A characterization of Sturmian sequences by indistinguishable asymptotic pairs

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Sebastián Barbieri*, Sébastien Labbé† and Štěpán Starosta‡

* Universidad de Santiago de Chile
† CNRS, LaBRI, Université de Bordeaux
‡ Czech Technical University in Prague

One World Combinatorics on Words Seminar
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Outline

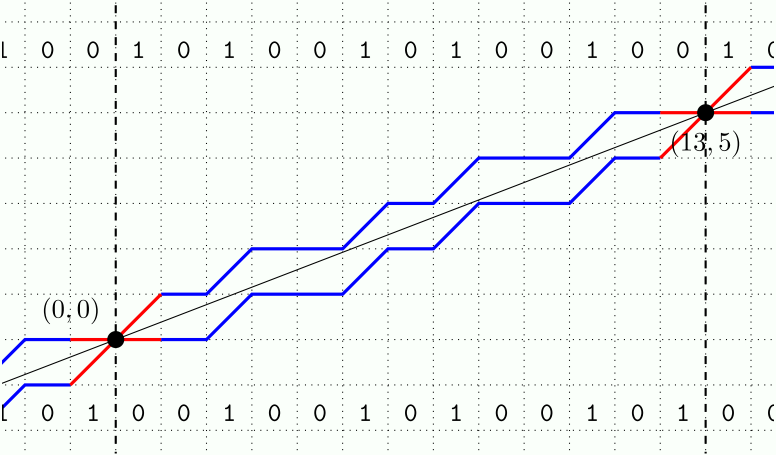
- Sturmian words
  Mechanical words, Christoffel words, Pirillo's theorem
- Terminology
  Symbolic dynamics, Asymptotic pairs, Pattern discrepancy, Indistinguishable asymptotic pairs
- Results
  Theorem A, Theorem B, Theorem C

Mechanical words (Morse, Hedlund, 1940)

Let $\alpha \in [0,1]$ and $c_\alpha, c'_\alpha : \mathbb{Z} \to \{0,1\}$ be the configurations

$$c'_\alpha(n) = \lceil \alpha(n+1) \rceil - \lceil \alpha n \rceil \qquad \text{(upper mechanical word)}$$

$$c_\alpha(n) = \lfloor \alpha(n+1) \rfloor - \lfloor \alpha n \rfloor \qquad \text{(lower mechanical word)}$$

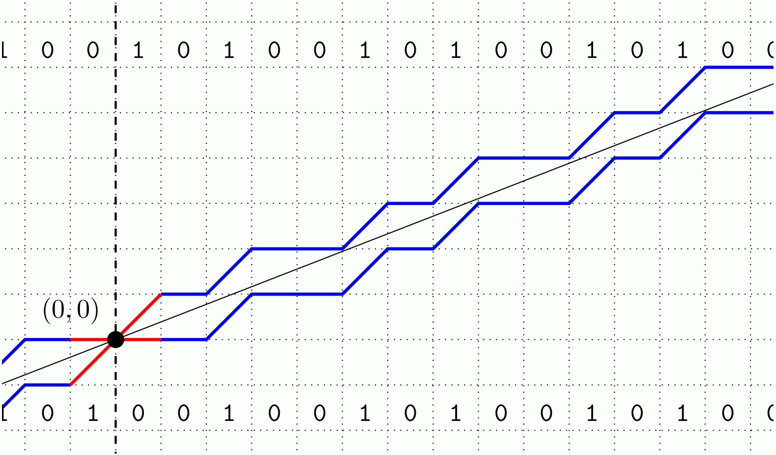


Sturmian words (Morse, Hedlund, 1940)

If $\alpha \in [0,1] \setminus \mathbb{Q}$, then the mechanical words are **not periodic** :

$c'_\alpha(n) = \lceil \alpha(n+1) \rceil - \lceil \alpha n \rceil$ (**upper** characteristic Sturmian word)

$c_\alpha(n) = \lfloor \alpha(n+1) \rfloor - \lfloor \alpha n \rfloor$ (**lower** characteristic Sturmian word)



