Markov partitions for $\mathbb{Z}^2$-actions

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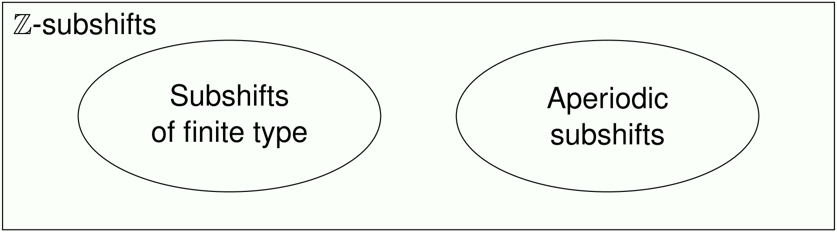
Outline

- Partitions for $\mathbb{Z}$-actions
  Symbolic representation of irrational rotations by Sturmian sequences
  Markov partitions for hyperbolic toral automorphisms
- Markov partitions for $\mathbb{Z}^2$-actions
  Example 1
  Example 2
- Open Questions
  Few questions and related results from literature

An incompatibility for 1-dimensional subshifts



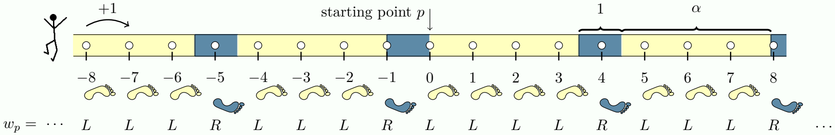
- Positive entropy
- $\exists$ periodic configurations
- Ex : Hyp. autom. of $\mathbb{T}^2$
- $\exists$ Markov partition

- Entropy can be zero
- No periodic configurations
- Ex : Irrat. rotation on $\mathbb{T}$
- No partition is Markov

Lind and Marcus (1995), Pytheas Fogg (2002)

Irrational rotations on the circle

The rotation $x \mapsto x + 1$ on the circle $\mathbb{R}/(1+\alpha)\mathbb{Z}$ can be represented symbolically by a partition of the circle into two intervals.



For $\alpha > 0$, let $X_\alpha = \overline{\{w_p \in \{L, R\}^{\mathbb{Z}} | p \in \mathbb{R}\}}$.

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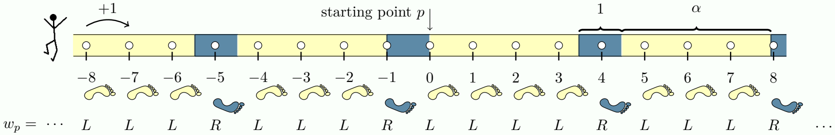


For $\alpha > 0$, let $X_\alpha = \overline{\{w_p \in \{L,R\}^{\mathbb{Z}} | p \in \mathbb{R}\}}$. If $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, then

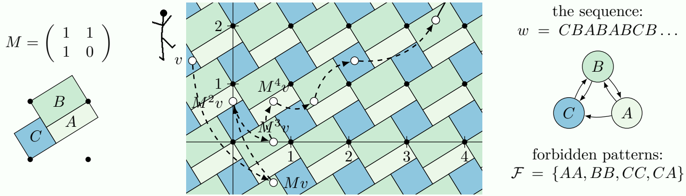
(C1) the partition gives a **symbolic representation**, i.e., every sequence $w \in X_\alpha$ is obtained from a unique starting point in $\mathbb{R}/(1+\alpha)\mathbb{Z}$, *...and thus we have a factor map $X_\alpha \to \mathbb{R}/(1+\alpha)\mathbb{Z}$,*

(C2) X_α is the set of sequences in $\{L,R\}^{\mathbb{Z}}$ whose language has $n+1$ patterns of length n for all $n \geq 0$ and whose letter frequencies exist and have ratio α (**combinatorial independent definition**).

Morse-Hedlund, Coven-Hedlund, Pytheas Fogg, Lothaire.

