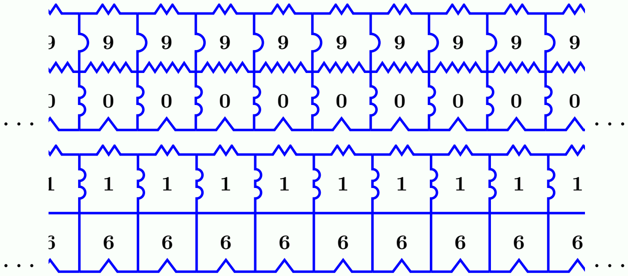
is an aperiodic set of 11 Wang tiles.

Jeandel, Emmanuel, Michaël Rao. « An Aperiodic Set of 11 Wang Tiles ». Advances in Combinatorics, 2021, 18614. https://doi.org/10.19086/aic.18614

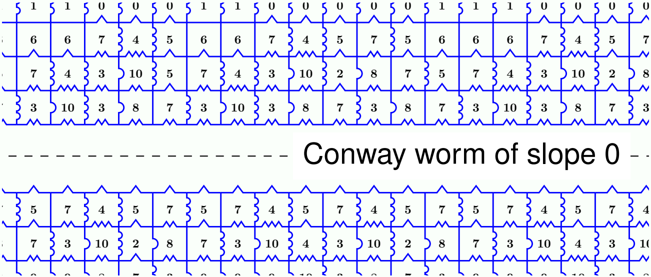
Equivalent geometrical shapes :



A Conway worm of slope 0 in Jeandel-Rao WS



are the **positive** and **negative resolutions** of



Result #3

Theorem (L., Mann, McLoud-Mann, 2022)

A minimal subshift of the Jeandel-Rao Wang shift contains exactly **4 nonexpansive directions** whose slopes are

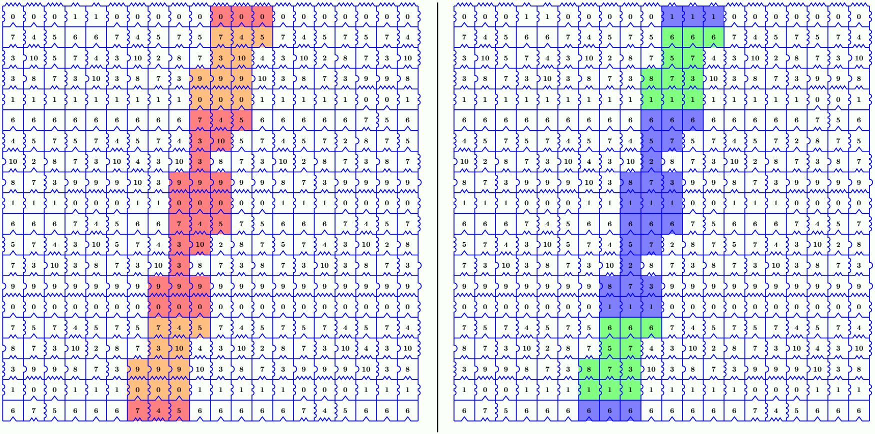
$$\left\{0, \quad \varphi+3, \quad -3\varphi+2, \quad -\varphi+\frac{5}{2}\right\}.$$

Nonexpansive directions come from slopes in the Markov Partition :

slope in the Markov Partition	slope of nonexpansive direction
0	0
∞	$\varphi+3$
φ	$-3\varphi+2$
φ^2	$-\varphi+\frac{5}{2}$

