Pavages apériodiques en dimension 1

de la combinatoire à la géométrie en passant par l'algèbre (2e partie)

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16 novembre 2020

Outline

- Cut and Project Scheme

  Cut and Project Scheme, the $\star$-map, Window and Model set

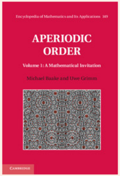
- Examples

  Geometric Sturmian tilings of $\mathbb{R}$, Symbolic Sturmian on $\mathbb{Z}$, Jeandel-Rao tilings

Baake & Grimm, aperiodicorder.org

— a book series on aperiodic order, published by Cambridge University Press —

Volume 1: A Mathematical Invitation (August 2013)



Quasicrystals are non-periodic solids that were discovered in 1982 by Dan Shechtman, Nobel Prize Laureate in Chemistry 2011. The underlying mathematics, known as the theory of aperiodic order, is the subject of this comprehensive multi-volume series.

This first volume provides a graduate-level introduction to the many facets of this relatively new area of mathematics. Special attention is given to methods from algebra, discrete geometry and harmonic analysis, while the main focus is on topics motivated by physics and crystallography. In particular, the authors provide a systematic exposition of the mathematical theory of kinematic diffraction.

Numerous illustrations and worked-out examples help the reader to bridge the gap between theory and application. The authors also point to more advanced topics to show how the theory interacts with other areas of pure and applied mathematics.

Robert V. Moody's review in the Mathematical Intelligencer

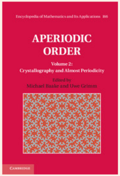
Aernout van Enter's review in Zentralblatt für Mathematik

Jean-Pierre Gazeau's review in Mathematical Reviews (MathSciNet)

Have a look at the table of contents and read Roger Penrose's foreword

Download Addenda and Corrigenda (PDF)

Volume 2: Crystallography and Almost Periodicity (November 2017)



This second volume begins to develop the theory in more depth. A collection of leading experts in the field, among them Robert V. Moody, introduce and review important aspects of this rapidly-expanding field.

The volume covers various aspects of crystallography, generalising appropriately from the classical case to the setting of aperiodically ordered structures. A strong focus is placed upon almost periodicity, a central concept of crystallography that captures the coherent repetition of local motifs or patterns, and its close links to Fourier analysis, which is one of the main tools available to characterise such structures.

The book opens with a foreword by Jeffrey C. Lagarias on the wider mathematical perspective and closes with an epilogue on the emergence of quasicrystals from the point of view of physical sciences, written Peter Kramer, one of the founders of the field on the side of theoretical and mathematical physics.

Cut and project scheme

A **cut and project scheme** (CPS) is a triple $(\mathbb{R}^d, H, \mathcal{L})$ where

- $\mathbb{R}^d$ is the **physical space**,
- H, the **internal space**, is a locally compact Abelian group,
- $\mathcal{L}$ is a **lattice** in $\mathbb{R}^d \times H$,
- π natural **projection** on $\mathbb{R}^d$ such that $\pi|_{\mathcal{L}}$ is injective,
- π_{int} natural **projection** on H such that $\pi_{\text{int}}(\mathcal{L})$ is dense in H.

$$
\begin{array}{ccccc}
H & \xleftarrow{\pi_{\text{int}}} & \mathbb{R}^d \times H & \xrightarrow{\pi} & \mathbb{R}^d \\
\cup \text{ dense} & & \cup & & \cup \\
\pi_{\text{int}}(\mathcal{L}) & \longleftarrow & \mathcal{L} & \xrightarrow{1-1} & \pi(\mathcal{L})
\end{array}
$$

A CPS is called **Euclidean** when $H = \mathbb{R}^m$ for some $m \in \mathbb{N}$.

