Pavages apériodiques et codage de $\mathbb{Z}^2$-actions sur le tore

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Outline

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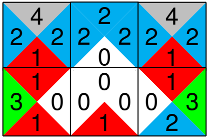
Wang tiles

A **Wang tile** is a square tile with a color on each border

Tile set T : a finite collection of such tiles.
A tiling of the plane : an assignment

$$\mathbb{Z} \times \mathbb{Z} \rightarrow T$$

of tiles on infinite square lattice so that the contiguous edges of adjacent tiles have the same color.



Note : rotation not allowed.

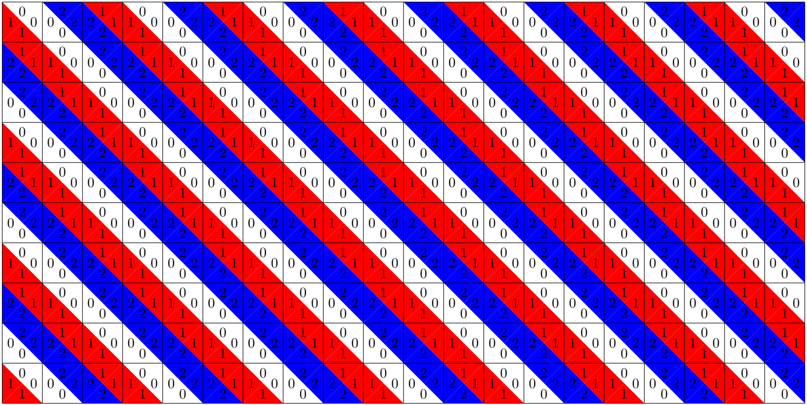
Eternity II puzzle (2007)

- A puzzle which involves **placing 256 square puzzle** pieces into a 16 by 16 grid while **matching adjacent edges**
- A **2 million prize** (for the first solution found before 2011)
- **No solution ever found**
- Only $256! \times 4^{256} \approx 1.15 \times 10^{661}$ possibilities to check.
- Existence of a tiling $\mathbb{Z} \times \mathbb{Z} \to T$ is **undecidable** in general...

Source : `https://en.wikipedia.org/wiki/Eternity_II_puzzle`

Periods

A tiling is called **periodic** if it is invariant under some non-zero translation of the plane.



A Wang tile set that admits a periodic tiling also admits a **doubly periodic** tiling : a tiling with a horizontal and a vertical period.

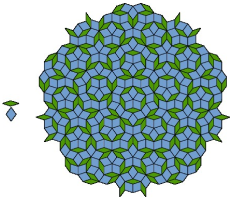
Aperiodicity

Conjecture (Wang 1961)

A Wang tile set **which tiles** the plane **admits a periodic tiling**.

A tile set is **aperiodic** if it tiles the plane, but no tiling is periodic

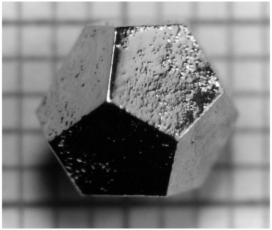
- 1966 (Berger) : There **exists an aperiodic** set of Wang tiles
- 1976 (Penrose) : discovered an aperiodic set of **two tiles**

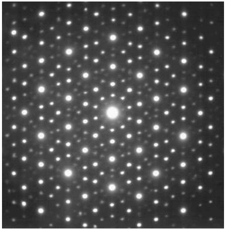


Quasicrystals

- 1982 (Shechtman) : observed that aluminium-manganese alloys produced a **quasicrystals structure**
- 2011 : Dan Shechtman receives **Nobel Prize** in Chemistry

A Ho-Mg-Zn **icosahedral quasicrystal** formed as a pentagonal dodecahedron and its **electron diffraction** pattern :





"*His discovery of quasicrystals revealed a **new principle for packing of atoms and molecules***", stated the Nobel Committee that "*led to a **paradigm shift** within chemistry*".

Source : `https://en.wikipedia.org/wiki/Quasicrystal`

Outline

Sur quelle rue de Bordeaux se trouve cette porte ?