Conway worm of slope $\varphi + 3$

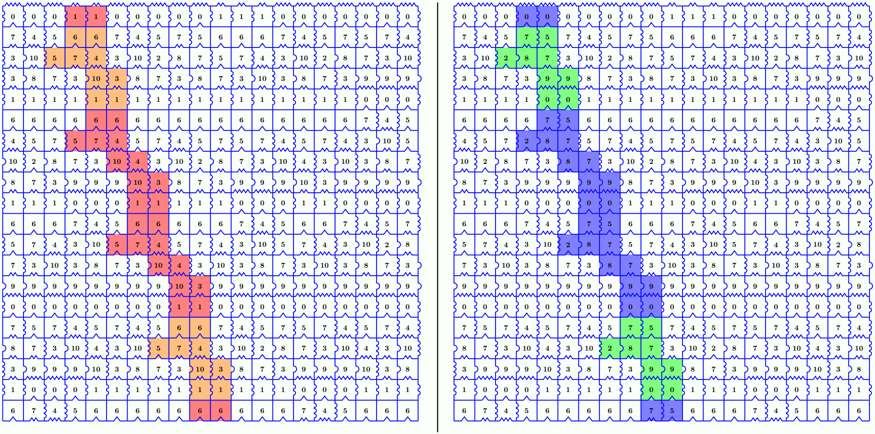
Two tilings of a 20×20 square illustrating the Conway worms of slope $\varphi + \mathbf{3}$:



The difference between the left and the right images is shown with a colored background.

Conway worm of slope $-3\varphi + 2$

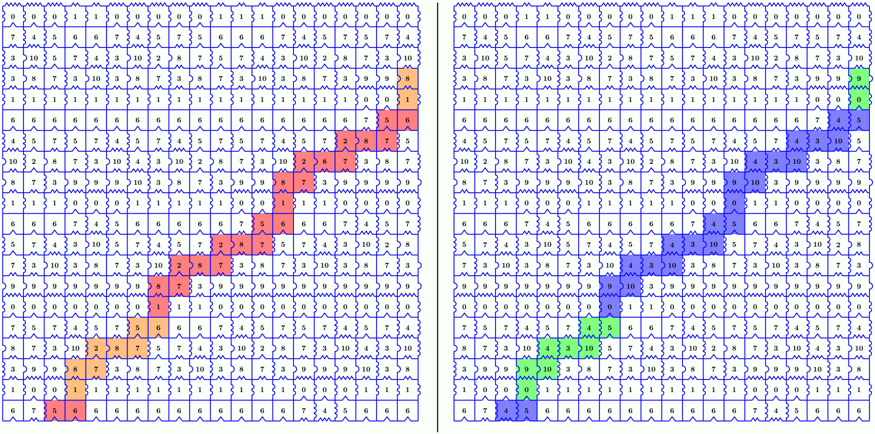
Two tilings of a 20×20 square illustrating the Conway worms of slope $-\mathbf{3}\varphi + \mathbf{2}$:



The difference between the left and the right images is shown with a colored background.

Conway worm of slope $-\varphi + \frac{5}{2}$

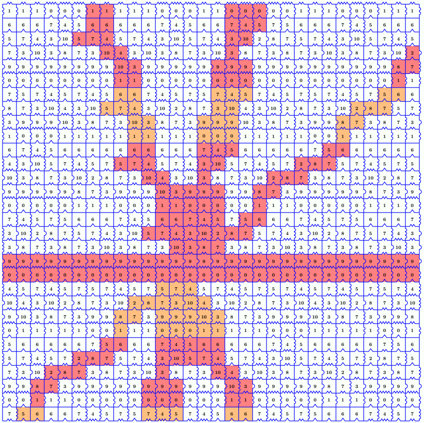
Two tilings of a 20×20 square illustrating the Conway worms of slope $-\varphi + \frac{5}{2}$:

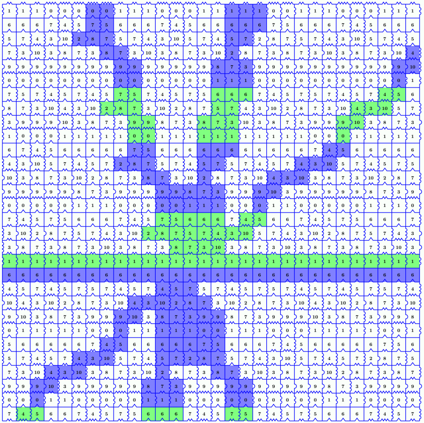


The difference between the left and the right images is shown with a colored background.

