

Pavages apériodiques et codage de \mathbb{Z}^2 -actions sur le tore

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LaBRI, Bordeaux, France



Outline

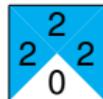
- 1 Wang tiles, aperiodicity and quasicrystals
- 2 The search for small aperiodic set of Wang tiles
- 3 Jeandel-Rao aperiodic Wang tile set
- 4 Motivation : Walking on a torus
- 5 Symbolic dynamics
- 6 Links

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Wang tiles

A **Wang tile** is a square tile with a color on each border

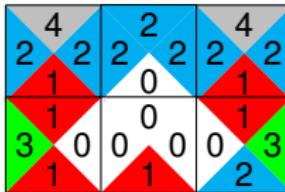


Tile set T : a finite collection of such tiles.

A tiling of the plane : an assignment

$$\mathbb{Z} \times \mathbb{Z} \rightarrow T$$

of tiles on infinite square lattice so that the **contiguous** edges of adjacent tiles have the **same** color.



Note : rotation not allowed.

Eternity II puzzle (2007)

- A puzzle which involves **placing 256 square puzzle** pieces into a 16 by 16 grid while **matching adjacent edges**
- A **2 million prize** (for the first solution found before 2011)
- **No solution ever found**
- Only $256! \times 4^{256} \approx 1.15 \times 10^{661}$ possibilities to check.
- Existence of a tiling $\mathbb{Z} \times \mathbb{Z} \rightarrow T$ is **undecidable** in general...



Source : https://en.wikipedia.org/wiki/Eternity_II_puzzle

Periods

A tiling is called **periodic** if it is invariant under some non-zero translation of the plane.

1	0	0	0	2	2	2	1	1	0	0	0	2	2	2	1	1	0	0	0	2	2	2	1	1	0	0	0	2	2			
1	1	0	0	0	2	2	2	1	1	0	0	2	2	2	1	1	0	0	0	2	2	2	1	1	0	0	0	2	2			
2	2	1	1	0	0	2	2	1	1	0	0	2	2	2	1	1	0	0	0	2	2	2	1	1	0	0	0	2	2			
0	2	2	1	1	0	0	2	2	1	1	0	0	2	2	2	1	1	0	0	2	2	2	1	1	0	0	0	2	2			
0	0	2	1	1	0	0	2	2	1	1	0	0	2	2	2	1	1	0	0	2	2	2	1	1	0	0	0	2	2			
1	1	0	0	2	2	1	1	0	0	2	2	2	1	1	0	0	2	2	2	1	1	0	0	0	2	2	1	0	0	2	2	
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0	2	2	1	1	0	0	2	2	1	1	0	0	2	2	2	1	1	0	0	2	2	2	1	1	0	0	0	2	2			
0	0	2	1	1	0	0	2	2	1	1	0	0	2	2	2	1	1	0	0	2	2	2	1	1	0	0	0	2	2			
1	1	0	0	2	2	1	1	0	0	2	2	2	1	1	0	0	2	2	2	1	1	0	0	0	2	2	1	0	0	2	2	
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0	2	2	1	1	0	0	2	2	1	1	0	0	2	2	2	1	1	0	0	2	2	2	1	1	0	0	0	2	2			
0	0	2	1	1	0	0	2	2	1	1	0	0	2	2	2	1	1	0	0	2	2	2	1	1	0	0	0	2	2			
1	1	0	0	2	2	1	1	0	0	2	2	2	1	1	0	0	2	2	2	1	1	0	0	0	2	2	1	1	0	2	2	

A Wang tile set that admits a periodic tiling also admits a **doubly periodic** tiling : a tiling with a horizontal and a vertical period.

Aperiodicity

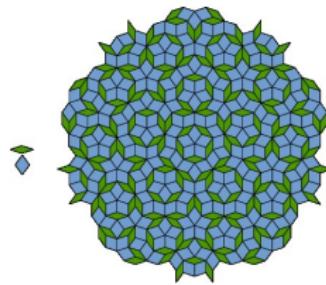
A tile set is **finite** if there is no tiling of the plane with this set.