Hyperbolic automorphisms of the torus

The hyperbolic automorphism $v \mapsto Mv$ with $M = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ on the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ can be represented symbolically by a partition of the torus into three rectangles. E.g., starting at position $v = \left(\frac{-7}{10}, \frac{14}{10}\right)^T$:



Let $\mathcal{X}_{\mathcal{P},M} = \overline{\{w_v \in \{A,B,C\}^{\mathbb{Z}} | v \in \mathbb{T}^2\}}$.

Hyperbolic automorphisms of the torus

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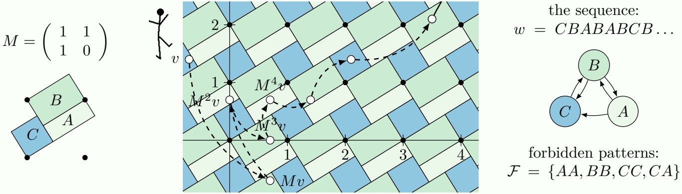


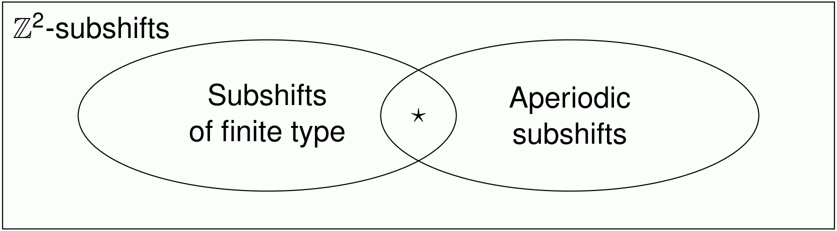
Let $\mathcal{X}_{\mathcal{P},M} = \overline{\{w_v \in \{A,B,C\}^{\mathbb{Z}} | v \in \mathbb{T}^2\}}$. The partition of $\mathbb{T}^2$ is a **Markov partition** for the automorphism (see Lind, Marcus, 1995) if :

(C1) the partition gives a **symbolic representation**, i.e., every sequence in $\mathcal{X}_{\mathcal{P},M}$ is obtained from a unique starting point in $\mathbb{T}^2$,

(C2') the set $\mathcal{X}_{\mathcal{P},M}$ is a **shift of finite type** (SFT), i.e., there exists a finite set $\mathcal{F}$ of patterns such that $\mathcal{X}_{\mathcal{P},M}$ is the set of sequences in $\{A,B,C\}^{\mathbb{Z}}$ which avoids the patterns in $\mathcal{F}$.

Adler and Weiss (1970), Sinaĭ (1968), Bowen (1975).

Is compatible for 2-dimensional subshifts



⋆ : Existence of 2-dim. aperiodic subshifts of finite type :

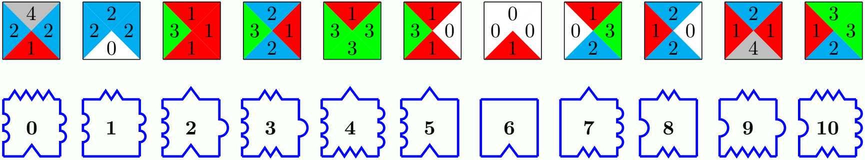
Berger (1966), Knuth (1969), Robinson (1971), Kari (1996), Jeandel-Rao (2015).