The $\star$-map of a CPS $(\mathbb{R}^d, H, \mathcal{L})$

$$\begin{tikzcd}
H & \mathbb{R}^d \times H \arrow[l, "\pi_{\mathrm{int}}"'] \arrow[r, "\pi"] & \mathbb{R}^d \\
\cup \text{ dense} & \cup & \cup \\
\pi_{\mathrm{int}}(\mathcal{L}) & \mathcal{L} \arrow[l] \arrow[r, "1-1"] & \pi(\mathcal{L}) \arrow[ll, bend left, "\star"]
\end{tikzcd}$$

There exists a map $\pi(\mathcal{L}) \to \pi_{\mathrm{int}}(\mathcal{L})$, called **star map**, defined as

$$x \mapsto x^\star := \pi_{\mathrm{int}}\left((\pi|_{\mathcal{L}})^{-1}(x) \right)$$

where $(\pi|_{\mathcal{L}})^{-1}(x)$ is the unique point in the set $\mathcal{L} \cap \pi^{-1}(x)$.

The lattice $\mathcal{L}$ can be viewed as a **diagonal embedding** of $L := \pi(\mathcal{L})$:

$$\mathcal{L} = \{(x, x^\star) \mid x \in L\}.$$

Model set

$$
\begin{tikzcd}
W \subset H & \mathbb{R}^d \times H \arrow[l, "\pi_{\mathrm{int}}"'] \arrow[r, "\pi"] & \mathbb{R}^d \\
\cup \ \text{dense} & \cup & \cup \\
\pi_{\mathrm{int}}(\mathcal{L}) & \mathcal{L} \arrow[l] \arrow[r, "1-1"] & \pi(\mathcal{L}) \supset \curlywedge(W) \arrow[ll, bend left, "\star"]
\end{tikzcd}
$$

For a given **acceptance set** (or **window** or **coding set**) $W \subset H$ in the internal space,

$$\curlywedge(W) := \{x \in L \mid x^{\star} \in W\}$$

is the projection set within the CPS, where $L = \pi(\mathcal{L})$.

Definition

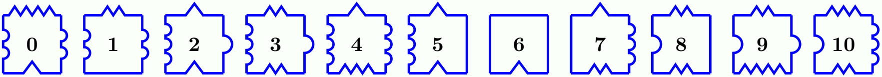
If $W \subset H$ is a relatively compact set with non-empty interior, any translate $t + \curlywedge(W)$ of the projection set, $t \in \mathbb{R}^d$, is called a **model set**.

Euclidean model sets with $H = \mathbb{R}^m$

$$
\begin{tikzcd}
W & \subset & \mathbb{R}^m & \mathbb{R}^d \times \mathbb{R}^m \arrow[l, "\pi_{\mathrm{int}}"'] \arrow[r, "\pi"] & \mathbb{R}^d & & \\
 & & \cup \text{ dense} & \cup & \cup & & \\
 & & \pi_{\mathrm{int}}(\mathcal{L}) & \mathcal{L} \arrow[l] \arrow[r, "1-1"] & \pi(\mathcal{L}) \arrow[ll, bend left, "\star"] & \supset & \curlywedge(W)
\end{tikzcd}
$$

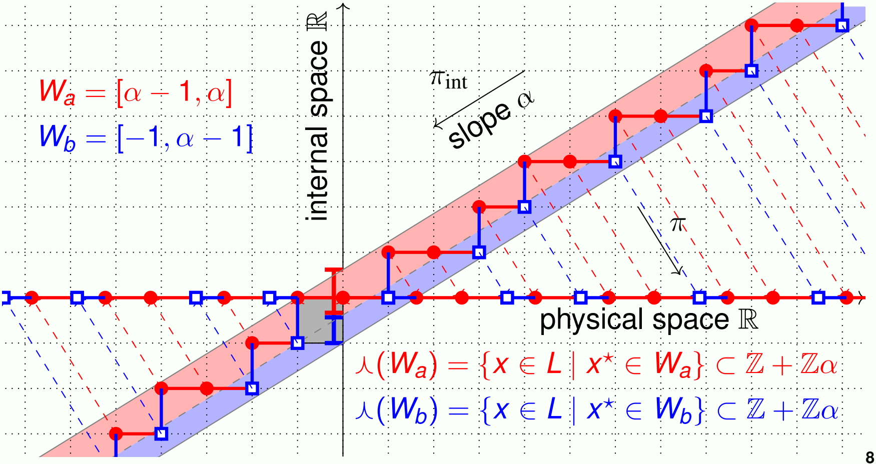
This talk :

d	m	Examples
1	1	**Geometric Sturmian** tilings of $\mathbb{R}$
1	1	**Sturmian** sequences on $\mathbb{Z}$
2	2	**Jeandel-Rao** aperiodic tilings