Discoveries of aperiodic Wang tile sets (< 2000)

Image credit : `http://chippewa.canalblog.com/archives/2010/06/04/18115718.html`

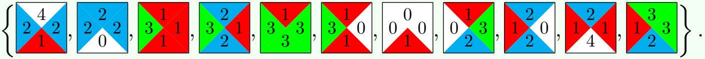
- 1966 (Berger) : 20426 tiles (lowered down later to 104)
- 1968 (Knuth) : 92 tiles
- 1971 (Robinson) : 56 tiles
- 1971 (Ammann) : 16 tiles
- 1987 (Grunbaum) : 24 tiles
- 1996 (Kari) : 14 tiles
- 1996 (Culik) : (same method) 13 tiles

Discoveries of aperiodic Wang tile sets (2015)

Theorem (Jeandel, Rao, 2015)

- All sets of ≤ 10 tiles are **periodic** or do not tile the plane.
- The following 11 Wang tile set is **aperiodic** :

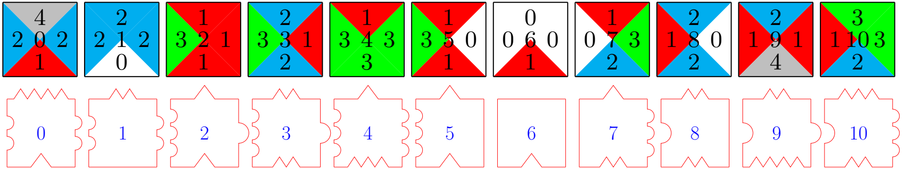
$\mathcal{T} =$



Their algorithm is pictured below :

Image credit : Le Bagger 288, `http://i.imgur.com/YH9xX.jpg`

JR’s 11 tiles ... as closed topological disks using base-1 rep. of $\mathbb{N}$:



152244 valid 7×7 solutions but only 483 ($+\varepsilon$) extend to $\mathbb{Z} \times \mathbb{Z}$.

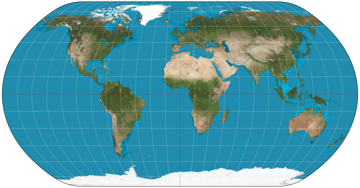
(Solution found by Pauline Hubert and Antoine Abram, Sep 2018)

Aram Dermenjian's solution (Montréal, Sep 2018)

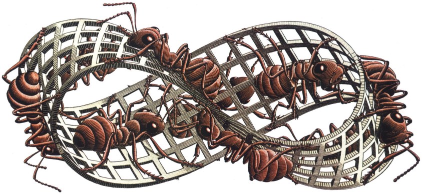
Jeandel and Rao proved aperiodicity by showing that tilings are given by horizontal strips of height 4 or 5 and the possible sequences $x \in \{4,5\}^{\mathbb{Z}}$ are exactly those whose language is given by the Fibonacci morphism $5 \mapsto 54, 4 \mapsto 5$.

Outline

Question



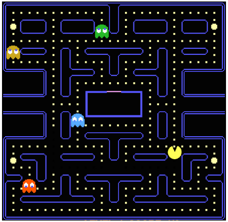
Source : Natural Earth Projection (wikipedia)



Source : `https://i.imgur.com/R2eAvWi.jpg`

Question

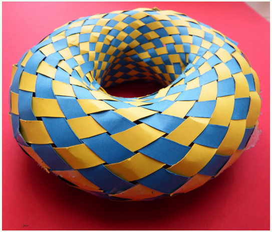
On **which surface** are we walking ?



Answer

The result, in simple terms

Walking on **Jeandel-Rao** tilings is like walking on a **2-torus** :



Consequences :

- list the **patterns**, prédire their **occurrences**, **frequencies** and **repetivity** ;
- description as **cut and project** schemes ;
- **diffraction** properties as a quasicrystals,

Definitions (§6.5 Lind-Marcus for $\mathbb{Z}^2$-actions)