A tile set is **aperiodic** if it tiles the plane, but no tiling is periodic

Conjecture (Wang 1961)

Every set of Wang tiles is either **finite or periodic**.

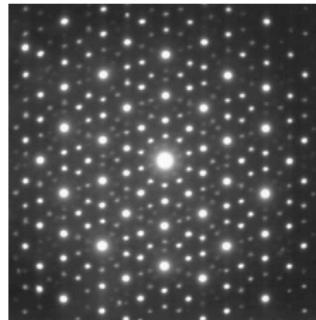
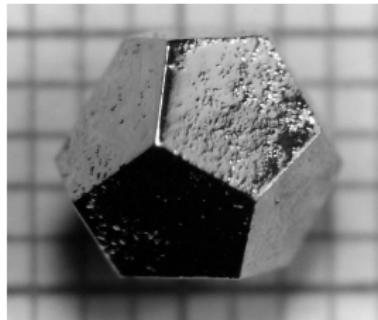
- 1966 (Berger) : There **exists an aperiodic** set of Wang tiles
- 1976 (Penrose) : discovered an aperiodic set of **two tiles**



Quasicrystals

- 1982 (Shechtman) : observed that aluminium-manganese alloys produced a **quasicrystals structure**
- 2011 : Dan Shechtman receives **Nobel Prize** in Chemistry

A Ho-Mg-Zn **icosahedral quasicrystal** formed as a pentagonal dodecahedron and its **electron diffraction** pattern :



*"His discovery of quasicrystals revealed a **new principle for packing of atoms and molecules**", stated the Nobel Committee that "led to a **paradigm shift** within chemistry".*

Source : <https://en.wikipedia.org/wiki/Quasicrystal>

The Russian meteorite (2016)



Quanta magazine

Physics Mathematics Biology Computer Science All Articles



Natalie Wolchover

Senior Writer

ABSTRACTIONS BLOG

A Quasicrystal's Shocking Origin

By blasting a stack of minerals with a four-meter-long gun, scientists have found a new clue about the backstory of a very strange rock.



July 8, 2016

*"They **loaded the minerals** found in the rock into a chamber, and then, using a four-meter-long propellant gun, **fired a projectile** into the stack of ingredients. [...] The findings [...] indicate that the **quasicrystals** in the Russian meteorite did indeed **form during a shock event**."*

Source (2016) : <https://www.quantamagazine.org/a-quasicrystals-shocking-origin-20160708/>

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Sur quelle rue de Bordeaux se trouve cette porte ?



Sur quelle rue de Bordeaux se trouve cette porte ? **Rue Gratiolet**



Discoveries of aperiodic Wang tile sets (< 1990)

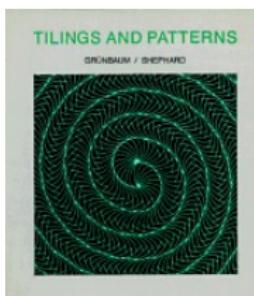


Image credit : <http://chippewa.canalblog.com/archives/2010/06/04/18115718.html>

- 1966 (Berger) : 20426 tiles (lowered down later to 104)
- 1968 (Knuth) : 92 tiles
- 1971 (Robinson) : 56 tiles
- 1971 (Ammann) : 16 tiles
- 1987 (Grunbaum) : 24 tiles
- ...

The search for small aperiodic Wang tile sets

The reduction in the number of Wang tiles in an aperiodic set from over 20,000 to 16 has been a notable achievement. Perhaps the minimum possible number has now been reached. If, however, further reductions are possible then it seems certain that new ideas and methods will be required. The discovery of such remains one of the outstanding challenges in this field of mathematics. One can, of course, look at the problem from the opposite point of view. Is it possible to prove that, for example, 15 tiles are not enough? It is difficult to see how any such proof could be constructed, and the only result we know in this direction is an unpublished theorem of Robinson that no aperiodic set of *four* Wang tiles can exist.