The 4 Conway worms in Jeandel-Rao WS

Tilings of a 30×30 square illustrating the four Conway worms :





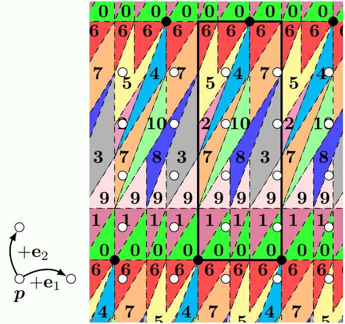
The difference between both images is shown with a colored background.

It reminds of the cartwheel tiling in the context of Penrose tilings.

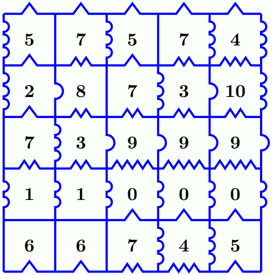
Proof is based on previous results

Recall that Jeandel-Rao tilings can be generated by coding the orbit of a $\mathbb{Z}^2$-action on a torus partitioned into polygonal atoms $P_0..P_{10}$:
The $\mathbb{Z}^2$**-action** on $\boldsymbol{T} = \mathbb{R}^2/\langle(\varphi, 0), (1, \varphi+3)\rangle_{\mathbb{Z}}$ with $\varphi = \frac{1+\sqrt{5}}{2}$ is :

$$\begin{array}{rccc} R: & \mathbb{Z}^2 \times \boldsymbol{T} & \to & \boldsymbol{T} \\ & (\mathbf{n}, \mathbf{x}) & \mapsto & \mathbf{x} + \mathbf{n}. \end{array}$$



$$\xrightarrow[\text{repr.}]{\text{symb.}} \begin{pmatrix} 5 & 7 & 5 & 7 & 4 \\ 2 & 8 & 7 & 3 & 10 \\ 7 & 3 & 9 & 9 & 9 \\ 1 & 1 & 0 & 0 & 0 \\ 6 & 6 & 7 & 4 & 5 \end{pmatrix} \xrightarrow[\text{tiling}]{\text{Wang}}$$



Markov partitions for toral $\mathbb{Z}^2$-rotations featuring Jeandel-Rao Wang shift and model sets. Annales Henri Lebesgue, *4 :283–324, 2021.*

Results on Jeandel-Rao Wang shift $\mathcal{X}_{\mathcal{P},R} \subsetneq \Omega_0$

Theorem

- $\mathcal{P}$ gives a **symbolic representation** of $(\mathbb{R}^2/\Gamma, \mathbb{Z}^2, R)$
- there exists an **almost 1-1 factor** map $f : \mathcal{X}_{\mathcal{P},R} \to \mathbb{R}^2/\Gamma$
- $(\mathbb{R}^2/\Gamma, \mathbb{Z}^2, R)$ is the **maximal equicontinuous factor** of $(\mathcal{X}_{\mathcal{P},R}, \mathbb{Z}^2, \sigma)$.
- $\mathcal{X}_{\mathcal{P},R}$ is a **proper minimal, aperiodic and uniquely ergodic** subshift of the Jeandel-Rao Wang shift, i.e., $\mathcal{X}_{\mathcal{P},R} \subsetneq \Omega_0$.
- The **measure-preserving** dynamical system $(\mathcal{X}_{\mathcal{P},R}, \mathbb{Z}^2, \sigma, \nu)$ is **isomorphic** to $(\mathbb{R}^2/\Gamma, \mathbb{Z}^2, R, \lambda)$ where
  - ν is the unique shift-invariant probability measure on $\mathcal{X}_{\mathcal{P},R}$
  - λ is the Haar measure on $\mathbb{R}^2/\Gamma$.
- Occurrences of patterns in $\mathcal{X}_{\mathcal{P},R}$ is a **4-to-2 C&P set**.

Markov partitions for toral $\mathbb{Z}^2$-rotations featuring Jeandel-Rao Wang shift and model sets, *Annales Henri Lebesgue* 4 (2021) 283-324. `doi:10.5802/ahl.73`

Results on Jeandel-Rao Wang shift $\mathcal{X}_{\mathcal{P},R} \subsetneq \Omega_0$

We define

$$\Delta_{\mathcal{P},R} := \bigcup_{\boldsymbol{n}\in\mathbb{Z}^2} \bigcup_{a\in\{0,1,\dots,10\}} R^{\boldsymbol{n}}(\partial P_a)$$

the image under the $\mathbb{Z}^2$-action of the **boundary** of the partition $\mathcal{P}$.

