Structure of Jeandel-Reck aperiodic Wang tiling

Based on a study split into articles: GD, DC6, A4H, IMA
+ 1 chapter arxiv 2012.03892

The first 10 minutes of the talk, we distribute tiles and we let people play with them while thinking about the following questions:

Q1: For $n \in \{3, 4, 5\}$, find a valid $n \times n$ pattern with the tiles.

Q2: What do you observe?

Q3: Can we find arbitrarily large $n \times n$ patterns?

Q4: (Domino Problem) Can we find a valid configuration $\mathbb{Z}^2 \rightarrow \{0, 1, \ldots, 10\}$?

- $Q4$ yes $\Rightarrow$ $Q3$ yes
- $Q3$ yes $\Rightarrow$ $Q4$ yes

**Example (Wang 1966)**

$$\begin{array}{ccc}
1 & 2 & 3 \\
4 & 5 & 6
\end{array} \rightarrow 
\begin{array}{ccc}
A & B & C \\
1 & 2 & 3
\end{array}$$

Conjecture (Wang 1966)

**Conjecture is true**

If conjecture is true, then $Q4$ is decidable

**Theorem (Berger, 1966)**

Conjecture is false. Domino Problem is undecidable.

**Proof**

1. Embed any Turing machine in a set of Wang tiles and reduce DP to the Halting Problem
2. A set of 20424 Wang tiles which is aperiodic (i.e. tile the plane but not periodically)


To prove the JC theorem, show "Aperiodic Order" book.
Symbolic Dynamical Systems

$M$ is compact metric space
$R : Z^d \times M \to M$ group action

Dynamical system: $Z^d \times M$

Topological Partition $P = \{ P_x \}_{x \in M}$

EX1: Hyperbolic toral automorphisms
$M = \mathbb{R}^2 / \mathbb{Z}^2$, $d=1$
$R : Z \times \mathbb{T}^2 \to \mathbb{T}^2$
$k, x \mapsto (1/k)x$

EX2: Irrational rotation on circle
$M = \mathbb{R} / \mathbb{Z}$, $R : x \mapsto x + \alpha \mod \mathbb{Z}$
$d=1$
$P = \{ \alpha - n \} \mod \mathbb{Z}$

EX3: Total $Z^2$ rotation
$M = \mathbb{R}^2 / \mathbb{Z}^2$, $d=2$, $R_o : \mathbb{R}^2 \to \mathbb{R}^2$
$R_0 : \mathbb{R}^2 \to \mathbb{R}^2$

$P_0$ is partition of $\mathbb{R}^2$ shown

Let $S \in Z^d$ finite support, we say that a pattern $w \in \mathbb{A}^S$ is allowed for $P, R$ if Coding Region $(w) = \bigcap_{k \in \mathbb{Z}} R_k(Pw_k) \neq \emptyset$

Language $L(P, R) = \{ w \in \mathbb{A}^S | w \text{ is allowed for } P, R \}$
The symbolic dynamical system corresponding to $P, R$ is

$Z^d \to X_{P, R} = \{ \omega \in \mathbb{A}^{Z^d} | |\omega| \leq L(P, R), \forall S \in Z^d \}$

Def: the partition $P$ gives a symbolic representation of $Z^d \times M$

If $\forall \omega \in X_{P, R}$, $f(\omega) = \bigcap_{k \in \mathbb{Z}} R^{-k}(Pw_k)$ contains exactly 1 point

This yields a factor map $X_{P, R} \to M$
$f \downarrow \bigcap_{k \in \mathbb{Z}} R^{-k} \times M$

Def $P$ is a Markov Partition for $Z^d \times M$ if

$P$ gives a symbolic representation of $Z^d \times M$

$X_{P, R}$ is a subshift of finite type

(i.e., finite set $F$ of patterns) s.t.
$X_F = \{ \omega \in \mathbb{A}^{Z^d} | \omega | S \in F, \forall S \in Z^d \}$

Ref: S. Lind, Marcus 1995 (d=1)
Hochimann, chapter 2016
Einsiedler, Schmidt, 1993
Hyperbolic toral automorphisms admit Markov Partitions

Since 1968
 Adler-Weiss, 1970
 Bowen, 1975

No, Sturmian Words (coding of) are not SFT

Moore-Heard, 1938
 Pytheas Fogg, 2002
 Lothaire, 2002

Thm (AHL) \( \mathcal{P}_0 \) gives a symbolic rep. of \( \mathbb{Z}^2 \to \mathbb{R}^2_{\mathcal{P}_0} \)

- \( X_{\mathcal{P}_0, \mathcal{R}_0} \subseteq \mathcal{R}_0 \) is a minimal aperiodic subshift of the Jeandel-Rao Wang shift

Next question is \( X_{\mathcal{P}_0, \mathcal{R}_0} \neq \mathcal{R}_0 \) ?

Strategy:
- Compute the subshift structure of \( \mathcal{R}_0 \) (done in DC6 with SageMath, proof to be reproduced in HEGL lab)
- Compute the subshift structure of \( X_{\mathcal{P}_0, \mathcal{R}_0} \) using 2-dim Random Induction (done in SIM with SageMath, proof to be reproduced in HEGL lab)

Thm (DC6-SIM) \( \exists \) a subshift \( \mathcal{X}_0 \) of finite type \( \subseteq \mathcal{R}_0 \) s.t. \( \mathcal{X}_0 = X_{\mathcal{P}_0, \mathcal{R}_0} \)

Corollary \( \mathcal{P}_0 \) is a Markov Partition for \( \mathbb{Z}^2 \to \mathbb{R}^2_{\mathcal{P}_0} \)

Unfortunately, \( \mathcal{R}_0 \setminus X_{\mathcal{P}_0, \mathcal{R}_0} \neq \emptyset \) because of fault lines

Conjecture \( \mathcal{R}_0 \setminus X_{\mathcal{P}_0, \mathcal{R}_0} \) has 0 measure for every shift-invariant probability measure on \( \mathcal{R}_0 \)

Next step is to understand the global picture.