From Sturmian to Wang shifts

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Outline

1. Sturmian shifts
2. Wang shifts

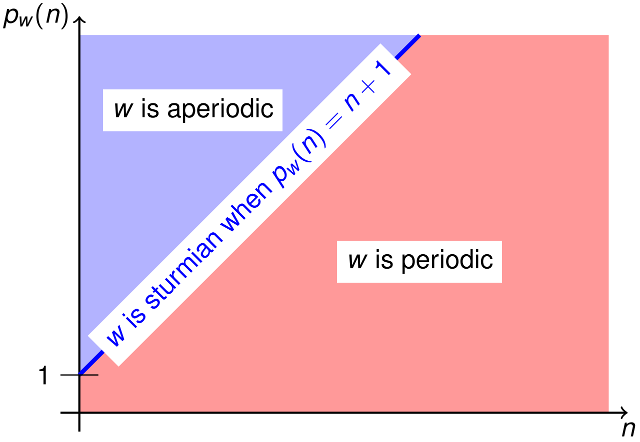
Outline

Factor complexity and sturmian sequences

Let $w \in \mathcal{A}^{\mathbb{Z}}$. The **factor complexity** is a function $p_w(n) : \mathbb{N} \to \mathbb{N}$ counting the number of factors of length n in the sequence w.

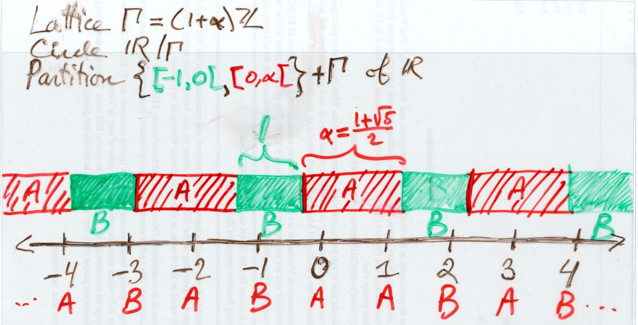
$$w = ...000100\boxed{\mathbf{0100}}01001000100010001001000010001001...$$

$$Fact_w(4) = \{0001, 0010, \mathbf{0100}, 1000, 1001\} \implies p_w(4) = 5$$



An easy way to construct Sturmian sequences

An easy way to construct Sturmian sequences



Theorem (Morse, Hedlund, 1940 ; Coven, Hedlund 1970)

$w \in \{a,b\}^{\mathbb{Z}}$ is the coding of an **irrational rotation** R_α with $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ if and only if w is a **sturmian sequence**.

- $\Longrightarrow$: the easy direction : counting number of intervals
- $\Longleftarrow$: **more difficult** : a **common substitutive structure**
  - Sturmian sequences in S_α can be desubstituted indefinitely :

$$S_\alpha \xleftarrow{\omega_0} S_{f(\alpha)} \xleftarrow{\omega_1} S_{f^2(\alpha)} \xleftarrow{\omega_2} S_{f^3(\alpha)} \dots$$

  - Coding $C_\alpha \subset \{a,b\}^{\mathbb{Z}}$ of R_α can be desubstituted (Rauzy induction) :

$$C_\alpha \xleftarrow{\omega_0} C_{f(\alpha)} \xleftarrow{\omega_1} C_{f^2(\alpha)} \xleftarrow{\omega_2} C_{f^3(\alpha)} \dots$$

    with $\omega_i \in \{\tau_a, \tau_b\}$ for every i where $\tau_a = (b \mapsto ab, a \mapsto a)$ and $\tau_b = (a \mapsto ba, b \mapsto b)$ where f is the Farey map.
  - Continued fraction expansion of α
  - Ostrowski expansion of the starting point $x_0 \in \mathbb{T}$ with respect to α.

Details in : Pytheas Fogg, 2002, chapter 6 written by Pierre Arnoux.