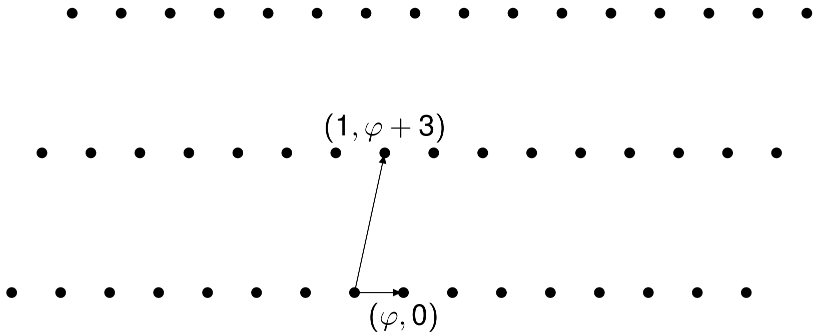
- A **topological partition** of a (compact) metric space M is a finite collection $\mathcal{P} = \{P_a\}_{a\in\mathcal{A}}$ of disjoint open sets whose closures $\overline{P_a}$ together cover M in the sense that $M = \cup_{a\in\mathcal{A}}\overline{P_a}$.
- Let $(M, \mathbb{Z}^2, R)$ be a **dynamical system** with $\mathbb{Z}^2$-action R on M.
- If $S \subset \mathbb{Z}^2$, a **pattern** $w : S \to \mathcal{A}$ is **allowed** for $\mathcal{P}, R$ if

$$\bigcap_{\mathbf{k}\in S} R^{-\mathbf{k}}(P_{w_{\mathbf{k}}}) \neq \varnothing.$$

- Let $\mathcal{L}_{\mathcal{P},R}$ be the collection of all allowed patterns for $\mathcal{P}, R$.
- $\mathcal{X}_{\mathcal{P},R}$ is the **symbolic dyn. system corresponding to $\mathcal{P}, R$**.
  It is the unique subshift $\mathcal{X}_{\mathcal{P},R} \subset \mathcal{A}^{\mathbb{Z}^2}$ whose language is $\mathcal{L}_{\mathcal{P},R}$.
- $\mathcal{P}$ gives a **symbolic representation** of $(M, \mathbb{Z}^2, R)$ if for every $w \in \mathcal{X}_{\mathcal{P},R}$ the intersection $\cap_{\mathbf{k}\in\mathbb{Z}^2} R^{-\mathbf{k}}\overline{P_{w_{\mathbf{k}}}}$ consists of exactly one point $m \in M$.
- $\mathcal{P}$ is a **Markov partition** for $(M, \mathbb{Z}^2, R)$ if $\mathcal{P}$ gives a symbolic representation of $(M, \mathbb{Z}^2, R)$ and $\mathcal{X}_{\mathcal{P},R}$ is a shift of finite type.

The torus $\mathbb{R}^2/\Gamma$ and the $\mathbb{Z}^2$-action R

Let $\varphi = \frac{1+\sqrt{5}}{2}$. Consider the **lattice** $\Gamma = \langle(\varphi, 0), (1, \varphi + 3)\rangle_{\mathbb{Z}}$.



We consider the **action** of $\mathbb{Z}^2$ on the **torus** $\mathbb{R}^2/\Gamma$:

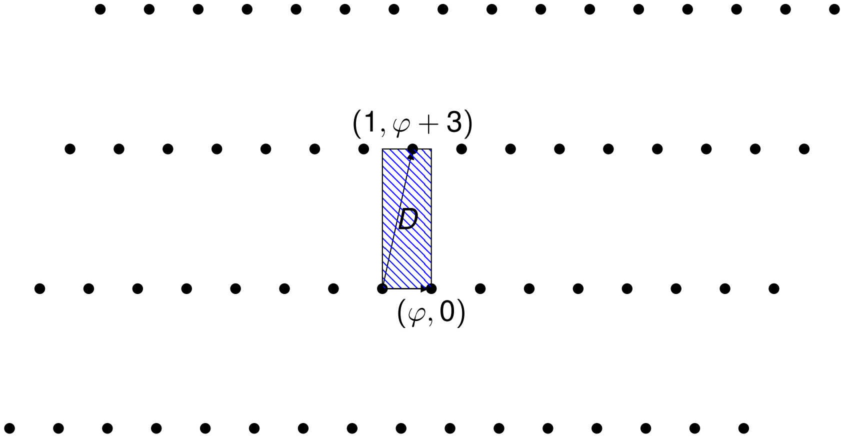
$$\begin{array}{rccl} R: & \mathbb{Z}^2 \times \mathbb{R}^2/\Gamma & \to & \mathbb{R}^2/\Gamma \\ & (\mathbf{n}, \mathbf{x}) & \mapsto & R^{\mathbf{n}}(\mathbf{x}) := \mathbf{x} + \mathbf{n} \end{array}$$

for every $\mathbf{n} \in \mathbb{Z}^2$.