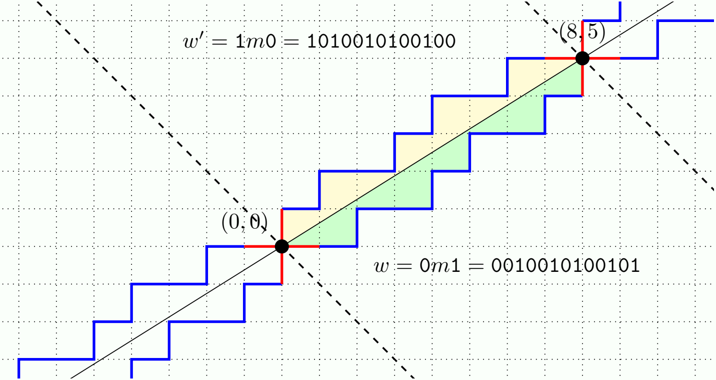
Christoffel words

If $\alpha \in [0, 1] \cap \mathbb{Q}$, then the mechanical words are **periodic** :

$c'_\alpha(n) = {}^\infty w'^\infty$ where w' is the **upper** Christoffel word of slope p/q,

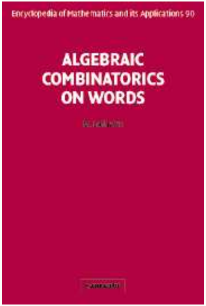
$c_\alpha(n) = {}^\infty w^\infty$ where w is the **lower** Christoffel word of slope p/q,

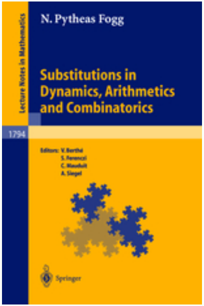
where $\alpha = p/(p+q)$ with $p, q \in \mathbb{Z}_{\geq 0}$ coprime integers.

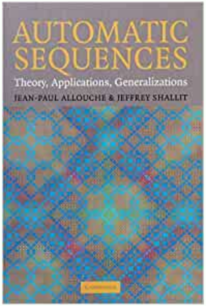


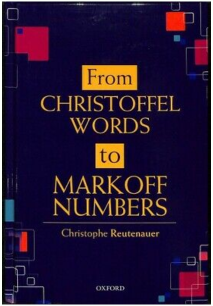
Moreover $w \leq_{lex} p \leq_{lex} w'$ for all primitive period p of c_α and c'_α.

Books









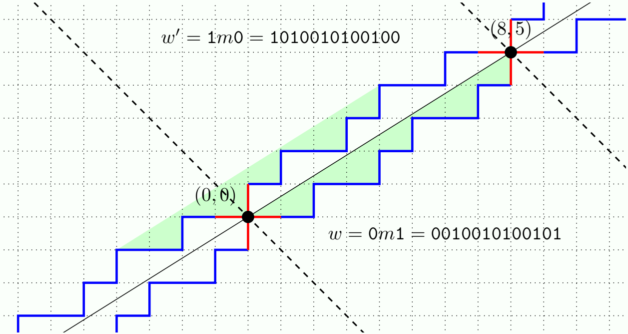
- Chapter 2 of Lothaire's book (2002), by Berstel and Séébold
- Chapter 6 of Pytheas Fogg's book (2002), by Arnoux
- Chapter 9 of Allouche and Shallit's book (2003)
- Christophe Reutenauer's book (2019)

Pirillo's theorem (2001)

Let $w = 0m1$ and $w' = 1m0$ for some $m \in \{0, 1\}^*$.

Theorem

The word w is a lower Christoffel word iff w and w' are conjugate.



A d-dimensional extension of Christoffel words

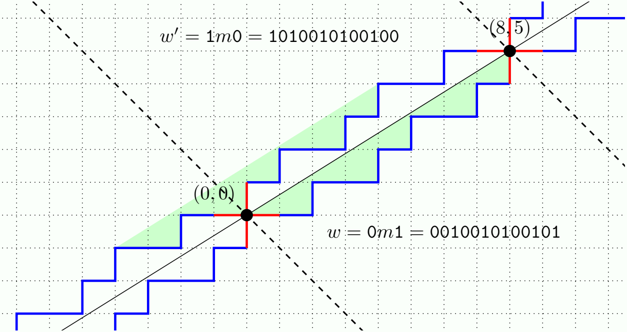
*S. Labbé, C. Reutenauer, Discrete Comput. Geom. **54** (2015) 152–181.*

Pirillo's theorem (restated for ${}^{\infty}w^{\infty}$)

Let $w = \mathtt{0}m\mathtt{1}$ and $w' = \mathtt{1}m\mathtt{0}$ for some $m \in \{\mathtt{0}, \mathtt{1}\}^*$.