Source : Grünbaum, Shephard, Tilings and patterns, 1987, p. 596.

Ammann set of 16 Wang tiles is self-similar

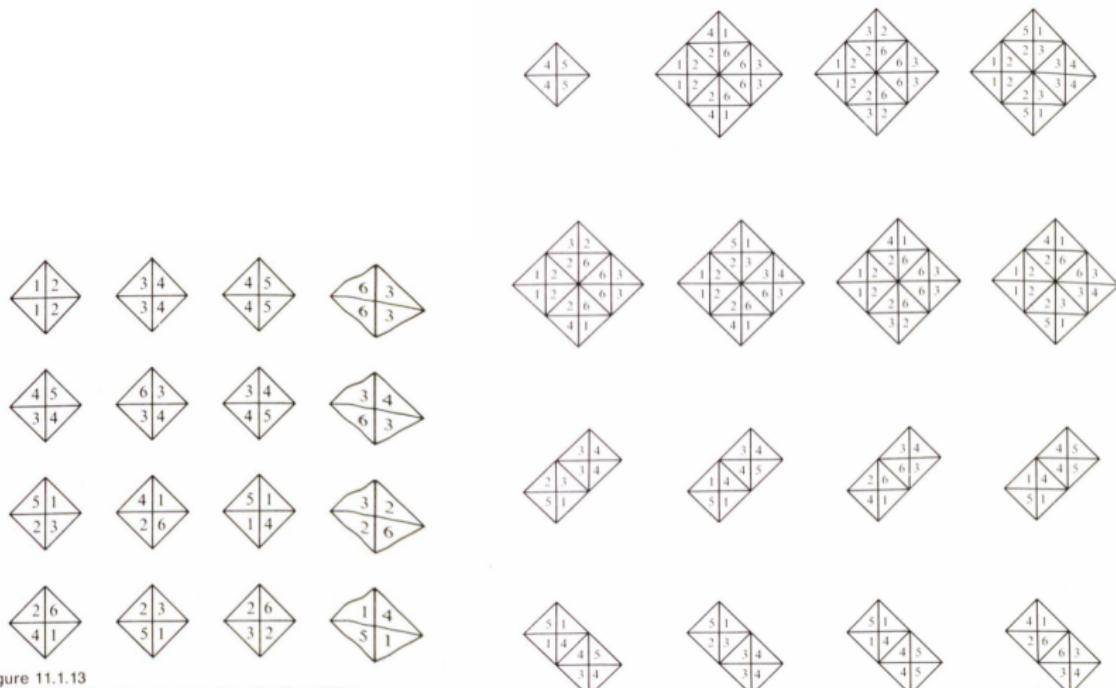


Figure 11.1.13

The 16 Wang tiles that correspond to the tiles of Figure 11.1.12.
These form the smallest known aperiodic set.

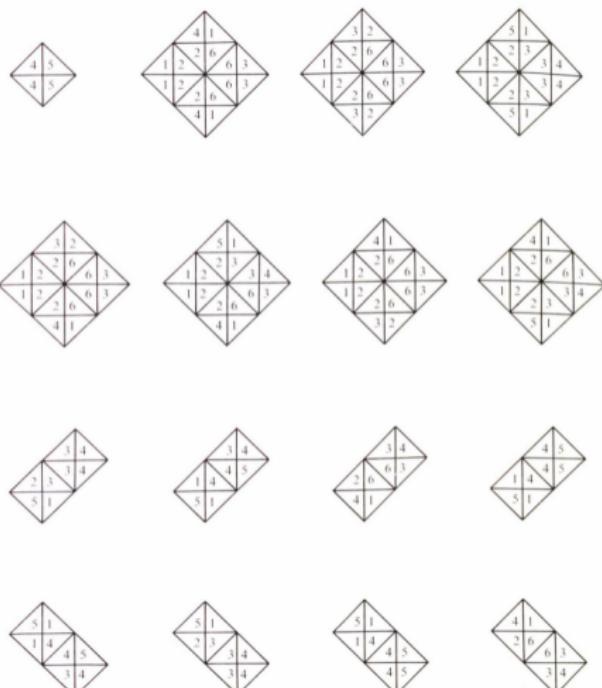


Figure 11.1.16

This diagram shows how the Wang tiles of Figure 11.1.13 can be “decomposed”.