A fundamental domain of $\mathbb{R}^2/\Gamma$

A **fundamental domain** of $\mathbb{R}^2/\Gamma$ is

$$D = [0, \varphi[\times [0, \varphi + 3[.$$



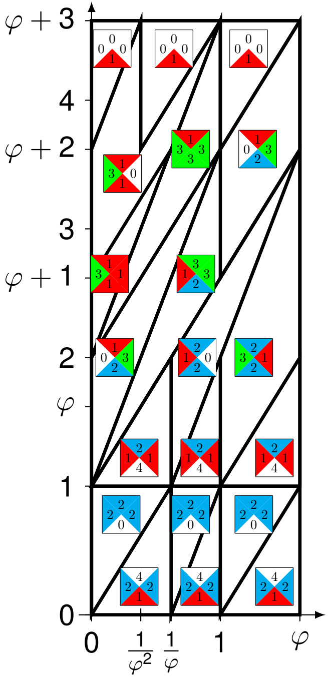
The partition $\mathcal{P}$ of $\mathbb{R}^2/\Gamma$

Let

- $\mathcal{T}$ be the Jeandel-Rao tile set,
- $\mathcal{P} = \{P_t\}_{t \in \mathcal{T}}$ be the partition of $\mathbb{R}^2/\Gamma$ shown on the figure,
- $R^{\mathbf{n}}(\mathbf{x}) = \mathbf{x} + \mathbf{n}$ the $\mathbb{Z}^2$-action,

Proposition

$\mathcal{X}_{\mathcal{P},R}$ is a subshift of the Wang shift $\Omega_{\mathcal{T}}$



Results on Jeandel-Rao tilings $\mathcal{X}_{\mathcal{P},R} \subsetneq \Omega_0$

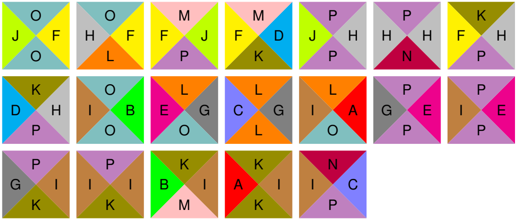
Theorem

- $\mathcal{P}$ gives a **symbolic representation** of $(\mathbb{R}^2/\Gamma, \mathbb{Z}^2, R)$
- there exists an **almost 1-1 factor** map $f : \mathcal{X}_{\mathcal{P},R} \to \mathbb{R}^2/\Gamma$
- $(\mathbb{R}^2/\Gamma, \mathbb{Z}^2, R)$ is the **maximal equicontinuous factor** of $(\mathcal{X}_{\mathcal{P},R}, \mathbb{Z}^2, \sigma)$.
- $\mathcal{X}_{\mathcal{P},R}$ is a **proper minimal, aperiodic and uniquely ergodic** subshift of the Jeandel-Rao Wang shift, i.e., $\mathcal{X}_{\mathcal{P},R} \subsetneq \Omega_0$.
- The **measure-preserving** dynamical system $(\mathcal{X}_{\mathcal{P},R}, \mathbb{Z}^2, \sigma, \nu)$ is **isomorphic** to $(\mathbb{R}^2/\Gamma, \mathbb{Z}^2, R, \lambda)$ where
  - ν is the unique shift-invariant probability measure on $\mathcal{X}_{\mathcal{P},R}$
  - λ is the Haar measure on $\mathbb{R}^2/\Gamma$.
- Occurences of patterns in $\mathcal{X}_{\mathcal{P},R}$ is a **4-to-2 C&P set**.

See : *A Markov partition for Jeandel-Rao aperiodic Wang tilings*, arXiv:1903.06137, Pi Day, 2019.

Outline

A self-similar aperiodic set $\mathcal{U}$ of 19 Wang tiles



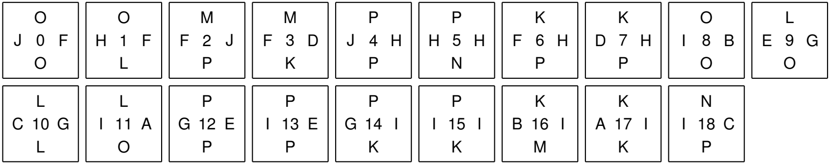
Theorem

The Wang shift $\Omega_{\mathcal{U}}$ is **self-similar**, **aperiodic** and **minimal**.