Theorem

${}^{\infty}w^{\infty} = c_\alpha$ is a lower mechanical word of slope $\alpha = p/(p+q)$ iff ${}^{\infty}w^{\infty}$ is a shift of ${}^{\infty}w'^{\infty}$.



Question : Let $\alpha \in [0,1] \setminus \mathbb{Q}$.

Does $\lim_{\frac{p}{p+q}\to\alpha}$ (Pirillo's theorem) exist ?

Symbolic dynamics

We consider

- a finite set Σ : the **alphabet**,
- the space of **configurations** $\Sigma^{\mathbb{Z}} = \{x \colon \mathbb{Z} \to \Sigma\}$,
- $\Sigma^{\mathbb{Z}}$ endowed with the **prodiscrete topology**,

$$x = \cdots \boxed{0}\,0010111011100.0110010000001101\,\boxed{100001}\cdots$$

$$y = \cdots \boxed{1}\,0010111011100.0110010000001101\,\boxed{011110}\cdots$$

- to a pattern $p \colon S \to \Sigma$ with finite support $S \subset \mathbb{Z}$, a **cylinder**

$$[p] = \left\{x \in \Sigma^{\mathbb{Z}} \colon x|_S = p\right\}.$$

- the **shift** action $\mathbb{Z} \overset{\sigma}{\curvearrowright} \Sigma^{\mathbb{Z}}$.

$$\sigma^{-1}(x) = \cdots 1000101110111 0.0011001000000110110000\cdots$$

$$x = \cdots 0001011101110 0.0110010000001101100001\cdots$$

$$\sigma(x) = \cdots 0010111011100 0.1100100000011011000010\cdots$$

$$\sigma^2(x) = \cdots 0101110111000 1.1001000000110110000100\cdots$$

Asymptotic pairs

Let $x, y \in \Sigma^{\mathbb{Z}}$ be two configurations $x, y \in \Sigma^{\mathbb{Z}}$, e.g.,

$$x = \cdots 0010111011\boxed{0}100.0\boxed{0001}01\boxed{10}110010000011\cdots$$
$$y = \cdots 0010111011\boxed{1}100.0\boxed{1110}01\boxed{01}110010000011\cdots$$

unequal at positions $F = \{-4\} \cup \{1, 2, 3, 4\} \cup \{7, 8\}$.

Definition

$x, y \in \Sigma^{\mathbb{Z}}$ are **asymptotic** if x and y differ in finitely many sites of $\mathbb{Z}$.

The set $F = \{n \in \mathbb{Z} : x_n \neq y_n\}$ is called the **difference set** of (x, y).

Pattern discrepancy

- Two asymptotic configurations $x, y \in \Sigma^{\mathbb{Z}}$ with difference set F.
- A pattern $p \colon S \to \Sigma$ with finite support $S \subseteq \mathbb{Z}$.

Goal : compare the # of occurrences of p in x and y : $|y|_p - |x|_p$.
Example : pattern $p = .1001$ with support $S = \{0, 1, 2, 3\}$

$$x = \cdots 10\underline{1001}0\underline{1001}0\boxed{\underline{1.0}}\,\underline{0\overline{1}}\overline{001}0\overline{1001}01\cdots$$

$$y = \cdots 10\underline{1001}0\underline{1001}\overline{0}\boxed{\overline{0.1}}\,0\overline{1001}0\overline{1001}01\cdots$$

with difference set $F = \{-1, 0\}$.

Definition

The ***p*-discrepancy** associated to (x, y) is given by

$$\Delta_p(x, y) = \sum_{n \in F - S} \mathbb{1}_{[p]}(\sigma^n y) - \mathbb{1}_{[p]}(\sigma^n x).$$

Note : $n \in \mathbb{Z} \setminus (F - S)$ if and only if $(n + S) \cap F = \varnothing$.

