On Jeandel-Rao aperiodic tilings

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Rencontre ANR CODYS
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Outline

1. Wang tiles, aperiodicity and quasicrystals
2. The search for small aperiodic set of Wang tiles
3. Substitutive structure of Jeandel-Rao aperiodic tilings
4. Walking on a torus
5. Extra stuff

Outline

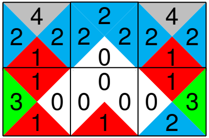
Wang tiles

A **Wang tile** is a square tile with a color on each border

Tile set T : a finite collection of such tiles.
A tiling of the plane : an assignment

$$\mathbb{Z} \times \mathbb{Z} \to T$$

of tiles on infinite square lattice so that the contiguous edges of adjacent tiles have the same color.



Note : rotation not allowed.

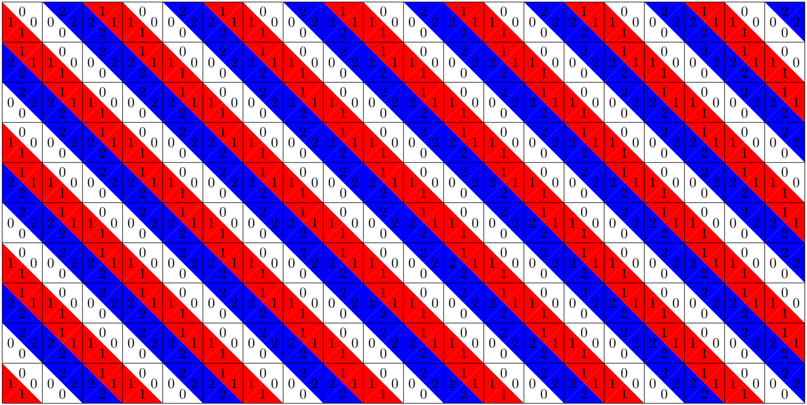
Eternity II puzzle (2007)

- A puzzle which involves **placing 256 square puzzle** pieces into a 16 by 16 grid while **matching adjacent edges**
- A **2 million prize** (for the first solution found before 2011)
- **No solution ever found**
- Only $256! \times 4^{256} \approx 1.15 \times 10^{661}$ possibilities to check.
- Existence of a tiling $\mathbb{Z} \times \mathbb{Z} \to T$ is **undecidable** in general...

Source : `https://en.wikipedia.org/wiki/Eternity_II_puzzle`

Periods

A tiling is called **periodic** if it is invariant under some non-zero translation of the plane.



A Wang tile set that admits a periodic tiling also admits a **doubly periodic** tiling : a tiling with a horizontal and a vertical period.

Aperiodicity

A tile set is **finite** if there is no tiling of the plane with this set.
A tile set is **aperiodic** if it tiles the plane, but no tiling is periodic

Conjecture (Wang 1961)

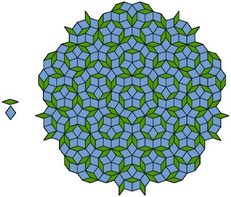
Every set of Wang tiles is either **finite or periodic**.

- 1966 (Berger) : There **exists an aperiodic** set of Wang tiles
- 1976 (Penrose) : discovered an aperiodic set of **two tiles**
- 1982 (Shechtman) : observed that aluminium-manganese alloys produced a **quasicrystals structure**
- 2011 : Dan Shechtman receives **Nobel Prize** in Chemistry

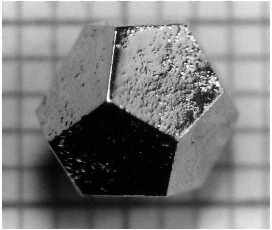
  "*His discovery of quasicrystals revealed a **new principle for packing of atoms and molecules***", stated the Nobel Committee that "*led to a **paradigm shift** within chemistry*".

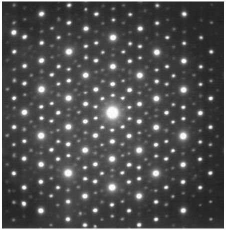
Quasicrystals

Penrose tiles and tiling :



A Ho-Mg-Zn **icosahedral quasicrystal** formed as a pentagonal dodecahedron and its **electron diffraction** pattern :





Source : `https://en.wikipedia.org/wiki/Quasicrystal`

Some notions and results (< 2000)

Definition

A discrete set X in $\mathbb{R}^d$ is a **Delone set** if it is uniformly discrete and relatively dense. It is called a **Meyer set** if the self-difference set $X - X$ a Delone set.