$$\Omega_{\mathcal{U}} \xleftarrow{\alpha\,:\,\square\mapsto\square,\,\square\mapsto\begin{smallmatrix}\square\\\square\end{smallmatrix}} \Omega_{\mathcal{V}} \xleftarrow{\beta\,:\,\square\mapsto\square,\,\square\mapsto\square\square} \Omega_{\mathcal{W}} \xleftarrow{\gamma\,:\,\square\mapsto\square} \Omega_{\mathcal{U}}$$

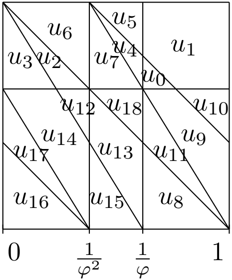
See : A self-similar aperiodic set of 19 Wang tiles, *Geom Dedicata* (2018) doi:10.1007/s10711-018-0384-8

The partition $\mathcal{P}_{\mathcal{U}}$ of $\mathbb{T}^2$



Let

- $\mathcal{U}$ be the above set of 19 tiles,
- $\mathcal{P}_{\mathcal{U}} = \{P_u\}_{u\in\mathcal{U}}$ be the partition of $\mathbb{T}^2$ shown on the figure,
- $R^{\mathbf{n}}(\mathbf{x}) = \mathbf{x} + \varphi^{-2}\mathbf{n}$ the $\mathbb{Z}^2$-action,



Proposition

$\mathcal{X}_{\mathcal{P}_{\mathcal{U}},R_{\mathcal{U}}}$ is a subshift of the Wang shift $\Omega_{\mathcal{U}}$

Results on $\mathcal{X}_{\mathcal{P}_\mathcal{U},R_\mathcal{U}} = \Omega_\mathcal{U}$

Theorem

- $\mathcal{P}_\mathcal{U}$ gives a **symbolic representation** of $(\mathbb{T}^2, \mathbb{Z}^2, R_\mathcal{U})$
- $\mathcal{X}_{\mathcal{P}_\mathcal{U},R_\mathcal{U}}$ is equal to the Wang shift $\Omega_\mathcal{U}$
- $\mathcal{P}_\mathcal{U}$ is a **Markov partition** for $(\mathbb{T}^2, \mathbb{Z}^2, R_\mathcal{U})$
- there exists an **almost 1-1 factor** map $f : \Omega_\mathcal{U} \to \mathbb{T}^2$
- $(\mathbb{T}^2, \mathbb{Z}^2, R_\mathcal{U})$ is the **maximal equicontinuous factor** of $(\Omega_\mathcal{U}, \mathbb{Z}^2, \sigma)$.
- The **measure-preserving** dynamical system $(\Omega_\mathcal{U}, \mathbb{Z}^2, \sigma, \nu)$ is **isomorphic** to $(\mathbb{T}^2, \mathbb{Z}^2, R, \lambda)$ where
  - ν is the unique shift-invariant probability measure on $\Omega_\mathcal{U}$
  - λ is the Haar measure on $\mathbb{T}^2$.
- Occurences of patterns in $\Omega_\mathcal{U}$ is a **4-to-2 C&P set**.

See : *A Markov partition for Jeandel-Rao aperiodic Wang tilings*, arXiv:1903.06137, Pi Day, 2019.

Outline

Factor complexity

Let $w \in \mathcal{A}^{\mathbb{Z}}$. The **factor complexity** is a function $p_w(n) : \mathbb{N} \to \mathbb{N}$ counting the number of factors of length n in the sequence w.

$$w = ...000100\boxed{\mathbf{0100}}010010001000100010010001001001...$$

$$Fact_w(4) = \{0001, 0010, \mathbf{0100}, 1000, 1001\} \quad \Longrightarrow \quad p_w(4) = 5$$