Folklore, stated in Ammann, Grünbaum, Shephard, 1992

If a tile set \mathcal{T} is **self-similar** and **has unique composition property**, then \mathcal{T} is **aperiodic**.

unique composition property : if in every tiling admitted by \mathcal{T} there is a **unique way** in which the tiles can be grouped into patches which lead to a tiling by **supertiles**

self-similar : if the markings on the supertiles, inherited from the original tiles, imply a matching condition for the supertiles which is **exactly equivalent** to that originally specified for the tiles,

Mossé 1992 (on \mathbb{Z}) ; Solomyak 1998 (in \mathbb{R}^d)

A self-similar tiling has the **unique composition property** if and only if it is **nonperiodic**.

Discoveries of aperiodic Wang tile sets (< 2000)



Image credit : <http://chippewa.canalblog.com/archives/2010/06/04/18115718.html>

- 1966 (Berger) : 20426 tiles (lowered down later to 104)
- 1968 (Knuth) : 92 tiles
- 1971 (Robinson) : 56 tiles
- 1971 (Ammann) : 16 tiles
- 1987 (Grunbaum) : 24 tiles
- 1996 (Kari) : 14 tiles
- 1996 (Culik) : (same method) 13 tiles

Note

A small aperiodic set of Wang tiles

Jarkko Kari*

Department of Computer Science, University of Iowa, MacLean Hall, Iowa City, Iowa 52242-1419, USA

Received 3 January 1995

Abstract

A new aperiodic tile set containing only 14 Wang tiles is presented. The construction is based on Mealy machines that multiply Beatty sequences of real numbers by rational constants.

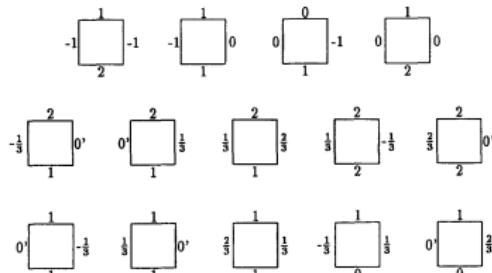


Fig. 1. Aperiodic set of 14 Wang tiles

Proposition 1. *The tile set T does not admit a periodic tiling.*

Proof. Assume that $f: \mathbb{Z}^2 \rightarrow T$ is a doubly periodic tiling with horizontal period a and vertical period b . For $i \in \mathbb{Z}$, let n_i denote the sum of colors on the upper edges of tiles $f(1, i), f(2, i), \dots, f(a, i)$. Because the tiling is horizontally periodic with period a , the ‘carries’ on the left edge of $f(1, i)$ and the right edge of $f(a, i)$ are equal. Therefore $n_{i+1} = q_i n_i$, where $q_i = 2$ if tiles of T_2 are used on row i and $q_i = 4$ if tiles of T_3 are used.

Because the vertical period of tiling f is b ,

$$B_1 = B_{1,1} = a_1 a_2 - a_3 a_4$$

and because two tiles with 0's on their upper edges cannot be next to each other, $n_1 \neq 0$. So $q_1 q_2 \dots q_b = 1$. This contradicts the fact that no non-empty product of 2's and 3's can be 1. \square

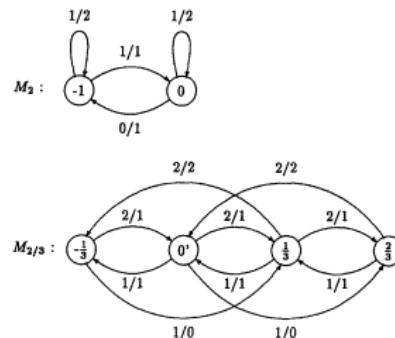


Fig. 2. Mcaly machine corresponding to the aperiodic tile set

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Discoveries of aperiodic Wang tile sets (2015)