Outline

1. Sturmian shifts

2. Wang shifts

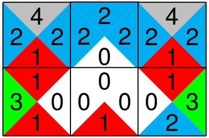
Wang tiles and Wang shifts

A **Wang tile** is a square tile with a color on each border

If T is **finite set** of such Wang tiles, a **tiling of the plane** is a map

$$\mathbb{Z} \times \mathbb{Z} \to T$$

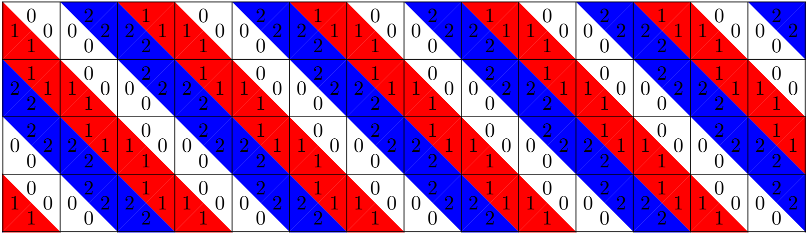
which is **valid** if contiguous edges of adjacent tiles have the **same** color.



A **Wang shift** is the set $\Omega_T = \{w : \mathbb{Z} \times \mathbb{Z} \to T \mid w \text{ is valid}\}$.
Note : rotations are not allowed.

Aperiodic set of Wang tiles

A tiling $\mathbb{Z} \times \mathbb{Z} \to T$ is called **periodic** if it is invariant under some non-zero translation of the plane.



A set T of Wang tiles is **aperiodic** if

- it **admits** a valid tiling $\mathbb{Z} \times \mathbb{Z} \to T$ and
- every valid tiling $\mathbb{Z} \times \mathbb{Z} \to T$ is **nonperiodic**.

Berger (1966)

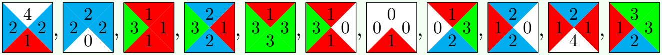
There **exists an aperiodic** set of Wang tiles.

Aperiodic Wang tile sets

- 1966 (Berger) : 20426 tiles (lowered down later to 104)
- 1968 (Knuth) : 92 tiles
- 1971 (Robinson) : 56 tiles
- 1971 (Ammann) : 16 tiles
- 1987 (Grunbaum) : 24 tiles
- 1996 (Kari) : 14 tiles
- 1996 (Culik) : (same method) 13 tiles

Theorem (Jeandel, Rao, 2015)

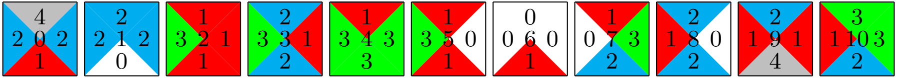
- All sets of $\leq$ 10 tiles are **periodic** or do not tile the plane.
- The following set of 11 Wang tiles is **aperiodic** :

$\mathcal{T}_0 = \{$ 

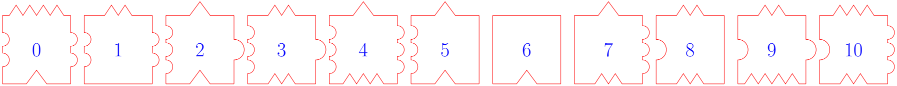
$\}$.

Laser cut version of Jeandel-Rao's 11 tiles

We represent the 11 Jeandel-Rao's tiles $\mathcal{T}_0$



as follows :