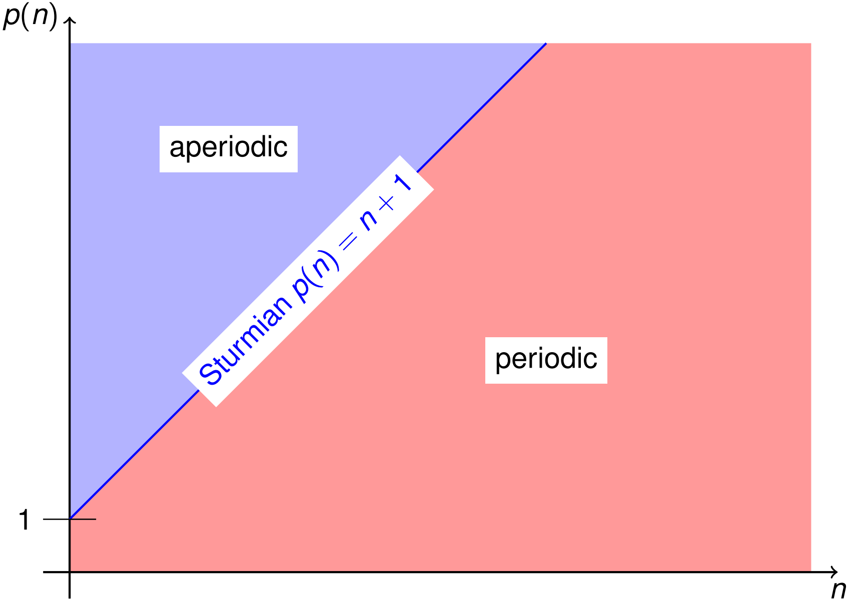
Lemma

An infinite word $w \in \mathcal{A}^{\mathbb{Z}}$ that has $p_w(n) \leq n$ factors of length n is **periodic**.

Definition

A **sturmian** word is an infinite word having exactly $p_w(n) = n+1$ factors of length n.

Words of complexity $p(n) = n + 1$



Theorem (Morse, Hedlund, 1940 ; Coven, Hedlund 1970)

$w \in \{a,b\}^{\mathbb{Z}}$ is the coding of an **irrational rotation** R_α with $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ if and only if w is a **sturmian sequence**.

- $\Longrightarrow$: the easy direction : counting number of intervals
- $\Longleftarrow$: **more difficult** : a **common substitutive structure**
  - Sturmian sequences in S_α can be desubstituted indefinitely :

$$S_\alpha \xleftarrow{\omega_0} S_{f(\alpha)} \xleftarrow{\omega_1} S_{f^2(\alpha)} \xleftarrow{\omega_2} S_{f^3(\alpha)} \dots$$

  - Coding $C_\alpha \subset \{a,b\}^{\mathbb{Z}}$ of R_α can be desubstituted (Rauzy induction) :

$$C_\alpha \xleftarrow{\omega_0} C_{f(\alpha)} \xleftarrow{\omega_1} C_{f^2(\alpha)} \xleftarrow{\omega_2} C_{f^3(\alpha)} \dots$$

    with $\omega_i \in \{\tau_a, \tau_b\}$ for every i where $\tau_a = (b \mapsto ab, a \mapsto a)$ and $\tau_b = (a \mapsto ba, b \mapsto b)$ where f is the Farey map.
  - Continued fraction expansion of α
  - Ostrowski expansion of the starting point $x_0 \in \mathbb{T}$ with respect to α.

Details in : Pytheas Fogg, 2002, chapter 6 written by Pierre Arnoux.

Outline

Open Questions

- Find the **10 line proof** for the aperiodicity of Jeandel-Rao tilings.
- Can we generalize Jeandel-Rao tilings to **other Pisot numbers** ?
- (done) I think the fundamental domain can be identified with the **window** of a cut-and-project set with dimension $2+2$.
- What are the structure of the **other aperiodic tile sets** of cardinality 11 found by Jeandel-Rao ?
- Does there **exists an aperiodic self-similar** Wang tile set of cardinality less than 16 ?
- Generalize the **characterization of primitive sequences** by Durand (1998) to Wang shifts.
- Describe the space of all aperiodic Wang shifts of low complexity (are **continued fraction** algorithms involved like for Sturmian sequences ?)

Open Questions

- Understand all of this in terms of **shape-symmetric** multidim. sequences (Maes, 1999) and S-automatic sequences using

Theorem (Charlier, Kärki, Rigo, 2010)

Let $d \geq 1$. The d-dimensional infinite word x is **S-automatic** for some abstract numeration system $S = (L, \Sigma, <)$ where $\epsilon \in L$ if and only if x is the **image by a coding of a shape-symmetric** d-dimensional infinite word.

- Prove that $\Omega_0 \setminus X_0$ is of measure 0.
- When is a SFT **metrically isomorphic** to the coding of a $\mathbb{Z}^2$-action on a torus ?
- When is the coding of a $\mathbb{Z}^2$-action on a torus **metrically isomorphic** to a SFT ?