*"The notion of Delone sets as fundamental objects of study in crystallography was introduced by the Russian school in the 1930's ; in particular, by Boris Delone [...]. One can think about a Delone set as an idealized model of an **atomic structure** of a material [...]"*

Source : Boris Solomyak, `arxiv:1802.02370`

Theorem (Lagarias, Meyer)

Let X be a Meyer set in $\mathbb{R}^d$ such that $\eta X \subseteq X$ for a real number $\eta > 1$, then η is a **Pisot number** or a **Salem number**.

Other important results relates them to **cut-and-project set**, **regular tetrahedron packing**, **meteorites**, etc.

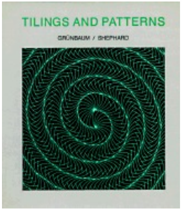
Outline

Discoveries of aperiodic Wang tile sets (< 1990)

Image credit : `http://chippewa.canalblog.com/archives/2010/06/04/18115718.html`

- 1966 (Berger) : 20426 tiles (lowered down later to 104)
- 1968 (Knuth) : 92 tiles
- 1971 (Robinson) : 56 tiles
- 1971 (Ammann) : 16 tiles
- 1987 (Grunbaum) : 24 tiles
- ...

The search for small aperiodic Wang tile sets



The reduction in the number of Wang tiles in an aperiodic set from over 20,000 to 16 has been a notable achievement. Perhaps the minimum possible number has now been reached. If, however, further reductions are possible then it seems certain that new ideas and methods will be required. The discovery of such remains one of the outstanding challenges in this field of mathematics. One can, of course, look at the problem from the opposite point of view. Is it possible to prove that, for example, 15 tiles are not enough? It is difficult to see how any such proof could be constructed, and the only result we know in this direction is an unpublished theorem of Robinson that no aperiodic set of *four* Wang tiles can exist.

Source : Grünbaum, Shephard, Tilings and patterns, 1987, p. 596.

Ammann set of 16 Wang tiles

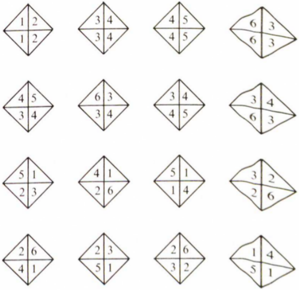


Figure 11.1.13
The 16 Wang tiles that correspond to the tiles of Figure 11.1.12. These form the smallest known aperiodic set.

11.1 APERIODIC SETS OF WANG TILES

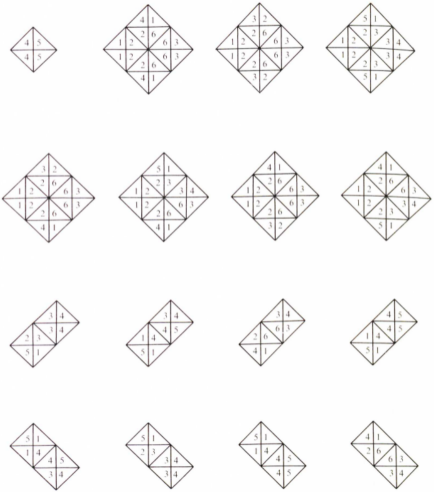


Figure 11.1.16
This diagram shows how the Wang tiles of Figure 11.1.13 can be "decomposed".

11.1 APERIODIC SETS OF WANG TILES 597

Grünbaum, Shephard, Tilings and patterns, 1987, p. 595-597.

Unique composition property in $\mathbb{R}^d$

Informally, two conditions that imply aperiodicity :

Ammann, Grünbaum, Shephard, 1992

Let $\mathcal{T}$ be a tile set. If

(a) in every tiling admitted by $\mathcal{T}$ there is a **unique way** in which the tiles can be grouped into patches which lead to a tiling by **supertiles** ; and

(b) the markings on the supertiles, inherited from the original tiles, imply a matching condition for the supertiles which is **exactly equivalent** to that originally specified for the tiles,

then $\mathcal{T}$ is **aperiodic**.

Mossé 1992 (on $\mathbb{Z}$) ; Solomyak 1998 (in $\mathbb{R}^d$)

A self-similar tiling has the **unique composition property** if and only if it is **nonperiodic**.