1D vs 2D

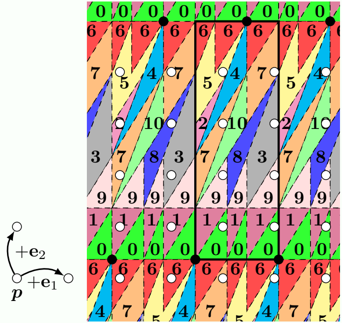
No partition of $\mathbb{T}$ is Markov for an aperiodic $\mathbb{Z}$-action, but their existence **can not be excluded** for free/zero entropy $\mathbb{Z}^2$-actions

Note : Einsiedler and Schmidt (1997) considered Markov partitions for $\mathbb{Z}^d$-actions by automorphisms

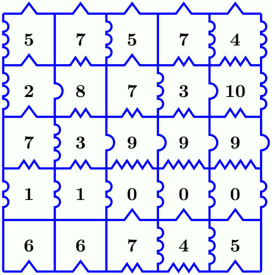
Jeandel-Rao tiles and associated partition



A toral partition allows to create tilings of the plane with these tiles :



$$\xrightarrow[\text{repr.}]{\text{symb.}} \begin{pmatrix} 5 & 7 & 5 & 7 & 4 \\ 2 & 8 & 7 & 3 & 10 \\ 7 & 3 & 9 & 9 & 9 \\ 1 & 1 & 0 & 0 & 0 \\ 6 & 6 & 7 & 4 & 5 \end{pmatrix} \xrightarrow[\text{tiling}]{\text{Wang}}$$



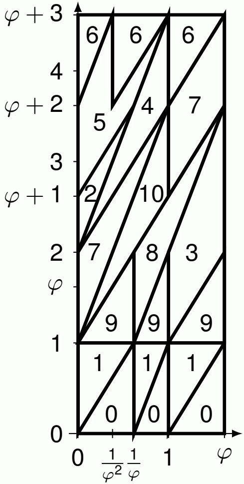
Example 1 : A Markov partition for a $\mathbb{Z}^2$-action

Let Γ be the **lattice** $\Gamma = \langle(\varphi, 0), (1, \varphi + 3)\rangle_{\mathbb{Z}}$ where $\varphi = \frac{1+\sqrt{5}}{2}$.

We consider the **$\mathbb{Z}^2$-action** on the **torus** $\mathbb{R}^2/\Gamma$:

$$
\begin{array}{rccc}
R: & \mathbb{Z}^2 \times \mathbb{R}^2/\Gamma & \to & \mathbb{R}^2/\Gamma \\
 & (\mathbf{n}, \mathbf{x}) & \mapsto & \mathbf{x} + \mathbf{n}.
\end{array}
$$

Let $\mathcal{P} = \{P_t\}_{t\in\{0,\dots,10\}}$ be the partition of the **fundamental domain** $D = [0, \varphi[\times [0, \varphi + 3[$ into 11 polygonal atoms shown on the right.



Theorem

$\mathcal{P}$ is a Markov partition for $\mathbb{Z}^2 \overset{R}{\curvearrowright} \mathbb{R}^2/\Gamma$.

Markov partitions for toral $\mathbb{Z}^2$-rotations featuring Jeandel-Rao Wang shift and model sets, *Annales Henri Lebesgue* 4 (2021) 283-324. `doi:10.5802/ahl.73`

Substitutive structure of Jeandel-Rao aperiodic tilings, *Discr. & Comput. Geom.* 65 (2021) 800-855. `doi:10.1007/s00454-019-00153-3`

Rauzy induction of polygon partitions and toral $\mathbb{Z}^2$-rotations, `arXiv:1906.01104v3`

Results on Jeandel-Rao Wang shift $\mathcal{X}_{\mathcal{P},R} \subsetneq \Omega_0$