Outline

Publications

- A self-similar aperiodic set of 19 Wang tiles,
  Geom Dedicata (2018) doi:10.1007/s10711-018-0384-8
- Substitutive structure of Jeandel-Rao aperiodic tilings
  arXiv:1808.07768
- A Markov partition for Jeandel-Rao aperiodic Wang tilings
  arXiv:1903.06137
- 2-dimensional induction of $\mathbb{Z}^2$-actions on $\mathbb{T}^2$,
  in preparation

Sage Jupyter notebooks reproducing the computations **available online** (link can be found in the arxiv comment section)

Some links

- Jupyter notebook .ipynb on nbviewer.jupyter.org/
- slabbe 0.5, Sage Optional Package,

```
sage -pip install slabbe
```

- Wooden laser-cut Jeandel-Rao tiles, blog post, Sep 7, 2018.
- Comparison of Wang tiling solver, blog post, Dec 12, 2018.

Special thanks to David Renault and the

- Université of Bordeaux FabLab, `https://www.iut.u-bordeaux.fr/cohabit/`

Regular tetrahedron packing arrangement (2009)

Letter

Disordered, quasicrystalline and crystalline phases of densely packed tetrahedra

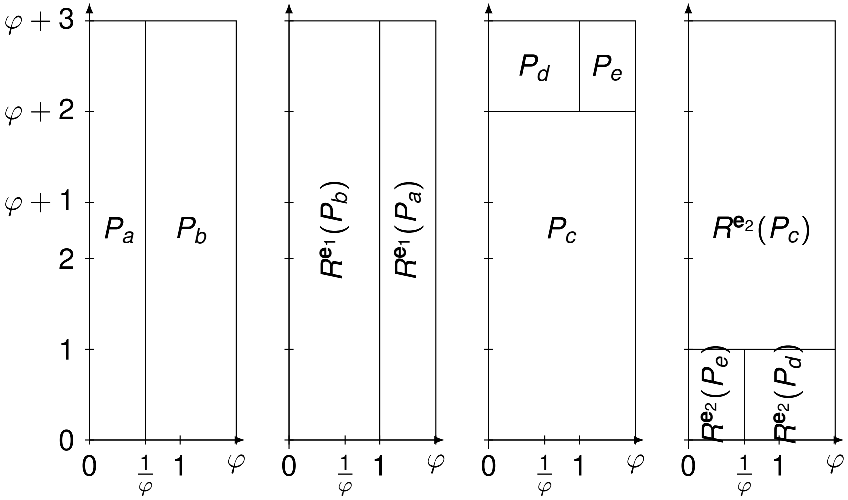
Amir Haji-Akbari, Michael Engel, Aaron S. Keys, Xiaoyu Zheng, Rolfe G. Petschek, Peter Palffy-Muhoray & Sharon C. Glotzer

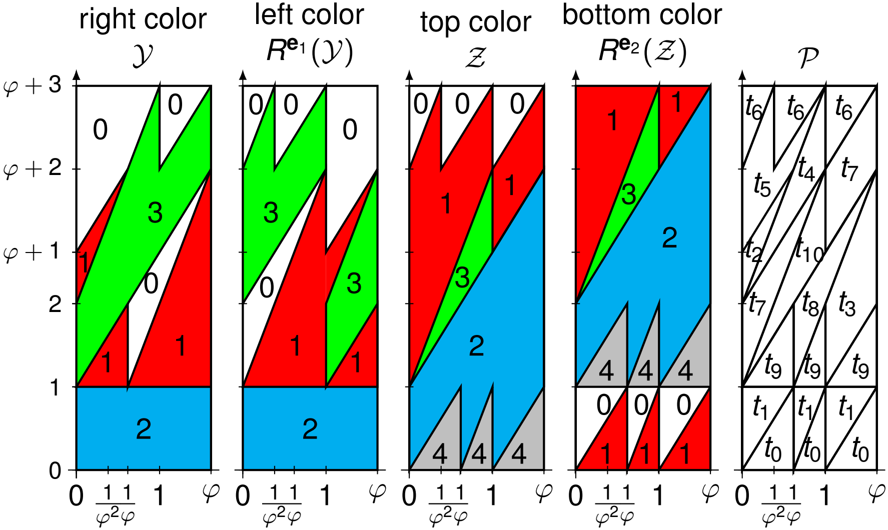
*"One of the simplest shapes for which the densest packing arrangement remains unresolved is the **regular tetrahedron** [...]. Using a novel approach involving thermodynamic computer simulations that allow the system to evolve naturally **towards high-density states**, Sharon Glotzer and colleagues have worked out the **densest ordered packing yet for tetrahedra**, a configuration with a packing fraction of 0.8324. Unexpectedly, the structure is a **dodecagonal quasicrystal**, [...] "*