Discoveries of aperiodic Wang tile sets (< 2000)

Image credit : `http://chippewa.canalblog.com/archives/2010/06/04/18115718.html`

- 1966 (Berger) : 20426 tiles (lowered down later to 104)
- 1968 (Knuth) : 92 tiles
- 1971 (Robinson) : 56 tiles
- 1971 (Ammann) : 16 tiles
- 1987 (Grunbaum) : 24 tiles
- 1996 (Kari) : 14 tiles
- 1996 (Culik) : (same method) 13 tiles

ELSEVIER

Discrete Mathematics 160 (1996) 259–264

DISCRETE MATHEMATICS

Note

A small aperiodic set of Wang tiles

Jarkko Kari*

Department of Computer Science, University of Iowa, MacLean Hall, Iowa City, Iowa 52242-1419, USA

Received 3 January 1995

Abstract

A new aperiodic tile set containing only 14 Wang tiles is presented. The construction is based on Mealy machines that multiply Beatty sequences of real numbers by rational constants.

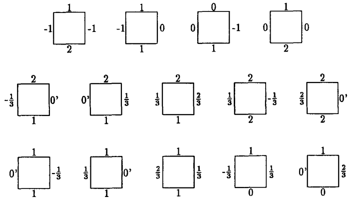


Fig. 1. Aperiodic set of 14 Wang tiles.

Proposition 1. *The tile set T does not admit a periodic tiling.*

Proof. Assume that $f: \mathbb{Z}^2 \to T$ is a doubly periodic tiling with horizontal period a and vertical period b. For $i \in \mathbb{Z}$, let n_i denote the sum of colors on the upper edges of tiles $f(1, i), f(2, i), \ldots, f(a, i)$. Because the tiling is horizontally periodic with period a, the 'carries' on the left edge of $f(1, i)$ and the right edge of $f(a, i)$ are equal. Therefore $n_{i+1} = q_i n_i$, where $q_i = 2$ if tiles of T_2 are used on row i and $q_i = \frac{2}{3}$ if tiles of $T_{2/3}$ are used.

Because the vertical period of tiling f is b,

$$n_1 = n_{b+1} = q_1 q_2 \ldots q_b \cdot n_1,$$

and because two tiles with 0's on their upper edges cannot be next to each other, $n_1 \neq 0$. So $q_1 q_2 \ldots q_b = 1$. This contradicts the fact that no non-empty product of 2's and $\frac{2}{3}$'s can be 1. □

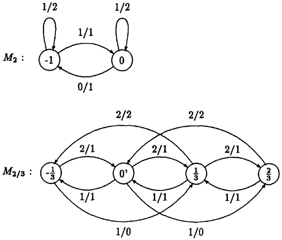


Fig. 2. Mealy machine corresponding to the aperiodic tile set.

Discoveries of aperiodic Wang tile sets (2015)

Theorem (Jeandel, Rao, 2015)

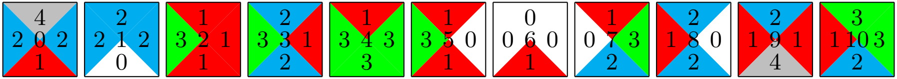
- All sets of ≤ 10 tiles are **periodic** or do not tile the plane.
- The following 11 Wang tile set is **aperiodic** :

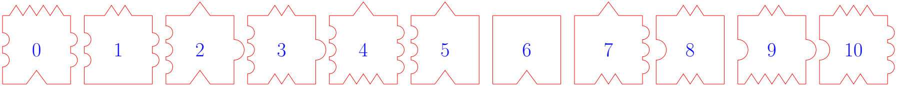
$\mathcal{T} = \{$  $\}$.

Their algorithm is pictured below :

Image credit : Le Bagger 288, `http://i.imgur.com/YH9xX.jpg`

JR’s 11 tiles ... as closed topological disks using base-1 rep. of $\mathbb{N}$:





152244 valid 7×7 solutions but only 483 $(+\varepsilon)$ extend to $\mathbb{Z} \times \mathbb{Z}$.

(Solution found by Pauline Hubert and Antoine Abram, Sep 2018)

Aram Dermenjian's solution

Jeandel and Rao proved aperiodicity by showing that tilings are given by horizontal strips of height 4 or 5 and the possible sequences $x \in \{4,5\}^{\mathbb{Z}}$ are exactly those whose language is given by the Fibonacci morphism $5 \mapsto 54, 4 \mapsto 5$.