and we denote the **Jeandel-Rao Wang shift** as

$$\Omega_0 = \{\text{valid tiling } \mathbb{Z}^2 \to \mathcal{T}_0\}.$$

An easy way to construct Jeandel-Rao tilings

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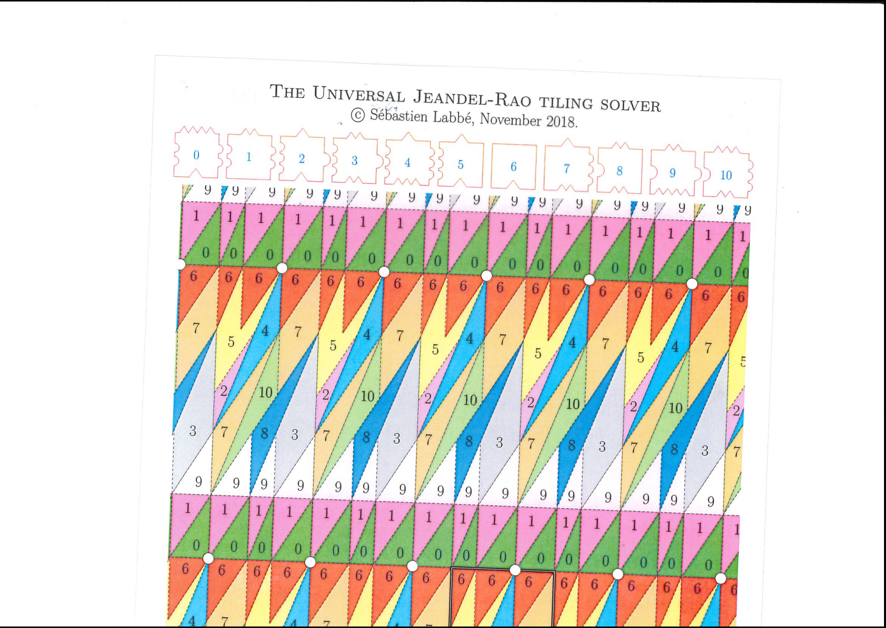
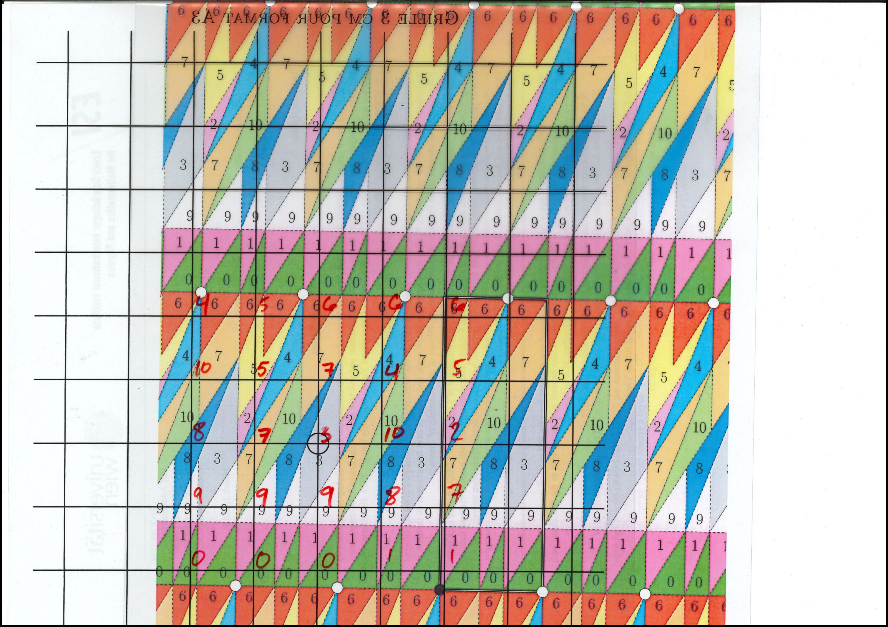


