Structure of Jeandel-Rao ~~aperiodic~~ hide it Wang tilings

Heidelberg
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Based on a study split into 4 articles: GD, DCG, AHL, JMD
+ 1 chapter arxiv:2012.03892

The first 10 minutes of the talk, we distribute tiles and we let people play with them while thinking about the following questions:

Q1: ~~For~~ $n\in\{3,4,5\}$, find a valid $n\times n$ pattern with the tiles.

Q2: What do you observe?

Q3: Can we find arbitrarily large $n\times n$ patterns?

Q4: (Domino Problem) Can we find a valid configuation $\mathbb{Z}^2\to\{0,1,\dots,10\}$? $\Omega_0=\{w:\mathbb{Z}^2\to\{0,\dots,10\}\mid w \text{ is valid}\}$ Is $\Omega_0=\phi$?

Q4 yes $\overset{\text{obvious}}{\Longrightarrow}$ Q3 yes, Q3 yes $\overset{\text{knuth vol 1}}{\Longrightarrow}$ Q4 yes

__EXAMPLE, (Wang 1961)__

$$1\,\underset{4}{\overset{3}{\boxed{A}}}\,2 \qquad 2\,\underset{3}{\overset{5}{\boxed{B}}}\,3 \qquad 3\,\underset{5}{\overset{4}{\boxed{C}}}\,1 \qquad \Rightarrow \qquad \begin{array}{c|ccc|c} & 3 & 5 & 4 & \\ \hline 1 & A & B & C & 1 \\ 3 & C & A & B & 3 \\ 2 & B & C & A & 2 \\ \hline & 3 & 5 & 4 & \end{array}$$

__Conjecture Wang 1961__ $T=\{c_i\,\underset{d_i}{\overset{b_i}{\square}}\,a_i \mid 1\leq i\leq K\}$ finite set of Wang tiles

$\exists$ valid configuration $w:\mathbb{Z}^2\to T$ $\iff$ $\exists$ valid configuration $w:\mathbb{Z}^2\to T$ which is periodic with periods $(p,0)$ and $(0,q)$ for some $p,q>0$.

__Remark (Wang, 1961)__ If conjecture is true, then Q4 is decidable

__Thm (Berger, 1966)__ Conjecture is false, Domino Problem is undecidable

__Proof #1__ Imbed any Turing machine in a set of Wang tiles and reduce DP to the Halting problem

__Proof #2__ $\exists$ set of 20426 Wang tiles which is aperiodic (ie tile the plane but never periodically)

Knuth: 92 tiles, Kari-Culik 13 tiles (1996), Jeandel-Rao (2015) 11 tiles

We can now illustrate the main theorem,

We provide the partition.

Show "Aperiodic Order" book.

Symbolic Dynamical Systems

Ref: Lind, Marcus 1995 ($d=1$); Hochmann, chapter, 2016; Einseidler, Schmidt, 1997

M: compact metric space

$R : \mathbb{Z}^d \times M \longrightarrow M$ group action

Dynamical system: $\mathbb{Z}^d \overset{R}{\curvearrowright} M$

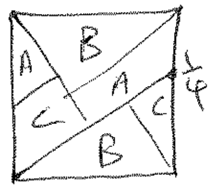
Topological Partition $\mathcal{P} = \{P_a\}_{a \in \mathcal{A}}$ of M $\left(\text{ie } P_a \text{ open sets disjoint s.t. } M = \bigcup_{a \in \mathcal{A}} \overline{P_a}\right)$

$\boxed{\text{EX1}}$ Hyperbolic toral automorphisms

$M = \mathbb{R}^2/\mathbb{Z}^2$, $d = 1$

$R : \mathbb{Z} \times \mathbb{T}^2 \longrightarrow \mathbb{T}^2$

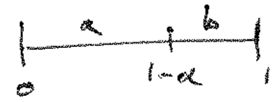
$k, x \longmapsto \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^k x$

$\mathcal{P} =$ 

$\boxed{\text{EX2}}$ Irrational rotation on circle

$M = \mathbb{R}/\mathbb{Z}$, $R : x \mapsto x + \alpha \mod \mathbb{Z}$

$d = 1$

$\mathcal{P} =$ 

$\boxed{\text{EX3}}$ Toral $\mathbb{Z}^2$-rotations

$M = \mathbb{R}^2/\Gamma_0$, $\Gamma_0 = \langle (\varphi, 0), (1, \varphi+3) \rangle_{\mathbb{Z}}$, $d = 2$, $R_0 : \mathbb{Z}^2 \times \mathbb{R}^2/\Gamma_0 \longrightarrow \mathbb{R}^2/\Gamma_0$