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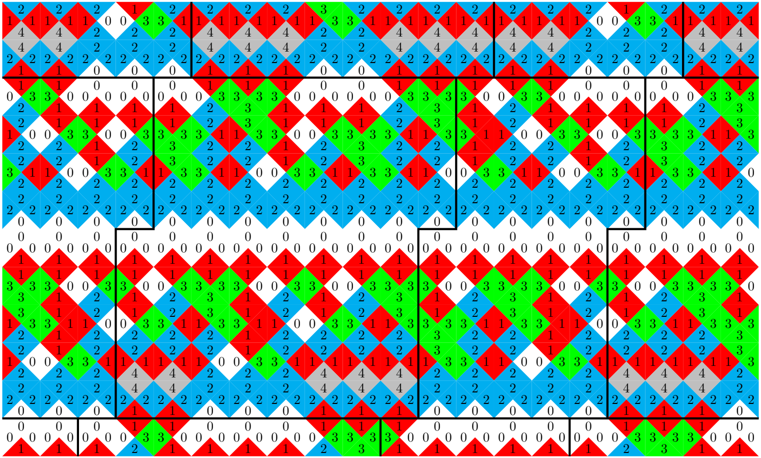
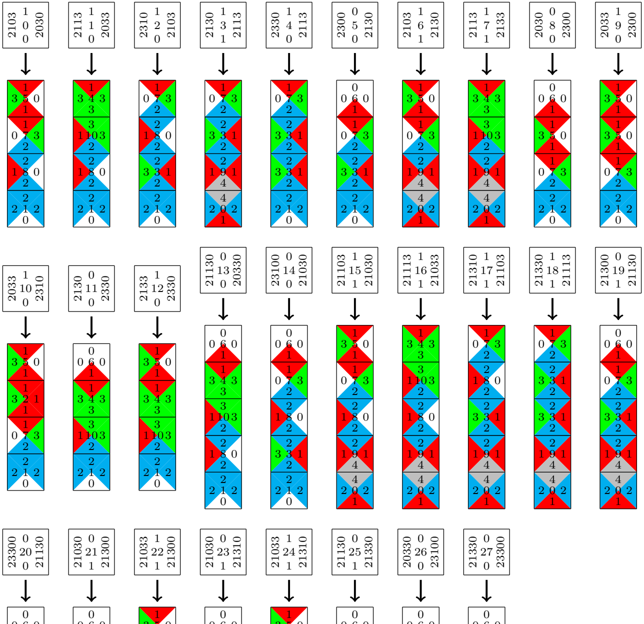
A decomposition into 19 self-similar supertiles

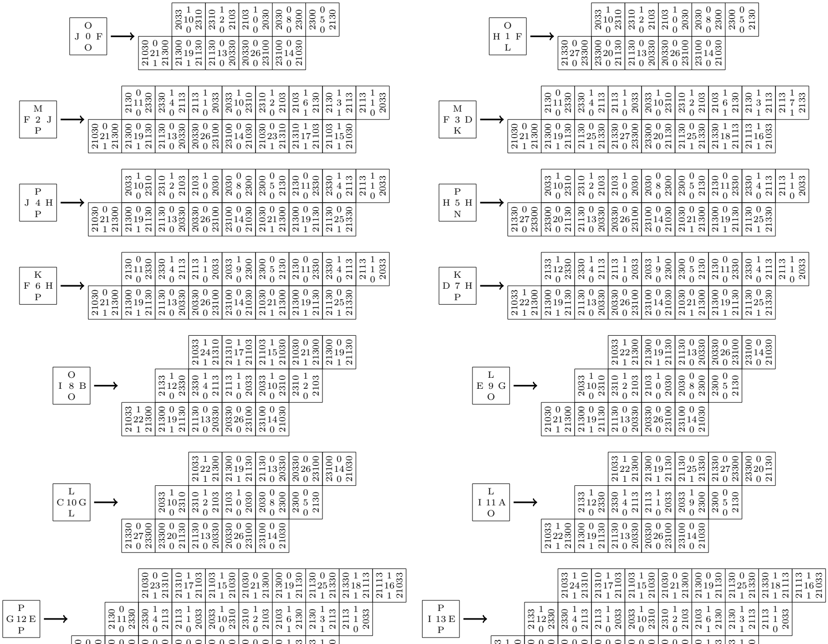
A finite part of a Jeandel-Rao aperiodic tiling in Ω_0. Any tiling in the minimal subshift X_0 of Ω_0 that we describe can be decomposed uniquely into 19 supertiles (two of size 45, six of size 72, four of size 70 and seven of size 112).