An easy way to construct Jeandel-Rao tilings



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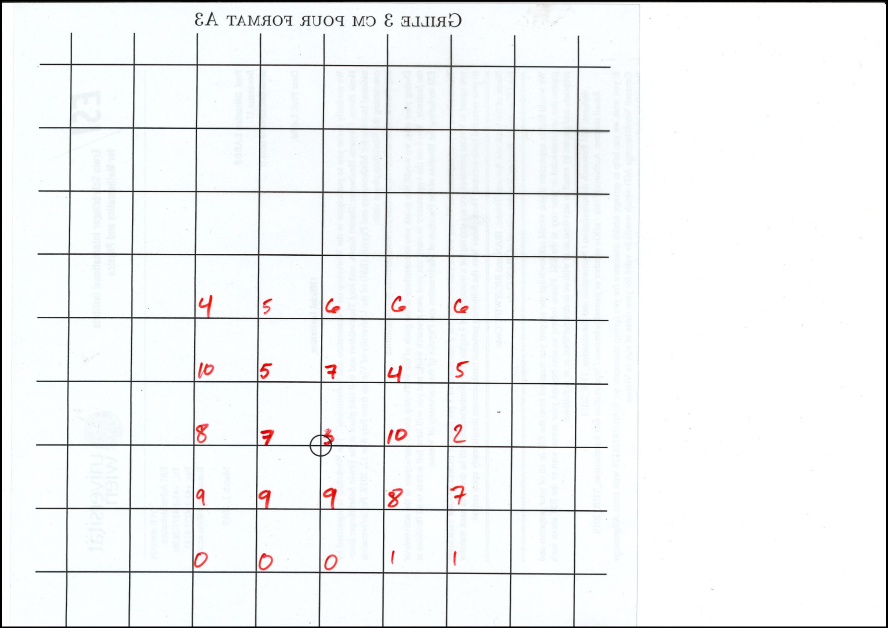
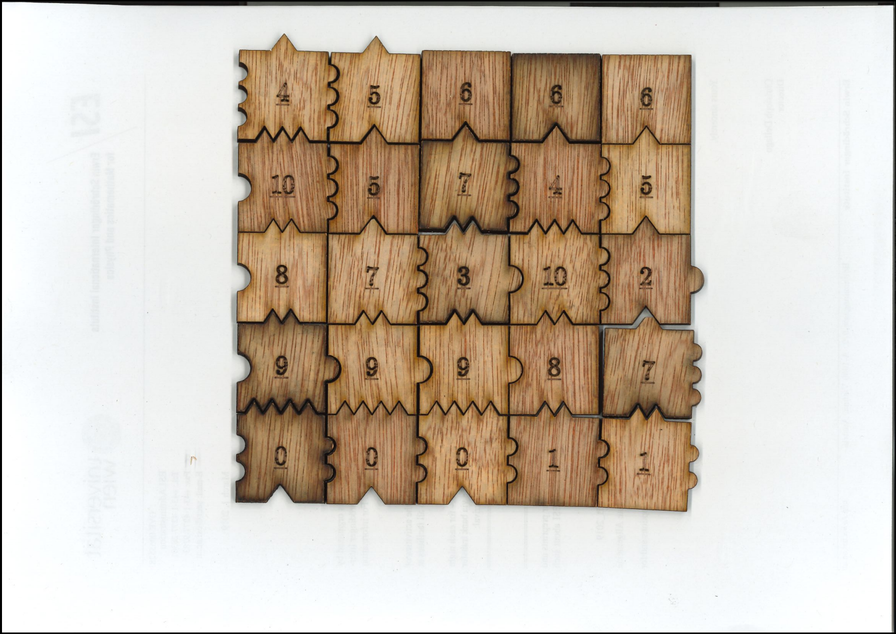


An easy way to construct Jeandel-Rao tilings



Definitions (§6.5 Lind-Marcus for $\mathbb{Z}^2$-actions)

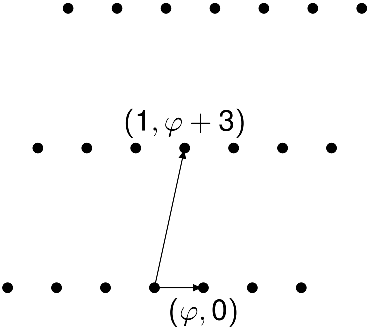
- A **topological partition** of a (compact) metric space M is a finite collection $\mathcal{P} = \{P_a\}_{a\in\mathcal{A}}$ of disjoint open sets whose closures $\overline{P_a}$ together cover M in the sense that $M = \cup_{a\in\mathcal{A}}\overline{P_a}$.
- Let $(M, \mathbb{Z}^2, R)$ be a **dynamical system** with $\mathbb{Z}^2$-action R on M.
- If $S \subset \mathbb{Z}^2$, a **pattern** $w : S \to \mathcal{A}$ is **allowed** for $\mathcal{P}, R$ if