Source (2009) : `doi:10.1038/nature08641` , `https://www.quantamagazine.org/digital-alchemist-sharon-glotzer-seeks-rules-of-emergence-20170`

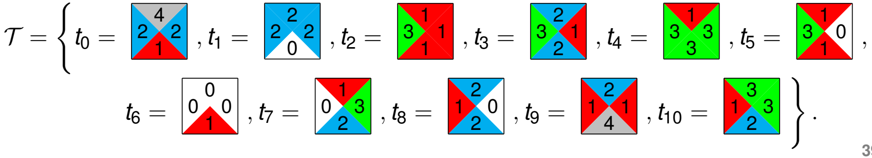
Polygon exchange transformations

Toral rotations $R^{\mathbf{e}_1}$ and $R^{\mathbf{e}_2}$ are **polygon exchange transformations** on the fundamental domain D.





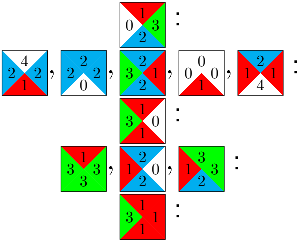
right color
$\mathcal{Y}$
left color
$R^{\mathbf{e}_1}(\mathcal{Y})$
top color
$\mathcal{Z}$
bottom color
$R^{\mathbf{e}_2}(\mathcal{Z})$
$\mathcal{P}$
$\varphi+3$
$\varphi+2$
$\varphi+1$
2
1
0
0
$\frac{1}{\varphi^2}$
$\frac{1}{\varphi}$
1
φ
t_0 t_1 t_2 t_3 t_4 t_5 t_6 t_7 t_8 t_9 t_{10}



$\mathcal{T} = \{ t_0 =$
4 2 2 1
$, t_1 =$
2 2 2 0
$, t_2 =$
1 3 1 1
$, t_3 =$
2 3 1 2
$, t_4 =$
1 3 3 3
$, t_5 =$
1 3 0 1
$,$
$t_6 =$
0 0 0 1
$, t_7 =$
1 0 3 2
$, t_8 =$
2 1 0 2
$, t_9 =$
2 1 1 4
$, t_{10} =$
3 1 3 2
$\}.$

Frequency of patterns

We deduce frequencies for every pattern, including the tile themselves :



$5/(12\varphi + 14) \approx 0.1496$

$1/(2\varphi + 6) \approx 0.1083$

$1/(5\varphi + 4) \approx 0.0827$

$1/(8\varphi + 2) \approx 0.0669$

$1/(18\varphi + 10) \approx 0.0256$

NOMBRES ALGÉBRIQUES ET SUBSTITUTIONS

PAR

G. RAUZY (*)

(1982)

- Let X be the subshift generated by the language of the **Tribonacci** substitution $1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 1$
- We have a **measurable isomorphism** with a rotation on $\mathbb{R}^2/\mathbb{Z}^2$:

$$\begin{tikzcd} X \arrow[r, "\sigma"] \arrow[d] & X \arrow[d] \\ \mathbb{R}^2/\mathbb{Z}^2 \arrow[r, "R_\xi"] & \mathbb{R}^2/\mathbb{Z}^2 \end{tikzcd}$$

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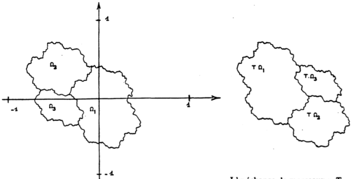

Le morcellement $\mathscr{M} = (\Omega_1, \Omega_2, \Omega_3)$.

L'« échange de morceaux » T défini sur Ω par : $Tx = x + \xi$ (modulo $\mathbf{Z}^2$).

Remarque. – Ω possède un centre de symétrie : le point $w/2$.

- **Open question (Pisot conjecture)** : is it true for every irreducible unimodular Pisot substitution ?