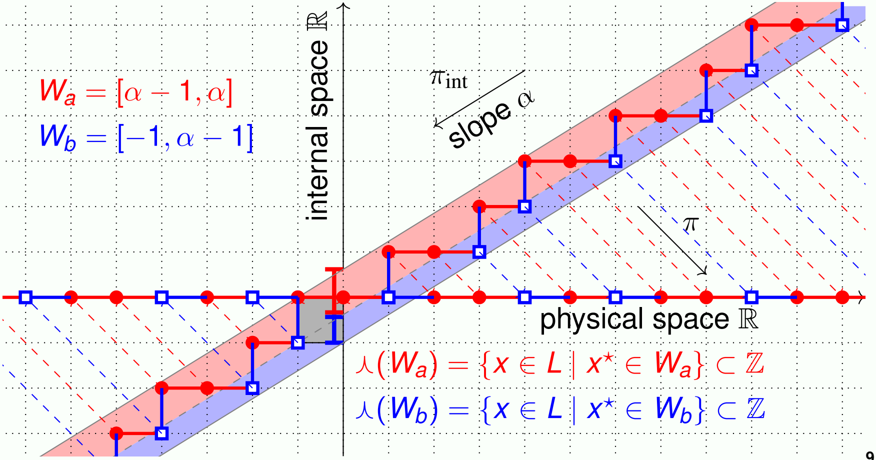
Geometric Sturmian with $\curlywedge(W) \subset \mathbb{Z} + \mathbb{Z}\alpha$

$$
\begin{tikzcd}
W \subset \mathbb{R} & \mathbb{R}\times\mathbb{R} \arrow[l, "\pi_{\mathrm{int}}"'] \arrow[r, "\pi"] & \mathbb{R} \\
-\alpha x_1 + x_2 & (x_1, x_2) \arrow[l, mapsto] \arrow[r, mapsto] & x_1 + \alpha x_2 \\
\pi_{\mathrm{int}}(\mathbb{Z}^2) \arrow[u, phantom, "\cup\ \text{dense}"] & \mathbb{Z}^2 \arrow[u, phantom, "\cup"] \arrow[l] \arrow[r, "1-1"] & \pi(\mathbb{Z}^2) \supset \curlywedge(W) \arrow[u, phantom, "\cup"] \arrow[ll, bend left, "\star"]
\end{tikzcd}
$$



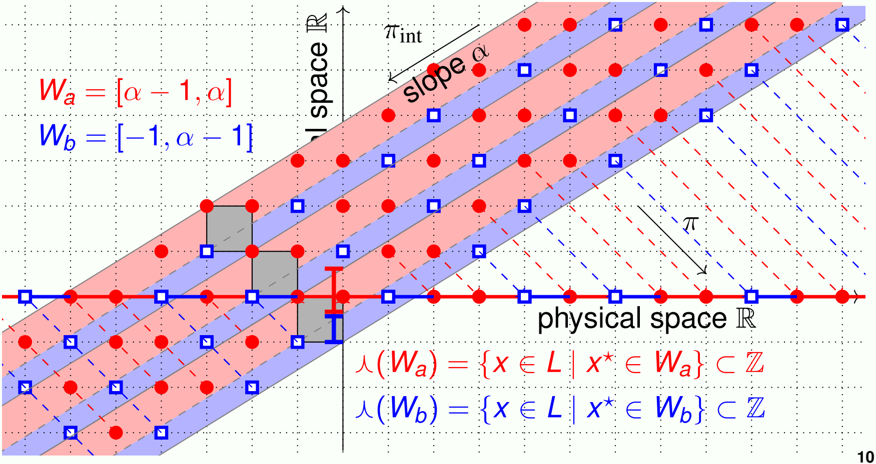
Sturmian with $\curlywedge(W) \subset \mathbb{Z}$