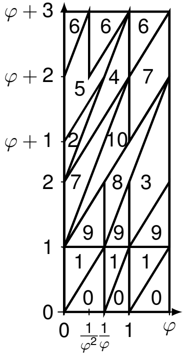
$$\bigcap_{\mathbf{k}\in S} R^{-\mathbf{k}}(P_{w_{\mathbf{k}}}) \neq \varnothing.$$

- Let $\mathcal{L}_{\mathcal{P},R}$ be the collection of all allowed patterns for $\mathcal{P}, R$.
- $\mathcal{X}_{\mathcal{P},R}$ is the **symbolic dyn. system corresponding to $\mathcal{P}, R$**.
  It is the unique subshift $\mathcal{X}_{\mathcal{P},R} \subset \mathcal{A}^{\mathbb{Z}^2}$ whose language is $\mathcal{L}_{\mathcal{P},R}$.
- $\mathcal{P}$ gives a **symbolic representation** of $(M, \mathbb{Z}^2, R)$ if for every $w \in \mathcal{X}_{\mathcal{P},R}$ the intersection $\cap_{\mathbf{k}\in\mathbb{Z}^2} R^{-\mathbf{k}}\overline{P_{w_{\mathbf{k}}}}$ consists of exactly one point $m \in M$.
- $\mathcal{P}$ is a **Markov partition** for $(M, \mathbb{Z}^2, R)$ if $\mathcal{P}$ gives a symbolic representation of $(M, \mathbb{Z}^2, R)$ and $\mathcal{X}_{\mathcal{P},R}$ is a shift of finite type.

The symbolic dynamical system $\mathcal{X}_{\mathcal{P}_0,R_0}$

Let $\varphi = \frac{1+\sqrt{5}}{2}$, the **lattice** $\Gamma_0 = \langle(\varphi,0),(1,\varphi+3)\rangle_{\mathbb{Z}}$ and the following topological **partition** $\mathcal{P}_0$ of $\mathbb{R}^2/\Gamma_0$.





We consider the **action** of $\mathbb{Z}^2$ on the **torus** $\mathbb{R}^2/\Gamma_0$:

$$R_0 : \begin{array}{ccc} \mathbb{Z}^2 \times \mathbb{R}^2/\Gamma_0 & \to & \mathbb{R}^2/\Gamma_0 \\ (\mathbf{n},\mathbf{x}) & \mapsto & R_0^{\mathbf{n}}(\mathbf{x}) := \mathbf{x}+\mathbf{n} \end{array}$$

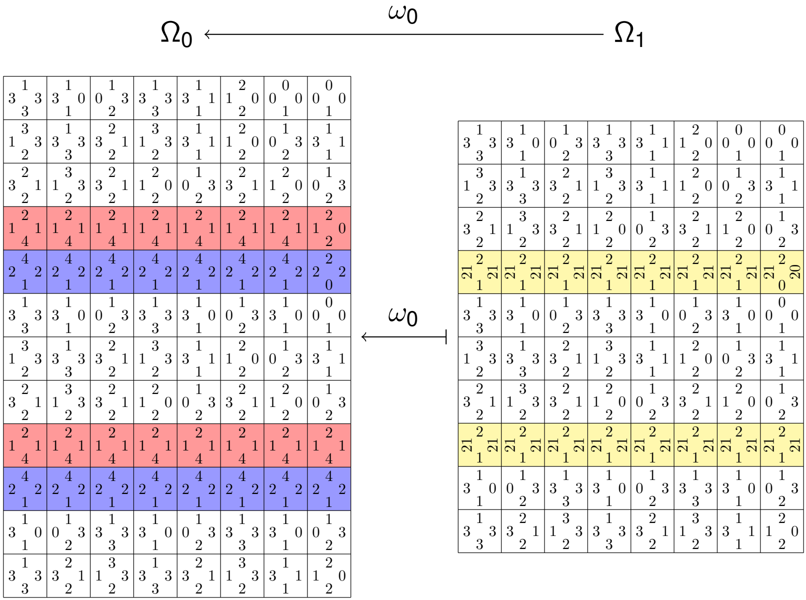
for every $\mathbf{n} \in \mathbb{Z}^2$.