Theorem (Jeandel, Rao, 2015)

- All sets of ≤ 10 tiles are **periodic** or do not tile the plane.
- The following 11 Wang tile set is **aperiodic** :

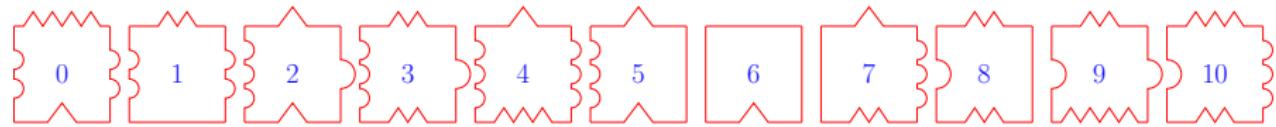
$$\mathcal{T} = \left\{ \begin{matrix} 2 & 4 & 2 \\ 2 & 1 & \\ \hline 1 & & \end{matrix}, \begin{matrix} 2 & 2 & 2 \\ 2 & 0 & \\ \hline 0 & & \end{matrix}, \begin{matrix} 3 & 1 & 1 \\ 3 & 1 & \\ \hline 1 & & \end{matrix}, \begin{matrix} 3 & 2 & 1 \\ 3 & 2 & \\ \hline 1 & & \end{matrix}, \begin{matrix} 3 & 1 & 3 \\ 3 & 3 & \\ \hline 1 & & \end{matrix}, \begin{matrix} 3 & 1 & 0 \\ 3 & 1 & \\ \hline 0 & & \end{matrix}, \begin{matrix} 0 & 0 & 0 \\ 0 & 1 & \\ \hline 1 & & \end{matrix}, \begin{matrix} 0 & 1 & 0 \\ 0 & 2 & \\ \hline 1 & & \end{matrix}, \begin{matrix} 1 & 2 & 0 \\ 1 & 2 & \\ \hline 0 & & \end{matrix}, \begin{matrix} 1 & 2 & 1 \\ 1 & 4 & \\ \hline 2 & & \end{matrix}, \begin{matrix} 1 & 3 & 3 \\ 1 & 2 & \\ \hline 2 & & \end{matrix} \right\} .$$

Their algorithm is pictured below :



Image credit : Le Bagger 288, <http://i.imgur.com/YH9xX.jpg>

JR's 11 tiles ... as closed topological disks using base-1 rep. of \mathbb{N} :



152244 valid 7×7 solutions but only 483 ($+\varepsilon$) extend to $\mathbb{Z} \times \mathbb{Z}$.



(Solution found by Pauline Hubert and Antoine Abram, Sep 2018)

Aram Dermenjian's solution (Montréal, Sep 2018)

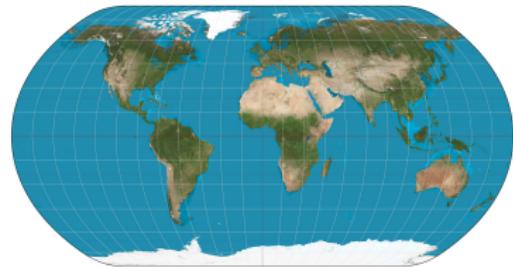


Jeandel and Rao proved aperiodicity by showing that tilings are given by horizontal strips of height 4 or 5 and the possible sequences $x \in \{4, 5\}^{\mathbb{Z}}$ are exactly those whose language is given by the Fibonacci morphism $5 \mapsto 54, 4 \mapsto 5$.

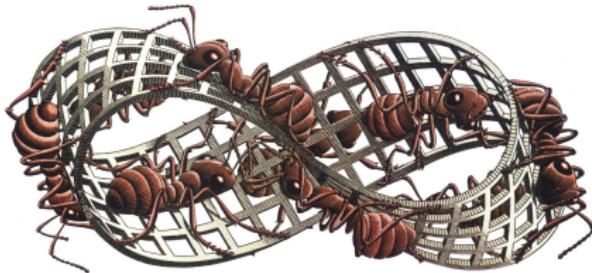
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Motivation



Source : Natural Earth Projection (wikipedia)



Source : <https://i.imgur.com/R2eAvWi.jpg>

Question

On **which surface** am I walking ?