$$\begin{tikzcd}
W \subset \mathbb{R} & \mathbb{R}\times\mathbb{R} \arrow[l, "\pi_{\mathrm{int}}"'] \arrow[r, "\pi"] & \mathbb{R} \\
-\alpha x_1 + x_2 & (x_1, x_2) \arrow[l, mapsto] \arrow[r, mapsto] & {\color{red} x_1 + x_2} \\
\cup\ \text{dense} & \cup & \cup \\
\pi_{\mathrm{int}}(\mathbb{Z}^2) & \mathbb{Z}^2 \arrow[l] \arrow[r, "\text{not } 1-1"] & \pi(\mathbb{Z}^2) \supset \curlywedge(W) \arrow[ll, bend left, "\star"]
\end{tikzcd}$$



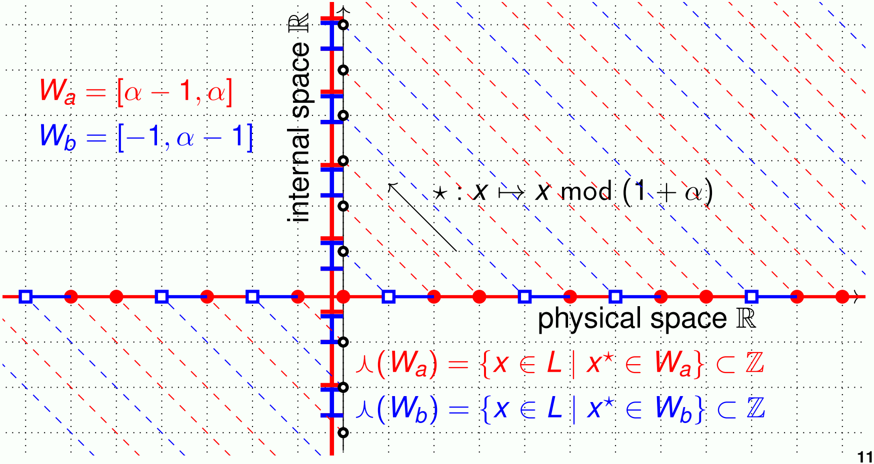
Sturmian with $\curlywedge(W) \subset \mathbb{Z}$ and $H = \mathbb{R}/(1+\alpha)\mathbb{Z}$

$$\begin{tikzcd}
W \subset \mathbb{R}/(1+\alpha)\mathbb{Z} & \mathbb{R}\times\mathbb{R}/\langle(1,-1)\rangle_{\mathbb{Z}} \arrow[l, "\pi_{\mathrm{int}}"'] \arrow[r, "\pi"] & \mathbb{R} \\
x_1 + x_2 & (x_1, x_2) \arrow[l, mapsto] \arrow[r, mapsto] & x_1 + x_2 \\[-10pt]
\cup \text{ dense} & \cup & \cup \\[-10pt]
\pi_{\mathrm{int}}(\mathbb{Z}^2) & \mathbb{Z}^2 \arrow[l] \arrow[r, "1-1"] & \pi(\mathbb{Z}^2) \supset \curlywedge(W) \arrow[ll, bend left, "\star"]
\end{tikzcd}$$



Sturmian with $\curlywedge(W) \subset \mathbb{Z}$: a trivial $\star$-map

$$
\begin{tikzcd}
W \subset \mathbb{R}/(1+\alpha)\mathbb{Z} & \mathbb{R} \times \mathbb{R}/\langle(1,-1)\rangle_{\mathbb{Z}} \arrow[l, "\pi_{\mathrm{int}}"'] \arrow[r, "\pi"] & \mathbb{R} \\
x_1 + x_2 & (x_1, x_2) \arrow[l, mapsto] \arrow[r, mapsto] & x_1 + x_2 \\
\pi_{\mathrm{int}}(\mathbb{Z}^2) \arrow[u, phantom, "\cup \text{ dense}"] & \mathbb{Z}^2 \arrow[u, phantom, "\cup"] \arrow[l] \arrow[r, "1-1"] & \pi(\mathbb{Z}^2) \supset \curlywedge(W) \arrow[u, phantom, "\cup"] \arrow[ll, bend left, "\star"]
\end{tikzcd}
$$