Results on Jeandel-Rao tilings $\mathcal{X}_{\mathcal{P}_0,R_0} \subsetneq \Omega_0$

Theorem

- $\mathcal{X}_{\mathcal{P}_0,R_0}$ is a **proper minimal, aperiodic and uniquely ergodic** subshift of the Jeandel-Rao Wang shift, i.e., $\mathcal{X}_{\mathcal{P}_0,R_0} \subsetneq \Omega_0$.
- $\mathcal{P}_0$ gives a **symbolic representation** of $(\mathbb{R}^2/\Gamma_0, \mathbb{Z}^2, R_0)$
- there exists an **almost 1-1 factor** map $f : \mathcal{X}_{\mathcal{P}_0,R_0} \to \mathbb{R}^2/\Gamma_0$
- $(\mathbb{R}^2/\Gamma_0, \mathbb{Z}^2, R_0)$ is the **maximal equicontinuous factor** of $(\mathcal{X}_{\mathcal{P}_0,R_0}, \mathbb{Z}^2, \sigma)$.
- The **measure-preserving** dynamical system $(\mathcal{X}_{\mathcal{P}_0,R_0}, \mathbb{Z}^2, \sigma, \nu)$ is **isomorphic** to $(\mathbb{R}^2/\Gamma_0, \mathbb{Z}^2, R_0, \lambda)$ where
  - ν is the unique shift-invariant probability measure on $\mathcal{X}_{\mathcal{P}_0,R_0}$
  - λ is the Haar measure on $\mathbb{R}^2/\Gamma_0$.
- Occurences of patterns in $\mathcal{X}_{\mathcal{P}_0,R_0}$ is a **4-to-2 C&P set**, more precisely a **regular** (generic or singular) **model set**.

Markers and recognizable 2-dim. morphisms



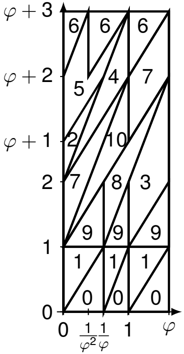
2-dim. Rauzy induction of $\mathbb{Z}^2$-PETs

$$\mathcal{X}_{\mathcal{P}_0,R_0} \xleftarrow{\quad\beta_0\quad} \mathcal{X}_{\mathcal{P}_1,R_1}$$

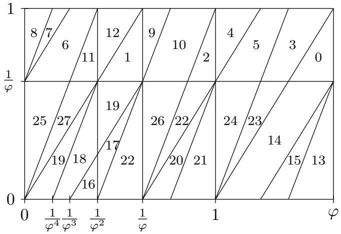
$\Gamma_0 = \langle(\varphi,0),(1,\varphi+3)\rangle_{\mathbb{Z}}$

$\Gamma_1 = \varphi\mathbb{Z}\times\mathbb{Z}$

$\mathcal{P}_0 =$



$\mathcal{P}_1 =$



$R_0^{(n_1,n_2)}(\mathbf{x}) := \mathbf{x} + n_1\mathbf{e}_1 + n_2\mathbf{e}_2 \mod \Gamma_0$

$R_1^{(n_1,n_2)}(\mathbf{x}) = \mathbf{x} + n_1\mathbf{e}_1 + n_2(\varphi^{-1},\varphi^{-2}) \mod \Gamma_1$