Theorem

- $\mathcal{P}$ gives a **symbolic representation** of $(\mathbb{R}^2/\Gamma, \mathbb{Z}^2, R)$
- there exists an **almost 1-1 factor** map $f : \mathcal{X}_{\mathcal{P},R} \to \mathbb{R}^2/\Gamma$
- $(\mathbb{R}^2/\Gamma, \mathbb{Z}^2, R)$ is the **maximal equicontinuous factor** of $(\mathcal{X}_{\mathcal{P},R}, \mathbb{Z}^2, \sigma)$.
- $\mathcal{X}_{\mathcal{P},R}$ is a **proper minimal, aperiodic and uniquely ergodic** subshift of the Jeandel-Rao Wang shift, i.e., $\mathcal{X}_{\mathcal{P},R} \subsetneq \Omega_0$.
- The **measure-preserving** dynamical system $(\mathcal{X}_{\mathcal{P},R}, \mathbb{Z}^2, \sigma, \nu)$ is **isomorphic** to $(\mathbb{R}^2/\Gamma, \mathbb{Z}^2, R, \lambda)$ where
  - ν is the unique shift-invariant probability measure on $\mathcal{X}_{\mathcal{P},R}$
  - λ is the Haar measure on $\mathbb{R}^2/\Gamma$.
- Occurrences of patterns in $\mathcal{X}_{\mathcal{P},R}$ is a **4-to-2 C&P set**.

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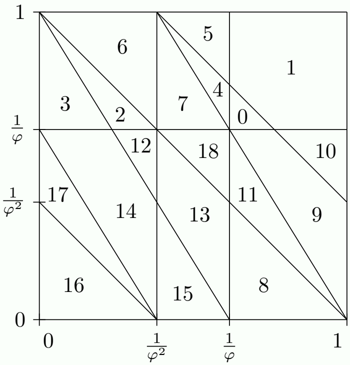
Example 2 : A Markov partition for a $\mathbb{Z}^2$-action

Let $\varphi = \frac{1+\sqrt{5}}{2}$. We consider the **$\mathbb{Z}^2$-action** on $\mathbb{T}^2$

$$\begin{array}{rccc} R_{\mathcal{U}}: & \mathbb{Z}^2 \times \mathbb{T}^2 & \to & \mathbb{T}^2 \\ & (\mathbf{n}, \mathbf{x}) & \mapsto & \mathbf{x} + \varphi^{-2}\mathbf{n}. \end{array}$$

Let $\mathcal{P}_{\mathcal{U}} = \{P_t\}_{t\in\{0,\dots,18\}}$ be the partition of the **fundamental domain** $D = [0,1[^2$ into 19 polygonal atoms shown on the right.



Theorem

$\mathcal{P}_{\mathcal{U}}$ is a Markov partition for $\mathbb{Z}^2 \overset{R_{\mathcal{U}}}{\curvearrowright} \mathbb{R}^2/\Gamma$.

A self-similar aperiodic set of 19 Wang tiles, *Geometriae Dedicata* 201 (2019) 81-109 doi:10.1007/s10711-018-0384-8

Three characterizations of a self-similar aperiodic 2-dimensional subshift, a chapter to appear in a book edited by N. Aubrun and M. Rao, arXiv:2012.03892

Open questions (starting with $d = 2$)

Question

Characterize $\mathbb{Z}^2$-rotations on $\mathbb{T}^2$ which admit a Markov partition.

Ideas :

- $\mathbb{Z}^2$-rotations must be given by computable numbers
- $\mathbb{Z}^2$-rotations must be given by algebraic integers

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When is the Markov partition of a $\mathbb{Z}^2$-action polygonal ?

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Question

Which polygonal partitions are Markov for a $\mathbb{Z}^2$-action ?

Vector subspaces characterized by patterns

A generic d-dimensional vector subspace $V \subset \mathbb{R}^n$ is said to be **characterized by patterns** if there is a finite set of (finite) patterns, called forbidden patterns, such that any projection tiling with generic physical space which does not contain any of these patterns is a canonical projection tiling with physical space V.

(*I think what they mean by generic is equivalent to* $V \cap \mathbb{Z}^n = \{0\}$)

Corollary 1 (Bédaride, Fernique, 2020)

Any generic d-dimensional vector subspace characterized by patterns is **algebraic**.