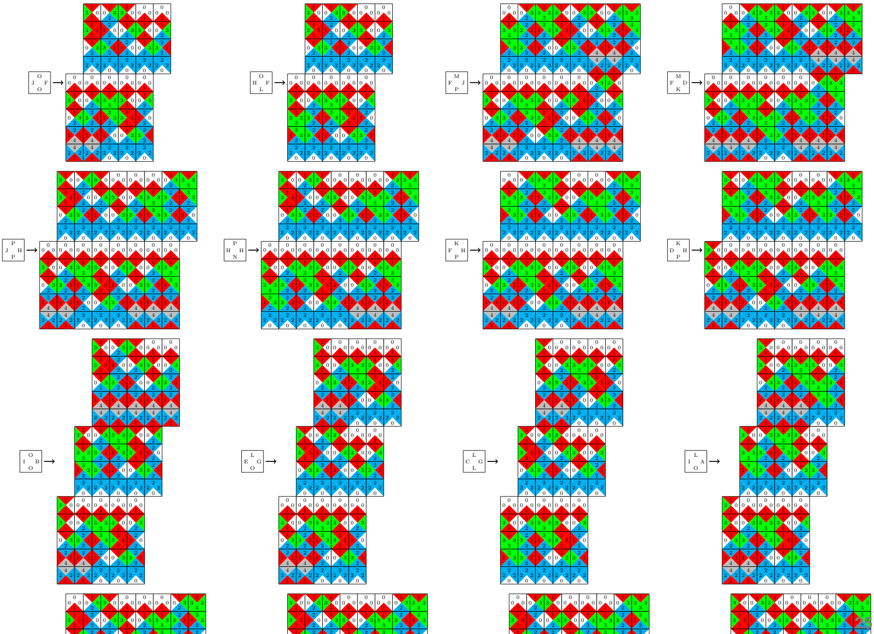
The morphism $\omega_0\,\omega_1\,\omega_2\,\omega_3 : \Omega_4 \to \Omega_0$



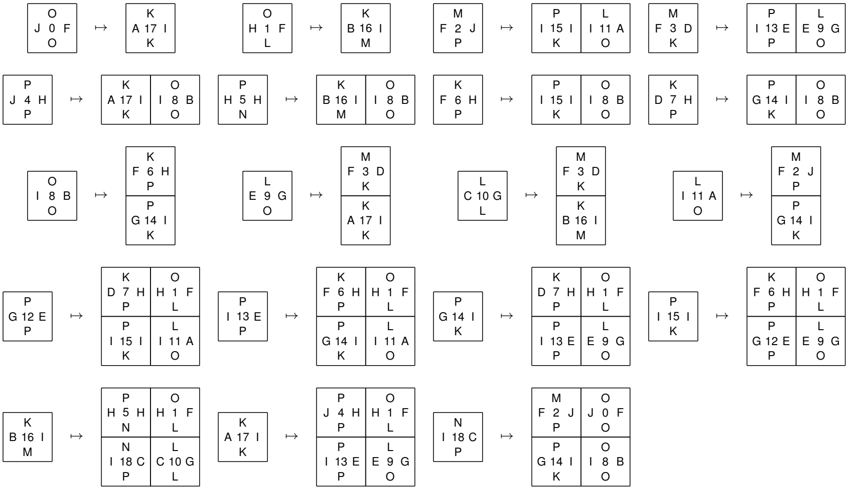
The morphism $\pi\,\eta\,\omega_6\,\omega_7\,\omega_8\,\omega_9\,\omega_{10}\,\omega_{11} : \Omega_{12} \to \Omega_4$



$\omega_0\,\omega_1\,\omega_2\,\omega_3\,\pi\,\eta\,\omega_6\,\omega_7\,\omega_8\,\omega_9\,\omega_{10}\,\omega_{11} : \Omega_{12} \to \Omega_0$



The morphism $\omega_{12} : \Omega_{12} \to \Omega_{12}$



$$
\begin{array}{ccccccccccccccc}
 & & & & & & & & \Omega_5 & \xleftarrow{\eta} & \Omega_6 \xleftarrow{\omega_6} \Omega_7 \xleftarrow{\omega_7} \Omega_8 \xleftarrow{\omega_8} \Omega_9 \xleftarrow{\omega_9} \Omega_{10} \xleftarrow{\omega_{10}} \Omega_{11} \xleftarrow{\omega_{11}} \Omega_{12} \circlearrowleft \omega_{12} \\
 & & & & & & & & \pi \downarrow & & \\
\Omega_0 & \xleftarrow{\omega_0} & \Omega_1 & \xleftarrow{\omega_1} & \Omega_2 & \xleftarrow{\omega_2} & \Omega_3 & \xleftarrow{\omega_3} & \Omega_4 & &
\end{array}
$$