Proof idea : a common substitutive structure

The symbolic dynamical system $\mathcal{X}_{\mathcal{P}_0,R_0}$ and the subshift $X_0 \subsetneq \Omega_0$ of the Jeandel-Rao Wang shift have a **common** substitutive structure :

$$
\begin{array}{ccccccccccccccccccccc}
\Omega_0 & \xleftarrow{\omega_0} & \Omega_1 & \xleftarrow{\omega_1} & \Omega_2 & \xleftarrow{\omega_2} & \Omega_3 & \xleftarrow{\omega_3} & \Omega_4 & & & & & & & & & & & & \circlearrowleft \omega_{\mathcal{U}} \\
\cup & & \cup & & \cup & & \cup & & \cup & & & & & & & & & & & & \\
X_0 & \xleftarrow{\omega_0} & X_1 & \xleftarrow{\omega_1} & X_2 & \xleftarrow{\omega_2} & X_3 & \xleftarrow{\omega_3} & X_4 & \xleftarrow{\pi} & \Omega_5 & \xleftarrow{\eta} & \Omega_6 & \xleftarrow{\omega_6} & \Omega_7 & \xleftarrow{\omega_7\omega_8\omega_9\omega_{10}\omega_{11}} & \Omega_{12} & \xleftarrow{\rho} & \Omega_{\mathcal{U}} \\
\| & & & & & & & & \| & & & & & & \| & & \| & & \| \\
\mathcal{X}_{\mathcal{P}_0,R_0} & & & \xleftarrow{\beta_0} & & & & & \mathcal{X}_{\mathcal{P}_1,R_1} & \xleftarrow{\beta_1} & \mathcal{X}_{\mathcal{P}_2,R_2} & \xleftarrow{\beta_2} & \mathcal{X}_{\mathcal{P}_3,R_3} & & & \xleftarrow{\beta_3\beta_4\beta_5\beta_6\beta_7} & \mathcal{X}_{\mathcal{P}_8,R_8} & \xleftarrow{\rho} & \mathcal{X}_{\mathcal{P}_{\mathcal{U}},R_{\mathcal{U}}} \\
& & & & & & & & & & & & & & & & & & \circlearrowright \omega_{\mathcal{U}}
\end{array}
$$

since

$$\omega_0\omega_1\omega_2\omega_3 = \beta_0$$

$$\pi\eta\omega_6 = \beta_1\beta_2$$

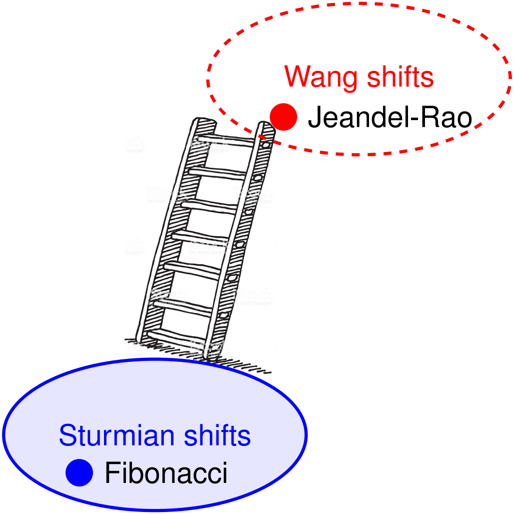
$$\omega_7 = \beta_3,\ \omega_8 = \beta_4,\ \omega_9 = \beta_5,\ \omega_{10} = \beta_6 \text{ and } \omega_{11} = \beta_7.$$

References

- A self-similar aperiodic set of 19 Wang tiles, *Geom Dedicata* (2018) doi:10.1007/s10711-018-0384-8
- Substitutive structure of Jeandel-Rao aperiodic tilings arXiv:1808.07768
- A Markov partition for Jeandel-Rao aperiodic Wang tilings arXiv:1903.06137
- Induction of $\mathbb{Z}^2$-actions and of partitions of the 2-torus arXiv:1906.01104

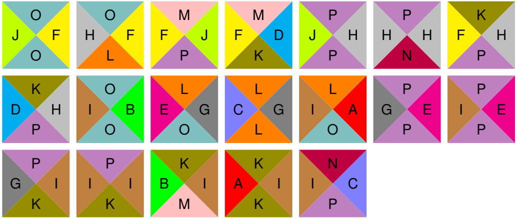
including Sage Jupyter notebooks reproducing the computations.