Indistinguishable asymptotic pairs

Let $x, y \in \Sigma^{\mathbb{Z}}$ be asymptotic configurations.

Definition

x, y are **indistinguishable** if $\Delta_p(x, y) = 0$ for every finite pattern p.

Example 1 : The **trivial** asymptotic pair (x, x) is indistinguishable.

Example 2 :

$$x = \cdots 00000000000000.\boxed{1}000000\boxed{0}000000000000\cdots$$
$$y = \cdots 00000000000000.\boxed{0}000000\boxed{1}000000000000\cdots$$

In both of these examples, x and y lie on the **same orbit** of $\mathbb{Z} \overset{\sigma}{\curvearrowright} \Sigma^{\mathbb{Z}}$.

Question : Can we find other examples ?

Indistinguishable asymptotic pairs

Let $x, y \in \Sigma^{\mathbb{Z}}$ be asymptotic configurations.

Definition

x, y are **indistinguishable** if $\Delta_p(x, y) = 0$ for every finite pattern p.

Non-Example 3, because $\Delta_1(x, y) = -7$:

$$x = \cdots 00000000000000.\boxed{1111111}\,1111111111111\cdots$$
$$y = \cdots 00000000000000.\boxed{0000000}\,1111111111111\cdots$$

Example 4, with $\Delta_{abcabc}(x, y) = 1 - 1 = 0$:

$$x = \cdots bcabcbc\underline{abcabc}bc\underline{abc.\,\boxed{abc}}\,bcabcbc\underline{abcabc}bcabcbc\cdots$$
$$y = \cdots bcabcbc\underline{abcabc}bcabc.\,\boxed{bc\underline{a}}\,\underline{bcabc}bc\underline{abcabc}bcabcbc\cdots$$

Theorem A

Theorem

Let $x, y \in \{\mathtt{0}, \mathtt{1}\}^{\mathbb{Z}}$ and assume that x is ***recurrent****.*
The pair (x, y) is an ***indistinguishable asymptotic pair*** *with difference set $F = \{-1, 0\}$ such that $x_{-1}x_0 = \mathtt{10}$ and $y_{-1}y_0 = \mathtt{01}$*

if and only if

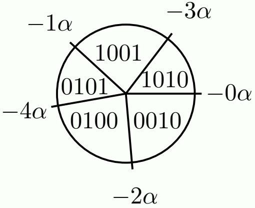
there exists $\alpha \in [0, 1] \setminus \mathbb{Q}$ such that $x = c_\alpha$ and $y = c'_\alpha$ are the lower and upper ***characteristic Sturmian words*** *of slope α.*

Idea of Proof of Theorem A

(Recall that x is recurrent)

x, y indistinguishable pair

x, y mechanical words ⟵ (Coven-Hedlund, 73') x, y have complexity $n+1$



$x = \cdots 101001010010\ \boxed{1.0}\ 010010100101 \cdots$

$y = \cdots 101001010010\ \boxed{0.1}\ 010010100101 \cdots$

Proposition

Let $x, y \in \Sigma^{\mathbb{Z}}$ be a non-trivial indistinguishable asymptotic pair whose difference set F is contained in an interval I. For every $n \geq 1$

$$n + 1 \leq \#\mathcal{L}_n(x) \leq n + \#I - 1.$$

Example using Christoffel words

Let $\mathtt{0}m\mathtt{1}$ be a lower Christoffel word of slope p/q with $p + q = n$.
The 2 words of length $2n$:

$$\mathtt{1}m\mathtt{1}.\mathtt{0}m\mathtt{1}$$

$$\mathtt{1}m\mathtt{0}.\mathtt{1}m\mathtt{1}$$

both contain $n + 1$ factors of size n (one occurence of each) :

```
sage: u = Word('10100101001010010010100101')
sage: v = Word('10100101001001010010100101')
sage: v.factor_set(13) == u.factor_set(13)
True
sage: len(u.factor_set(13))
14
```

But their binomial coefficients are not equal :

```
sage: u.number_of_subword_occurrences(Word('01'))
82
sage: v.number_of_subword_occurrences(Word('01'))
83
```