Proposition

There exist $(\mathcal{T}_i)_{0\leq i\leq 12}$ sets of Wang tiles, with their associated Wang shift $\Omega_i = \Omega_{\mathcal{T}_i}$, and there exist

(i) morphisms $\omega_i : \Omega_{i+1} \to \Omega_i$ that are recognizable and onto up to a shift for each $i \in \{0, ..., 12\} \setminus \{4, 5\}$;

(ii) $\pi : \Omega_5 \to \Omega_4$ is an embbeding ;

(iii) $\eta : \Omega_6 \to \Omega_5$ is a sheering topological conjugacy.

Proof is based on the notion of **markers** and is partly done with computer.

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & & \Omega_5 \xleftarrow{\eta} \Omega_6 \xleftarrow{\omega_6} \Omega_7 \xleftarrow{\omega_7} \Omega_8 \xleftarrow{\omega_8} \Omega_9 \xleftarrow{\omega_9} \Omega_{10} \xleftarrow{\omega_{10}} \Omega_{11} \xleftarrow{\omega_{11}} \Omega_{12} \circlearrowleft \omega_{12} \\
 & & & & & & & & \downarrow \pi \\
\Omega_0 & \xleftarrow{\omega_0} & \Omega_1 & \xleftarrow{\omega_1} & \Omega_2 & \xleftarrow{\omega_2} & \Omega_3 & \xleftarrow{\omega_3} & \Omega_4
\end{array}
$$

Theorem

Let $X_{12} = \Omega_{12}$ and $X_i = \overline{\omega_i(X_{i+1})}^{\sigma}$ for every i with $0 \le i \le 11$.

(i) $X_{12} = \Omega_{12}$ is self-similar, aperiodic and minimal,

(ii) $X_i = \Omega_i$ and Ω_i is aperiodic and minimal, $5 \le i \le 11$,

(iii) $X_i \subsetneq \Omega_i$ is an aperiodic and minimal proper subshift of Ω_i for every i with $0 \le i \le 4$.

Conjecture

$\Omega_0 \setminus X_0$ is of measure 0 for any shift-invariant probability measure on Ω_0 (it consists of sliding half-plane along a fault line of 0's or 1's).

Definition : Markers

Let $\mathcal{T}$ be a Wang tile set and let $\Omega_\mathcal{T}$ be its Wang shift. A nonempty proper subset $M \subset \mathcal{T}$ is called **markers in the direction $\mathbf{e}_1$** if

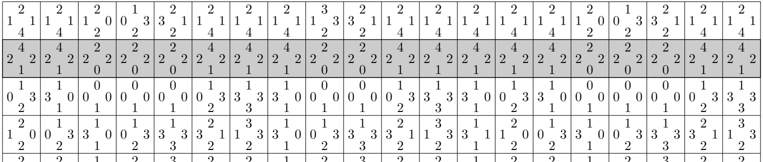
$$M \odot^1 M, \qquad M \odot^2 (\mathcal{T} \setminus M), \qquad (\mathcal{T} \setminus M) \odot^2 M$$

are forbidden in $\Omega_\mathcal{T}$. It is called **markers in the direction $\mathbf{e}_2$** if

$$M \odot^2 M, \qquad M \odot^1 (\mathcal{T} \setminus M), \qquad (\mathcal{T} \setminus M) \odot^1 M$$

are forbidden in $\Omega_\mathcal{T}$.

The markers in the direction $\mathbf{e}_1$ (resp. $\mathbf{e}_2$) appear as nonadjacent columns (resp. rows) of tiles in a tiling.



Proposition

Let $\mathcal{T}$ be a Wang tile set and let $\Omega_\mathcal{T}$ be its Wang shift. If there exists a subset $M \subset \mathcal{T}$ of markers in the direction $\mathbf{e}_i \in \{\mathbf{e}_1, \mathbf{e}_2\}$, then

(i) there exists a Wang tile set $\mathcal{S}_R$ and a 2-dimensional morphism $\omega_R : \Omega_{\mathcal{S}_R} \to \Omega_\mathcal{T}$ such that

$$\omega_R(\mathcal{S}_R) \subseteq (\mathcal{T} \setminus M) \cup \Big((\mathcal{T} \setminus M) \odot^i M\Big)$$

which is recognizable and onto up to a shift and

(ii) there exists a Wang tile set $\mathcal{S}_L$ and a 2-dimensional morphism $\omega_L : \Omega_{\mathcal{S}_L} \to \Omega_\mathcal{T}$ such that

$$\omega_L(\mathcal{S}_L) \subseteq (\mathcal{T} \setminus M) \cup \Big(M \odot^i (\mathcal{T} \setminus M)\Big)$$

which recognizable and onto up to a shift.