Next steps : continue the exploration



Outline

A self-similar aperiodic set $\mathcal{U}$ of 19 Wang tiles



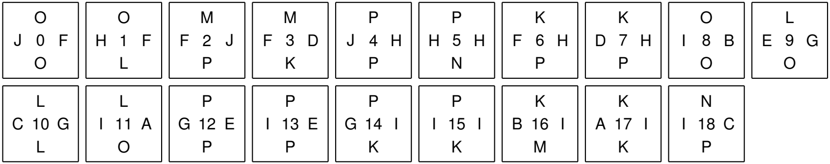
Theorem

The Wang shift $\Omega_{\mathcal{U}}$ is **self-similar**, **aperiodic** and **minimal**.

$$\Omega_{\mathcal{U}} \xleftarrow{\alpha\,:\,\square \mapsto \square,\ \square \mapsto \genfrac{}{}{0pt}{}{\square}{\square}} \Omega_{\mathcal{V}} \xleftarrow{\beta\,:\,\square \mapsto \square,\ \square \mapsto \square\square} \Omega_{\mathcal{W}} \xleftarrow{\gamma\,:\,\square \mapsto \square} \Omega_{\mathcal{U}}$$

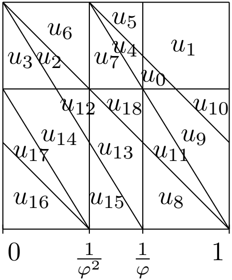
See : A self-similar aperiodic set of 19 Wang tiles, *Geom Dedicata* (2018) doi:10.1007/s10711-018-0384-8

The partition $\mathcal{P}_\mathcal{U}$ of $\mathbb{T}^2$



Let

- $\mathcal{U}$ be the above set of 19 tiles,
- $\mathcal{P}_\mathcal{U} = \{P_u\}_{u\in\mathcal{U}}$ be the partition of $\mathbb{T}^2$ shown on the figure,
- $R^\mathbf{n}(\mathbf{x}) = \mathbf{x} + \varphi^{-2}\mathbf{n}$ the $\mathbb{Z}^2$-action,



Proposition

$\mathcal{X}_{\mathcal{P}_\mathcal{U},R_\mathcal{U}}$ is a subshift of the Wang shift $\Omega_\mathcal{U}$

Results on $\mathcal{X}_{\mathcal{P}_\mathcal{U},R_\mathcal{U}} = \Omega_\mathcal{U}$

Theorem

- $\mathcal{P}_\mathcal{U}$ gives a **symbolic representation** of $(\mathbb{T}^2, \mathbb{Z}^2, R_\mathcal{U})$
- $\mathcal{X}_{\mathcal{P}_\mathcal{U},R_\mathcal{U}}$ is equal to the Wang shift $\Omega_\mathcal{U}$
- $\mathcal{P}_\mathcal{U}$ is a **Markov partition** for $(\mathbb{T}^2, \mathbb{Z}^2, R_\mathcal{U})$
- there exists an **almost 1-1 factor** map $f : \Omega_\mathcal{U} \to \mathbb{T}^2$
- $(\mathbb{T}^2, \mathbb{Z}^2, R_\mathcal{U})$ is the **maximal equicontinuous factor** of $(\Omega_\mathcal{U}, \mathbb{Z}^2, \sigma)$.
- The **measure-preserving** dynamical system $(\Omega_\mathcal{U}, \mathbb{Z}^2, \sigma, \nu)$ is **isomorphic** to $(\mathbb{T}^2, \mathbb{Z}^2, R, \lambda)$ where
  - ν is the unique shift-invariant probability measure on $\Omega_\mathcal{U}$
  - λ is the Haar measure on $\mathbb{T}^2$.
- Occurences of patterns in $\Omega_\mathcal{U}$ is a **4-to-2 C&P set**.

See : *A Markov partition for Jeandel-Rao aperiodic Wang tilings*, arXiv:1903.06137, Pi Day, 2019.