$\vec{n}, \vec{x} \longmapsto \vec{n} + \vec{x} \mod \Gamma_0$

$\mathcal{P}_0 =$ partition of $\mathbb{R}^2/\Gamma_0$ shown

Let $S \Subset \mathbb{Z}^d$ finite support, we say that a pattern $w \in \mathcal{A}^S$ is allowed for $\mathcal{P}, R$ if $\text{CodingRegion}(w) = \bigcap_{k \in S} R^{-k}(P_{w_k}) \neq \emptyset$

Language $\mathcal{L}_{\mathcal{P},R} = \{ w \in \mathcal{A}^S \mid w \text{ allowed for } \mathcal{P}, R \text{ and } S \Subset \mathbb{Z}^d \}$

The symb. dyn. syst. corresponding to $\mathcal{P}, R$ is

$$\mathbb{Z}^d \overset{\sigma}{\curvearrowright} \mathcal{X}_{\mathcal{P},R} = \{ w \in \mathcal{A}^{\mathbb{Z}^d} \mid w|_S \in \mathcal{L}_{\mathcal{P},R},\ \forall S \Subset \mathbb{Z}^d \}$$

Def The partition $\mathcal{P}$ gives a symbolic representation of $\mathbb{Z}^d \overset{R}{\curvearrowright} M$ if $\forall w \in \mathcal{X}_{\mathcal{P},R}$, $f(w) = \bigcap_{k \in \mathbb{Z}^d} R^{-k}(\overline{P_{w_k}})$ contains exactly 1 point.

This yield a factor map

$$\begin{array}{ccc} \mathcal{X}_{\mathcal{P},R} & \xrightarrow{f} & M \\ \downarrow \sigma & & \downarrow R \\ \mathcal{X}_{\mathcal{P},R} & \xrightarrow{f} & M \end{array}$$

(onto, continuous, & commutes the actions)

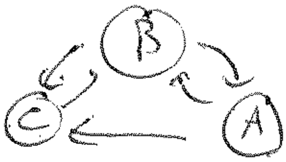
Def $\mathcal{P}$ is a Markov Partition for $\mathbb{Z}^d \overset{R}{\curvearrowright} M$ if

- $\mathcal{P}$ gives a symbolic representation of $\mathbb{Z}^d \overset{R}{\curvearrowright} M$
- $\mathcal{X}_{\mathcal{P},R}$ is a subshift of finite type

$\left(\text{ie } \exists \text{ finite set } F \text{ of patterns s.t. } \mathcal{X}_F = \{ w \in \mathcal{A}^{\mathbb{Z}^d} \mid w|_S \notin F,\ \forall S \Subset \mathbb{Z}^d \}\right)$

EX1 Hyperbolic toral automorphisms admit Markov Partitions

{ Sinai, 1968
Adler-Weiss, 1970
Bowen, 1975



EX2 N_0, Sturmian words (codings of irrational rotations) are not SFT

{ Morse-Hedlund, 1938
Pytheas Fogg, 2002
Lothaire, 2002

EX3 <u>Thm</u> (AHL) • $\mathcal{P}_0$ gives a symb. repr. of $\mathbb{Z}^2 \xrightarrow{R_0} \mathbb{R}^2/\Gamma_0$

• $\mathcal{X}_{\mathcal{P}_0,R_0} \subseteq \Omega_0$ is a minimal aperiodic subshift of the Jeandel-Rao Wang shift

Next question is $\mathcal{X}_{\mathcal{P}_0,R_0} \supseteq \Omega_0$?

<u>Strategy</u>: • Compute the subst. structure of Ω_0 (done in DCG with SageMath, proof to be reproduced in HEGL lab)

• Compute the subst. structure of $\mathcal{X}_{\mathcal{P}_0 R_0}$ using 2-dim Rauzy Induction (done in JMD with SageMath, proof to be reproduced in HEGL lab)

<u>Thm</u> (DCG + JMD) $\exists$ subshift X_0 of finite type $\subseteq \Omega_0$ s.t. $X_0 = \mathcal{X}_{\mathcal{P}_0 R_0}$

<u>Corollary</u> $\mathcal{P}_0$ is a Markov Partition for $\mathbb{Z}^2 \xrightarrow{R_0} \mathbb{R}^2/\Gamma_0$

Unfortunately, $\Omega_0 \setminus \mathcal{X}_{\mathcal{P}_0 R_0} \neq \emptyset$ because of fault lines

<u>Conjecture</u> $\Omega_0 \setminus \mathcal{X}_{\mathcal{P}_0,R_0}$ has 0 measure for every shift-invariant probability measure on Ω_0.

Next step is to understand the global picture.