Theorem B

Theorem

Let $x, y \in \{\mathtt{0}, \mathtt{1}\}^{\mathbb{Z}}$.
The pair (x, y) is an ***indistinguishable asymptotic pair*** *with difference set $F = \{-1, 0\}$ such that $x_{-1}x_0 = \mathtt{10}$ and $y_{-1}y_0 = \mathtt{01}$*

if and only if

there exists a monotone sequence $(\alpha_n)_{n\in\mathbb{N}}$ with $\alpha_n \in [0, 1] \setminus \mathbb{Q}$ s.t.

$$x = \lim_{n\to\infty} c_{\alpha_n} \quad \text{and} \quad y = \lim_{n\to\infty} c'_{\alpha_n}.$$

are the limits of ***characteristic Sturmian words*** *of slope α_n.*

If $\alpha = \lim_{n\to\infty} \alpha_n \in [0, 1] \setminus \mathbb{Q}$, then

$$x = \lim_{n\to\infty} c_{\alpha_n} = c_\alpha \quad \text{and} \quad y = \lim_{n\to\infty} c'_{\alpha_n} = c'_\alpha$$

and it corresponds to Theorem A.

Theorem B : limit towards a rational slope

Assume $\lim_{n\to\infty} \alpha_n = p/(p+q) \in [0,1] \cap \mathbb{Q}$, with $p, q \in \mathbb{Z}_{\geq 0}$ coprime.
If $p \neq 0$ and $q \neq 0$ and the limit is **from above**, then

$$\lim_{\alpha \to \frac{p}{p+q}^+} c_\alpha = {}^\infty(1m0)(1m1).(0m1)(0m1)^\infty$$

$$\lim_{\alpha \to \frac{p}{p+q}^+} c'_\alpha = {}^\infty(1m0)(1m0).(1m1)(0m1)^\infty$$

or the limit is **from below**, then

$$\lim_{\alpha \to \frac{p}{p+q}^-} c_\alpha = {}^\infty(0m1)(0m1).(0m0)(1m0)^\infty$$

$$\lim_{\alpha \to \frac{p}{p+q}^-} c'_\alpha = {}^\infty(0m1)(0m0).(1m0)(1m0)^\infty$$

Limit cases : when $p = 0$ and $q = 1$ or $p = 1$ and $q = 0$, then

$$\lim_{\alpha \to 0^+} c_\alpha = {}^\infty 01.00^\infty \qquad \lim_{\alpha \to 1^-} c_\alpha = {}^\infty 11.01^\infty$$

$$\lim_{\alpha \to 0^+} c'_\alpha = {}^\infty 00.10^\infty \qquad \lim_{\alpha \to 1^-} c'_\alpha = {}^\infty 10.11^\infty$$

Idea of Proof of Theorem B ($\Longrightarrow$)

Let $x, y \in \{0, 1\}^{\mathbb{Z}}$ and assume x is **not recurrent**.
If the pair (x, y) is an **indistinguishable asymptotic pair** with difference set $F = \{-1, 0\}$ such that $x_{-1}x_0 = 10$ and $y_{-1}y_0 = 01$,

then

$x = \sigma^k(y)$ for some $k \in \mathbb{Z}$.
If $k \geq 2$, then there exists $m \in \{0, 1\}^{k-2}$ s.t.

$$x = {}^{\infty}(1m0)(1m1).(0m1)(0m1)^{\infty}$$
$$y = {}^{\infty}(1m0)(1m0).(1m1)(0m1)^{\infty}$$

- $1m0$ appears in y intersecting the difference set F
- it must appear in x intersecting the difference set F
- Thus $1m0$ is a factor of $1m1.0m1$, but certainly not as a prefix
- Therefore $1m0$ is a factor of $m1.0m1$ and $0m1.0m1 = (0m1)^2$
- This implies that $1m0$ and $0m1$ are conjugate
- **Pirillo**'s Theorem $\Longrightarrow$ $0m1$ is a **lower Christoffel word** of slope p/q for some coprime integers $p, q \in \mathbb{Z}_{\geq 0}$ satisfying $p + q = k$.