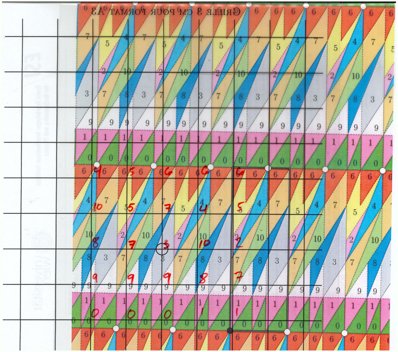


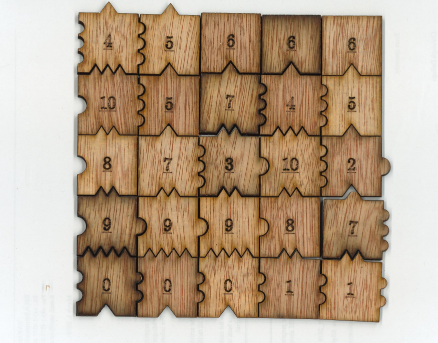
Jeandel-Rao tilings : $\curlywedge(W) \subset \mathbb{Z}^2$ and a trivial $\star$-map

With $\Lambda = \langle (1,-1,0,0), (0,0,1,-1) \rangle_{\mathbb{Z}}$ and $\Gamma = \langle (\varphi, 0), (1, \varphi+3) \rangle_{\mathbb{Z}}$

$$
\begin{tikzcd}
W \subset \mathbb{R}^2/\Gamma & \mathbb{R}^4/\Lambda \arrow[l, "\pi_{\text{int}}"'] \arrow[r, "\pi"] & \mathbb{R}^2 \\
(x_1+x_2, x_3+x_4) & (x_1, x_2, x_3, x_4) \arrow[l, mapsto] \arrow[r, mapsto] & (x_1+x_2, x_3+x_4) \\
\cup \text{ dense} & \cup & \cup \\
\pi_{\text{int}}(\mathbb{Z}^4) & \mathbb{Z}^4 \arrow[l] \arrow[r, "1-1"] & \pi(\mathbb{Z}^4) \supset \curlywedge(W) \arrow[ll, bend left, "\star"]
\end{tikzcd}
$$

provides the positions $\curlywedge(W_i) \subset \mathbb{Z}^2$ of the tile $i \in \{0, 1, \ldots, 10\}$:





to appear in Annales Henri Lebesgue

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During the review process, the title changed to :

Markov partitions for toral $\mathbb{Z}^2$-rotations featuring Jeandel-Rao Wang shift and model sets

Other Euclidean model sets with $H = \mathbb{R}^m$

$$\begin{tikzcd}
W \subset \mathbb{R}^m & \mathbb{R}^d \times \mathbb{R}^m \arrow[l, "\pi_{\text{int}}"'] \arrow[r, "\pi"] & \mathbb{R}^d \\
\pi_{\text{int}}(\mathcal{L}) \arrow[u, phantom, "\cup \text{ dense}"] & \mathcal{L} \arrow[u, phantom, "\cup"] \arrow[l] \arrow[r, "1-1"] & \pi(\mathcal{L}) \arrow[u, phantom, "\cup"] \arrow[ll, bend left, "\star"] \supset \curlywedge(W)
\end{tikzcd}$$

d	m	Examples
1	1	**Geometric Sturmian** tilings of $\mathbb{R}$ ✓
1	1	**Sturmian** sequences on $\mathbb{Z}$ ✓
2	2	**Jeandel-Rao** aperiodic tilings ✓
1	2	**Billiard** sequences in a cube (polygonal window)
1	m	Billiard sequences in a hypercube
1	2	**Tribonacci** sequence (window is the Rauzy fractal)
2	1	**Discrete planes**
d	1	Discrete hyperplanes
2	3	**Penrose** aperiodic tilings