Lemma

The set of points in $\boldsymbol{T}$ whose fibers under the factor map $f : \mathcal{X}_{\mathcal{P},R} \to \boldsymbol{T}$ are not singletons is $\{y \in \boldsymbol{T} : |f^{-1}(y)| > 1\} = \Delta_{\mathcal{P},R}$.

As a consequence :

Lemma

Let H be a nonexpansive half-space for the subshift $\mathcal{X}_{\mathcal{P},R}$. Then there exist $x, y \in \mathcal{X}_{\mathcal{P},R}$ such that $x|_{H\cap\mathbb{Z}^2} = y|_{H\cap\mathbb{Z}^2}$, $x \neq y$, and $f(x) = f(y) \in \Delta_{\mathcal{P},R}$.

Nonexpansive halfspace

The notion of nonexpansive direction can also be stated equivalently in terms of nonexpansive half-spaces.

Definition (nonexpansive halfspace)

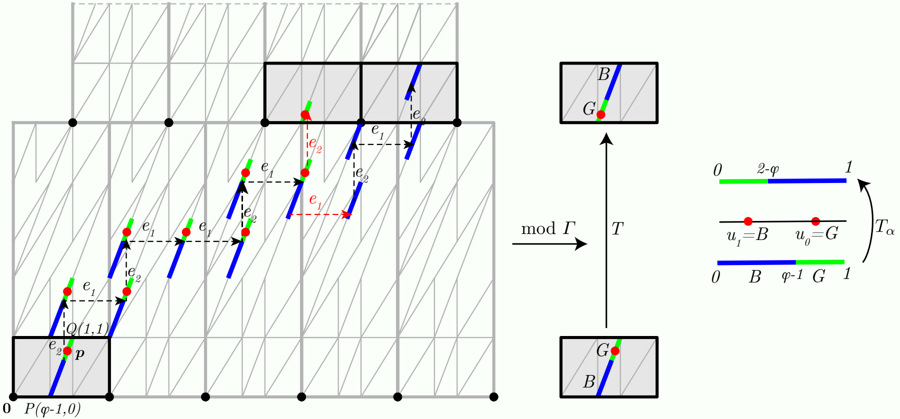
Let $X \subset \mathcal{A}^{\mathbb{Z}^d}$ be a subshift and σ be a $\mathbb{Z}^d$-action on X. We say that a half-space $H \subset \mathbb{R}^d$ is **nonexpansive** for σ if there exist $x, y \in X$ such that $x|_{\mathbb{Z}^d \cap H} = y|_{\mathbb{Z}^d \cap H}$ but $x \neq y$.

Lemma (Einsiedler, Lind, Miles, Ward, 2001)

A codimension 1 subspace V of $\mathbb{R}^d$ is nonexpansive for σ if and only if there is a half-space H whose boundary is V and which is nonexpansive for σ.

Manfred Einsiedler, Douglas Lind, Richard Miles, and Thomas Ward. Expansive subdynamics for algebraic $\mathbb{Z}^d$-actions. Ergodic Theory Dynam. Systems, *21(6) :1695–1729, 2001. (see Lemma 2.9)*

Idea of the proof



The $\mathbb{Z}^2$ action induces an exchange of the intervals (or rotation of) B and G. A point $\boldsymbol{p}$ on the segment $\overline{PQ}$ from $(\varphi - 1, 0)$ to $(1, 1)$ will return to the segment in a manner captured by a rotation of $\frac{1+\sqrt{5}}{2}$ on $[0, 1]$.

References

Recent preprints :

A Fibonacci's complement numeration system, with Jana Lepšová, (May 2022) arXiv:2205.02574

Nonexpansive directions in the Jeandel-Rao Wang shift, with Casey Mann and Jennifer McLoud-Mann, (June 2022), arXiv:2206.02414