Bédaride, Fernique. Canonical projection tilings defined by patterns *Geometriae Dedicata* 208 (2020) 157-175 `doi:10.1007/s10711-020-00515-9`

To learn about canonical projection tilings and model sets :

Baake, Grimm. *Aperiodic Order. Vol. 1*, Cambridge Univ. Press, 2013.
`doi:10.1017/CBO9781139025256`

When β-shift are of finite type

Recall that the β-shift S_β is the closure of the set of β-expansions of numbers in $[0, 1)$.

Theorem

The β-shift S_β is a coded symbolic dynamical system which is

- **sofic** if and only if
  $d_\beta(1)$ is eventually periodic, i.e., β is **Parry number**,
- **of finite type** if and only if
  $d_\beta(1)$ is finite, i.e., β is **simple Parry number**.

Parry numbers $\subset$ Perron number $\subset$ Algebraic integers

Ito and Takahashi 1974, Bertrand-Mathis 1986,
Blanchard 1989, CANT, Chapter 2 and Theorem 2.3.15

Fractal Markov Partitions

Theorem (Bowen, 1978)

The boundaries of the sets in a Markov partition for linear Anosov diffeomorphisms of $\mathbb{T}^3$ **cannot be smooth**.

Bowen, Rufus. Markov partitions are not smooth. Proceedings of the American Mathematical Society 71 (1978) 130-32. doi:10.2307/2042234

Example : a Markov partition for the automorphism $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ of $\mathbb{T}^3$ is :

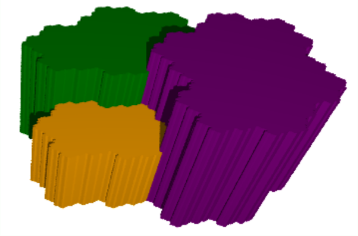


Image credit : Timo Jolivet's talk, *Toral automorphisms, Markov Partitions and fractals*, Japan, 2012.

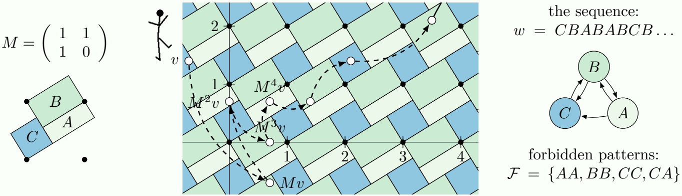
Smooth Markov Partitions

Theorem (Cawley, 1991)

The only hyperbolic toral automorphisms f for which there exist Markov partitions with piecewise **smooth boundary** are those for which a power f^k is linearly covered by a **direct product of automorphisms of the 2-torus**.

Cawley, Elise, Smooth Markov partitions and toral automorphisms. Ergodic Theory and Dynamical Systems 11 (1991) 633-51. doi:10.1017/S0143385700006404

$$M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$



Self-similarity of the 19 atoms example

The incidence matrix of the self-similarity for the partition $\mathcal{P}_{\mathcal{U}}$ is :

$$M = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{pmatrix}$$

with characteristic polynomial

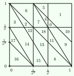
$$\chi_M(\lambda) = \lambda^3(\lambda-1)^4(\lambda+1)^4(\lambda^2-3x+1)(\lambda^2+\lambda-1)^3.$$

A self-similar aperiodic set of 19 Wang tiles, *Geometriae Dedicata* 201 (2019) 81-109
doi:10.1007/s10711-018-0384-8

Question

The Rauzy fractal, which codes some rotation $x \mapsto x + \xi$ on $\mathbb{T}^2$, is a section of the Markov partition  for the automorphism $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ of $\mathbb{T}^3$.

Question

What is the relation between the Markov partition $\mathcal{P}_{\mathcal{U}} =$ 

, which codes the $\mathbb{Z}^2$-rotation $R_{\mathcal{U}}$ on $\mathbb{T}^2$, and the Markov partition of $\mathbb{T}^8$ associated to the restriction of the action of the 19×19 matrix M to the 8-dimensional subspace on which it is an hyperbolic automorphism ?