There exists an algorithm which computes ω_R and ω_L.

Some links

- A self-similar aperiodic set of 19 Wang tiles, *Geom Dedicata* (2018) doi:10.1007/s10711-018-0384-8
- Substitutive structure of Jeandel-Rao aperiodic tilings arXiv:1808.07768
- Jupyter notebook .ipynb on nbviewer.jupyter.org/
- slabbe 0.4.3, Sage Optional Package,

```
sage -pip install slabbe
```

- Wooden laser-cut Jeandel-Rao tiles, blog post, Sep 7, 2018.

Special thanks to the

- Université of Bordeaux FabLab, `https://www.iut.u-bordeaux.fr/cohabit/`

Outline

Decimal expansion

- Let $A = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ be digits
- Consider the **multiplication by 10** on the circle $\mathbb{R}/\mathbb{Z} = [0, 1[$

$$\begin{array}{rcl} T_{10} : [0,1[& \to & [0,1[\\ x & \mapsto & 10x - \lfloor 10x \rfloor \end{array}$$

- Consider the shift map σ on $A^{\mathbb{N}}$:

$$\sigma : (a_0 a_1 a_2 \ldots) \mapsto (a_1 a_2 a_3 \ldots)$$

- We have a **measurable isomorphism** :

$$\begin{array}{ccc} A^{\mathbb{N}} & \xrightarrow{\sigma} & A^{\mathbb{N}} \\ \downarrow & & \downarrow \\ [0,1[& \xrightarrow{T_{10}} & [0,1[\end{array}$$

- Exponential complexity, positive entropy.

Factor complexity

Let $w \in \mathcal{A}^{\mathbb{Z}}$. The **factor complexity** is a function $p_w(n) : \mathbb{N} \to \mathbb{N}$ counting the number of factors of length n in the sequence w.

$$w = ...000100\boxed{\mathbf{0100}}0100100010001000100100010001001...$$

$$Fact_w(4) = \{0001, 0010, \mathbf{0100}, 1000, 1001\} \quad \Longrightarrow \quad p_w(4) = 5$$

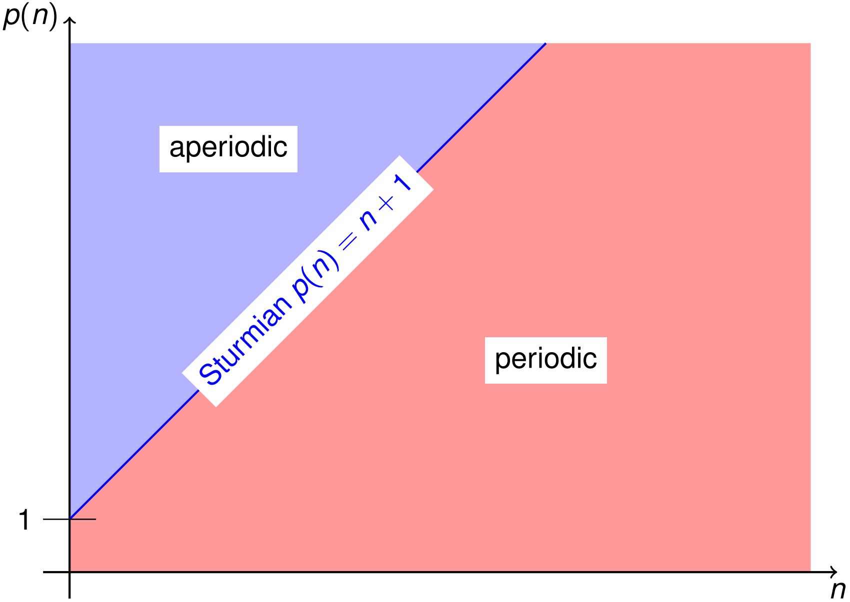
Lemma

An infinite word $w \in \mathcal{A}^{\mathbb{Z}}$ that has $p_w(n) \leq n$ factors of length n is **periodic**.

Definition

A **sturmian** word is an infinite word having exactly $p_w(n) = n+1$ factors of length n.