Theorem C

Theorem

Let Σ be a ***finite alphabet*** *and $x, y \in \Sigma^{\mathbb{Z}}$ a non-trivial asymptotic pair. Then x, y is* ***indistinguishable*** *if and only if either*

- *x is* ***recurrent*** *and there exists $\alpha \in [0,1] \setminus \mathbb{Q}$, a substitution $\varphi\colon \{\mathtt{0},\mathtt{1}\} \to \Sigma^+$ and an integer $m \in \mathbb{Z}$ such that*

$$\{x, y\} = \{\sigma^m \varphi(\sigma^1(c_\alpha)), \sigma^m \varphi(\sigma^1(c'_\alpha))\},$$

- *x is* ***not recurrent*** *and there exists a substitution $\varphi\colon \{\mathtt{0},\mathtt{1}\} \to \Sigma^+$ and an integer $m \in \mathbb{Z}$ such that*

$$\{x, y\} = \{\sigma^m \varphi({}^\infty\mathtt{0.1}\mathtt{0}^\infty), \sigma^m \varphi({}^\infty\mathtt{0.01}\mathtt{0}^\infty)\}.$$

Idea of Proof of Theorem C ($\Longleftarrow$)

Let $x, y \in \Sigma^{\mathbb{Z}}$ be an **asymptotic pair** such that their difference set F is contained in $[\![0, k-1]\!]$.

Lemma

Let $\varphi\colon \Sigma \to \Gamma^{+}$ be a ***substitution*** *on $\Sigma^{\mathbb{Z}}$.*
If (x, y) is ***indistinguishable****, then $(\varphi(x), \varphi(y))$ is* ***indistinguishable****.*

$$x = \cdots 010010.\boxed{01}0100101\cdots$$

$$y = \cdots 010010.\boxed{10}0100101\cdots$$

Applying $\varphi : 0 \mapsto abc, 1 \mapsto bc$:

$$\varphi(x) = \cdots bcabcbcabcabcbcabc.\boxed{abc}bcabcbcabcabcbcabcbc\cdots$$

$$\varphi(y) = \cdots bcabcbcabcabcbcabc.\boxed{bca}bcabcbcabcabcbcabcbc\cdots$$

Idea of Proof of Theorem C ($\Longrightarrow$)

Lemma

Assume $a \in \Sigma$ appears with bounded gaps in x.
*Let $D_a(x)$ be the **derived sequence** of x wrt return words to $a \in \Sigma$.*
*If (x, y) **indistinguishable**, then $(D_a(x), D_a(y))$ **indistinguishable**.*

Return words to letter c in x and y are cab and cb :

$$x = \cdots bcabcbcabcabcbcabc.\boxed{abc}bcabcbcabcabcbcabcbc\cdots$$

$$y = \cdots bcabcbcabcabcbcabc.\boxed{bca}bcabcbcabcabcbcabcbc\cdots$$

Replacing $cab \mapsto 0$, $cb \mapsto 1$, we obtain the **derived sequences** :

$$D_c(x) = \cdots 010010.\boxed{01}0100101\cdots$$

$$D_c(y) = \cdots 010010.\boxed{10}0100101\cdots$$

with a **smaller** difference set.

Thermodynamics and Gibbs theory

Gibbsian representations of continuous specifications : the theorems of Kozlov and Sullivan revisited.

S. Barbieri, R. Gómez, B. Marcus, T. Meyerovitch, S. Taati. arXiv:2001.03880

They defined the following **norm** on asymptotic configurations of $\Sigma^{\mathbb{Z}}$:

$$\|(x,y)\|^*_{\mathrm{NS}} = \sup_{\substack{S \subseteq \mathbb{Z} \\ S \text{ finite}}} \frac{1}{|S|} \sum_{p \in \Sigma^S} |\Delta_p(x,y)|.$$

- Every asymptotic pair induces an evaluation map on the space of **continuous cocycles** on the equiv. relation of asymptotic pairs.
- They show that this norm coincides with the **dual norm** in the space of linear functionals on the space of continuous cocycles.
- In other words, the asymptotic pairs which induce the null operator are precisely the **indistinguishable pairs**.
- Thus, our results provide a full characterization of which asymptotic pairs induce the **null operator**.

Ongoing

We are currently working to extend Theorem A from $\mathbb{Z}$ to $\mathbb{Z}^d$.

Extending Theorem B and Theorem C to $\mathbb{Z}^d$ seems more difficult.