GEORGE G. SZPIRO

La conjecture de Poincaré

Comment Grigori Perelman a résolu l'une des plus grandes énigmes mathématiques



Comment Grigori Perelman a résolu l'une des plus grandes énigmes mathématiques.

Pendant cent ans la conjecture de Poincaré a mobilisé sans succès les plus grands esprits scientifiques.

En 1904, Henri Poincaré soulevait la question suivante : «Imaginez une fourmi marchant sur une surface. Comment cet insecte peut-il savoir, sans s'élever au-dessus d'elle, si cette surface est plate ou s'il évolue sur une sphère ou sur toute autre forme ?»

En 2003, Grigori Perelman publie sur Internet trois

Walking on a torus

Conjecture

There exists a **measurable isomorphism** between the torus $\mathbb{R}^2/\mathbb{Z}^2$ and the Jeandel-Rao Wang shift Ω_0 :

$$\mathbb{R}^2/\mathbb{Z}^2 \rightarrow \Omega_0$$

Walking on Jeandel-Rao tilings is like walking on a **torus** :



Application : **uniform random generation** of patterns of any given shape $S \subseteq \mathbb{Z}^2$ appearing in Jeandel-Rao tilings.

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Decimal expansion

- Let $A = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ be digits
- Consider the **multiplication by 10** on the circle $\mathbb{R}/\mathbb{Z} = [0, 1[$

$$\begin{aligned}T_{10} : [0, 1[&\rightarrow [0, 1[\\x &\mapsto 10x - \lfloor 10x \rfloor\end{aligned}$$

- Consider the shift map σ on $A^{\mathbb{N}}$:

$$\sigma : (a_0 a_1 a_2 \dots) \mapsto (a_1 a_2 a_3 \dots)$$

- We have a **measurable isomorphism**:

$$\begin{array}{ccc}A^{\mathbb{N}} & \xrightarrow{\sigma} & A^{\mathbb{N}} \\ \downarrow & & \downarrow \\ [0, 1[& \xrightarrow{T_{10}} & [0, 1[\end{array}$$

- Exponential complexity, positive entropy.

Factor complexity

Let $w \in A^{\mathbb{Z}}$. The **factor complexity** is a function $p_w(n) : \mathbb{N} \rightarrow \mathbb{N}$ counting the number of factors of length n in the sequence w .