Words of complexity $p(n) = n + 1$



SYMBOLIC DYNAMICS II. STURMIAN TRAJECTORIES.*

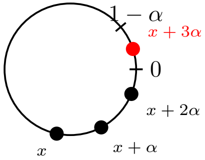
By Marston Morse and Gustav A. Hedlund. (1940)

- Let $S_\alpha \subset \{a, b\}^{\mathbb{Z}}$ be the set (subshift) of **Sturmian words** of slope $\alpha \in \mathbb{R} \setminus \mathbb{Q}$
- Consider the **rotation** on the circle $\mathbb{R}/\mathbb{Z} = [0, 1[$

$$\begin{array}{rcl} R_\alpha : [0,1[& \to & [0,1[\\ x & \mapsto & (x+\alpha) \quad \text{mod } 1 \end{array}$$

- We have a **measurable isomorphism** :

$$\begin{array}{ccc} S_\alpha & \xrightarrow{\sigma} & S_\alpha \\ \downarrow & & \downarrow \\ [0,1[& \xrightarrow{R_\alpha} & [0,1[\end{array}$$



- Sturmian words can be **desubstituted** by $0 \mapsto 0, 1 \mapsto 01$ or $0 \mapsto 1, 1 \mapsto 01$.

NOMBRES ALGÉBRIQUES ET SUBSTITUTIONS

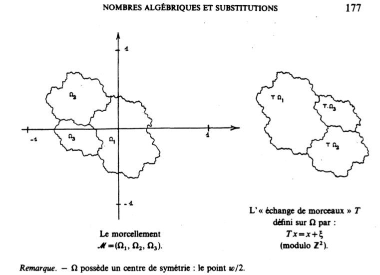
PAR

G. RAUZY (*)

(1982)

- Let X be the subshift generated by the language of the **Tribonacci** substitution $1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 1$
- We have a **measurable isomorphism** with a rotation on $\mathbb{R}^2/\mathbb{Z}^2$:

$$\begin{array}{ccc} X & \xrightarrow{\ \sigma\ } & X \\ \downarrow & & \downarrow \\ \mathbb{R}^2/\mathbb{Z}^2 & \xrightarrow{\ R_\xi\ } & \mathbb{R}^2/\mathbb{Z}^2 \end{array}$$



- **Open question (Pisot conjecture)** : is it true for every irreducible unimodular Pisot substitution ?

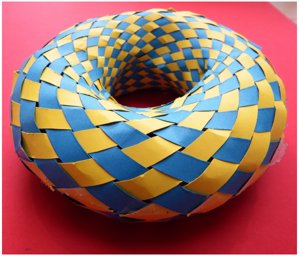
Walking on a torus

Conjecture

There exists a **measurable conjugacy** between the torus $\mathbb{R}^2/\mathbb{Z}^2$ and the Jeandel-Rao Wang shift Ω_0 :

$$\mathbb{R}^2/\mathbb{Z}^2 \to \Omega_0$$

Walking on Jeandel-Rao tilings is like walking on a **torus** :



Outline

Regular tetrahedron packing arrangement (2009)

Letter

Disordered, quasicrystalline and crystalline phases of densely packed tetrahedra

Amir Haji-Akbari, Michael Engel, Aaron S. Keys, Xiaoyu Zheng, Rolfe G. Petschek, Peter Palffy-Muhoray & Sharon C. Glotzer

"One of the simplest shapes for which the densest packing arrangement remains unresolved is the ***regular tetrahedron*** *[...]. Using a novel approach involving thermodynamic computer simulations that allow the system to evolve naturally* ***towards high-density states****, Sharon Glotzer and colleagues have worked out the* ***densest ordered packing yet for tetrahedra****, a configuration with a packing fraction of 0.8324. Unexpectedly, the structure is a* ***dodecagonal quasicrystal****, [...] "*

Source (2009) : `doi:10.1038/nature08641` ,
`https://www.quantamagazine.org/digital-alchemist-sharon-glotzer-seeks-rules-of-emergence-20170`

The Russian meteorite (2016)

"They ***loaded the minerals*** *found in the rock into a chamber, and then, using a four-meter-long propellant gun,* ***fired a projectile*** *into the stack of ingredients. [...] The findings [...] indicate that the* ***quasicrystals*** *in the Russian meteorite did indeed* ***form during a shock event****."*

Source (2016) : `https://www.quantamagazine.org/a-quasicrystals-shocking-origin-20160708/`