$w = \dots 000100 \boxed{0100} 0100100010001000100100010001001\dots$

$$Fact_w(4) = \{0001, 0010, \boxed{0100}, 1000, 1001\} \implies p_w(4) = 5$$

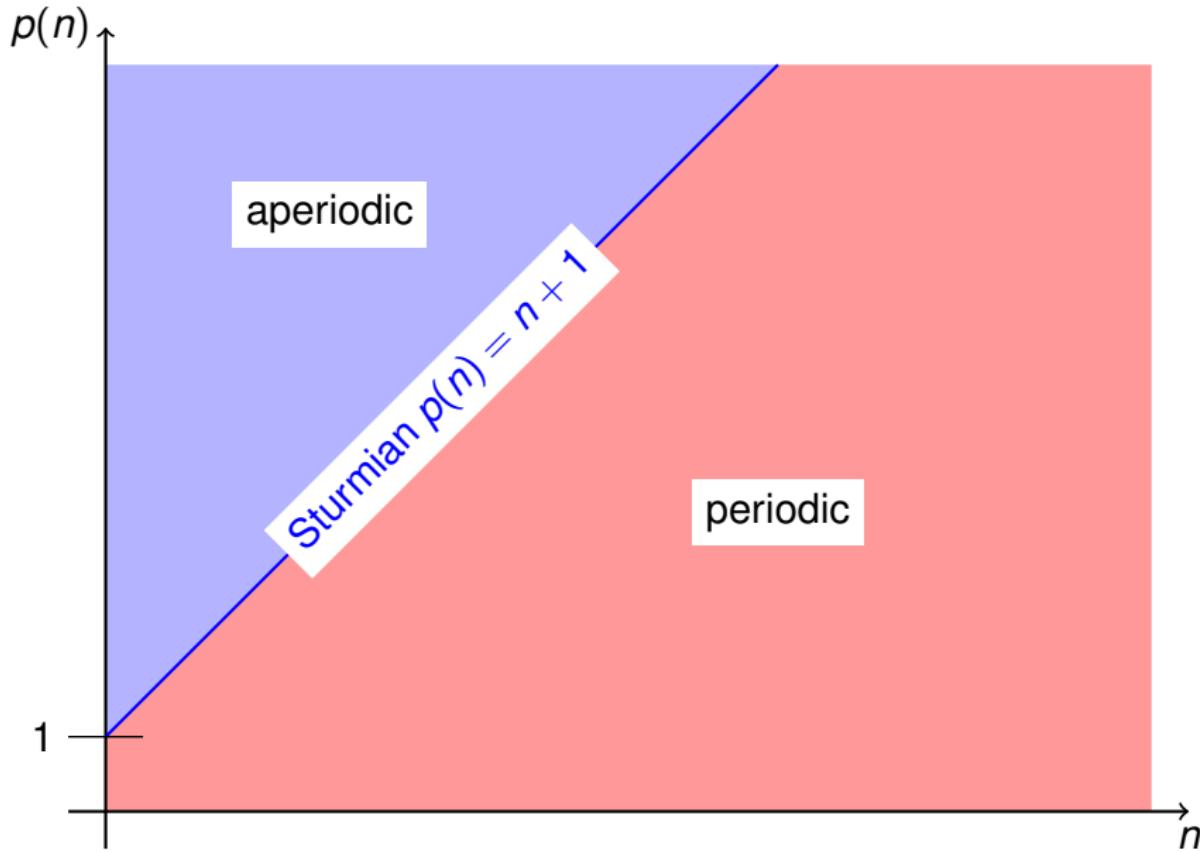
Lemma

An infinite word $w \in A^{\mathbb{Z}}$ that has $p_w(n) \leq n$ factors of length n is **periodic**.

Definition

A **sturmian** word is an infinite word having exactly $p_w(n) = n+1$ factors of length n .

Words of complexity $p(n) = n + 1$



SYMBOLIC DYNAMICS II. STURMIAN TRAJECTORIES.*

By MARSTON MORSE and GUSTAV A. HEDLUND.

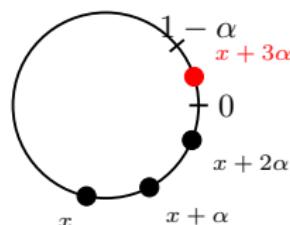
(1940)

- Let $S_\alpha \subset \{a, b\}^{\mathbb{Z}}$ be the set (subshift) of **Sturmian words** of slope $\alpha \in \mathbb{R} \setminus \mathbb{Q}$
- Consider the **rotation** on the circle $\mathbb{R}/\mathbb{Z} = [0, 1[$

$$R_\alpha : [0, 1[\rightarrow [0, 1[\\ x \mapsto (x + \alpha) \bmod 1$$

- We have a **measurable isomorphism** :

$$\begin{array}{ccc} S_\alpha & \xrightarrow{\sigma} & S_\alpha \\ \downarrow & & \downarrow \\ [0, 1[& \xrightarrow{R_\alpha} & [0, 1[\end{array}$$



- Sturmian words can be **desubstituted** by $0 \mapsto 0, 1 \mapsto 01$ or $0 \mapsto 1, 1 \mapsto 01$.

NOMBRES ALGÉBRIQUES
ET SUBSTITUTIONS

PAR

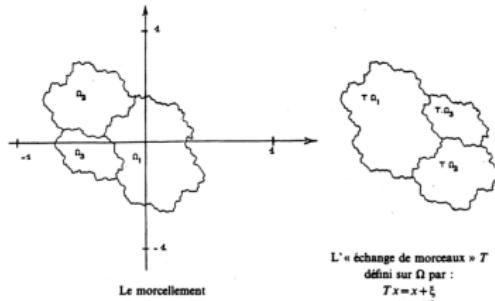
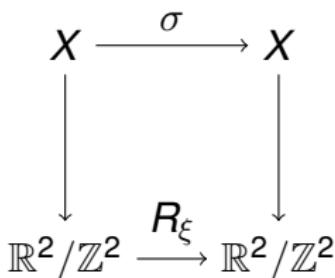
G. RAUZY (*)

(1982)

- Let X be the subshift generated by the language of the **Tribonacci** substitution $1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 1$
- We have a **measurable isomorphism** with a rotation on $\mathbb{R}^2/\mathbb{Z}^2$:

NOMBRES ALGÉBRIQUES ET SUBSTITUTIONS

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Remarque. — Ω possède un centre de symétrie : le point $w/2$.

- Open question (Pisot conjecture)** : is it true for every irreducible unimodular Pisot substitution ?

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Some links

- A self-similar aperiodic set of 19 Wang tiles, *Geom Dedicata* (2018) doi:10.1007/s10711-018-0384-8
- Substitutive structure of Jeandel-Rao aperiodic tilings arXiv:1808.07768
- Jupyter notebook .ipynb on nbviewer.jupyter.org/
- slabbe 0.4.3, Sage Optional Package,

```
sage -pip install slabbe
```
- Wooden laser-cut Jeandel-Rao tiles, blog post, Sep 7, 2018.
- Comparison of Wang tiling solver, blog post, Dec 12, 2018.

Special thanks to David Renault and the

- Université of Bordeaux FabLab,
<https://www.iut.u-bordeaux.fr/cohabit/>

Petite école de combinatoire : déc.-janv. 2019

- Séance 1** Définition et rappels historiques sur les pavages apériodiques et la découverte de Jeandel-Rao.
Construction de pavages de Jeandel-Rao par le codage de \mathbb{Z}^2 -orbites sur le tore (injectivité). La surjectivité nécessite plus de travail.
- Séance 2** Rappels sur les mots sturmiens. Conjugaison avec une rotation irrationnelle sur le cercle. Transformation induite.
Construction des mots sturmiens par substitutions. Lien avec les fractions continues. Autosimilarité.
- Séance 3** Mots et substitutions bidimensionnelles, substitutions de tuiles de Wang, notion de tuiles marqueurs. Algorithmes de désubstitutions de tuiles de Wang. Structure substitutive des pavages de Jeandel-Rao. Pavages autosimilaires.
- Séance 4** Codage de \mathbb{Z}^2 -actions et transformations induites sur le tore. Codage induits sur le tore. Désubstitutions de mots bidimensionnels. Codage autosimilaires.

Regular tetrahedron packing arrangement (2009)

Letter

Disordered, quasicrystalline and
crystalline phases of densely packed
tetrahedra

Amir Haji-Akbari, Michael Engel, Aaron S. Keys, Xiaoyu Zheng, Rolfe G. Petschek, Peter Palffy-Muhoray & Sharon C. Glotzer 

*"One of the simplest shapes for which the densest packing arrangement remains unresolved is the **regular tetrahedron** [...]. Using a novel approach involving thermodynamic computer simulations that allow the system to evolve naturally **towards high-density states**, Sharon Glotzer and colleagues have worked out the **densest ordered packing yet for tetrahedra**, a configuration with a packing fraction of 0.8324. Unexpectedly, the structure is a **dodecagonal quasicrystal**, [...] "*

Source (2009) : doi:10.1038/nature08641 ,

<https://www.quantamagazine.org/>

digital-alchemist-sharon-glotzer-seeks-rules-of-emergence-2017034/34