A self similar aperiodic set of 19 Wang tiles, Geom Dedicata, Aug 2018

Subst. structure of Jeandel-Rao aperiodic tilings, arxiv 1808.07768

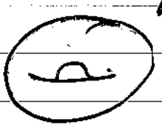
S. Labbé, CRM, Montréal, 27/09/2018

Jeandel-Rao 11 tiles: $T_0 = \{ \#0, \#1, \#2, \dots, \#10 \}$

Wang shift $\Omega_0 = \Omega_{T_0} = \{ \mathbb{Z}^2 \to T_0 : \text{valid Wang tilings} \}$

Known (JR, 2015) Ω_0 is aperiodic ie $\Omega_0 \neq \emptyset$, $\forall t \in \Omega_0$, t has no nontrivial period

Motivation



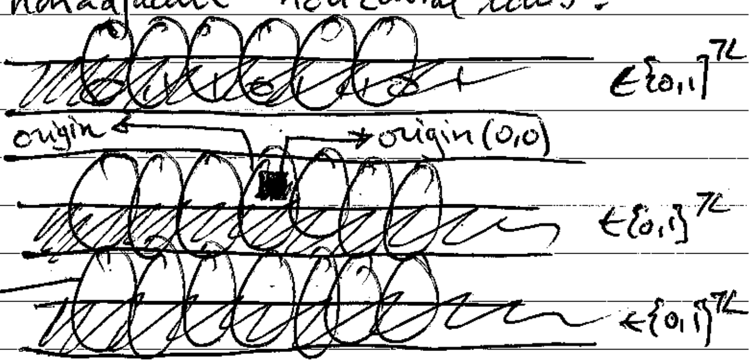
$\circlearrowleft \mathbb{Z}^2 \xleftrightarrow[\text{meas. conj.}]{} \Omega_0 \circlearrowleft \mathbb{Z}^2$

for $k \in \mathbb{Z}^2$, $t \in \Omega_0$, $\sigma^k t : \mathbb{Z}^2 \to T_0$, $x \mapsto t(x+k)$

Observations on T_0

	then $\notin \{0,1\}$	
then $\in \{0,1\}$	if $\in \{0,1\}$	then $\in \{0,1\}$
	then $\notin \{0,1\}$	

which means that tiles 0,1 form nonadjacent horizontal rows:



Desubstitute!

DEF T Wang tile set $\emptyset \neq M \subsetneq T$ is a subset of markers in direction e_2 if
$$M \odot^2 M, \quad M \odot^1 (T \setminus M), \quad (T \setminus M) \odot^2 M$$
are forbidden in Ω_T.

(notations from Charlier, Kärki, Rigo ~2005)

$\boxed{EX}$ $\{0,1\} \subset T_0$ is a subset of markers in direction e_2 for JR tiles

More observations

then $=9$
if $=0$

then $\in \{9,7,8,10\}$
if $=1$

$$\omega : \begin{cases} 0 \to \binom{9}{0}, \ 1 \to \binom{3}{1}, \ 2 \to \binom{7}{1}, \ 3 \to \binom{8}{1}, \ 4 \to \binom{10}{1}, \\ 5 \to 2, \ 6 \to 3, \ 7 \to 4, \ 8 \to 5, \ 9 \to 6, \ 10 \to 7, \ 11 \to 8 \\ \cancel{12 \to 9}, \ 12 \to 10 \end{cases}$$

$\boxed{\text{THM}}$ If $\exists$ markers $M \subset T$ in the direction e_i, then $\exists$ Wang tile set S_L and a 2-dim morphism

$$\omega_L : \Omega_{S_L} \to \Omega_T$$

s.t.

$$\omega_L(S_L) \subseteq (T|M) \cup (M \odot^i (T|M))$$

which is onto up to a shift

$$\left(\text{ie } \forall y \in \Omega_T \ \exists x \in \Omega_{S_L} \ \exists k \in \{0, e_i\} \text{ s.t. } y = \sigma^k \omega_L(x) \right)$$

and recognizable (ie assuming y_0 lies in the image of x_0 then x and k are unique).

$\boxed{\text{EX}}$

$$\omega : \begin{cases} \square = 0 \mapsto \begin{pmatrix} 9 \\ 0 \end{pmatrix} = \square, \quad \square = 1 \mapsto \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \square, \ \ldots, \\ \square = 5 \mapsto 2 = \square \end{cases}$$

In general ω has the form: $\begin{cases} u \boxminus^i v \mapsto u \odot^i v \\ u \mapsto u \end{cases}$ or

Algorithms (in 2nd paper)
- FINDMARKERS(T, i, r)
- FINDSUBSTITUTION(T, M, i, r)

$$\Omega_0 \xleftarrow{\omega_0} \Omega_1 \xleftarrow{\omega_1} \Omega_2 \xleftarrow{\omega_2} \Omega_3 \xleftarrow{\omega_3} \Omega_4$$

- 2 useless tiles in T_4
- No markers in T_4

π: $0,6 \to 0$, $1,5 \to 1$, $2 \to 2$, $3 \to 3$, $4 \to 4$ is an embedding!

$$\Omega_5 \xrightarrow{\pi} \Omega_4$$

$$\Omega_6 \xrightarrow{\eta} \Omega_5$$

$$\Omega_6 \xleftarrow{\omega_6} \Omega_7 \xleftarrow{\omega_7} \Omega_8 \xleftarrow{\omega_8} \Omega_9 \xleftarrow{\omega_9} \Omega_{10} \xleftarrow{\omega_{10}} \Omega_{11} \xleftarrow{\omega_{11}} \Omega_{12}$$

$$\Omega_{12} \xrightarrow{\sim} \Omega_U$$

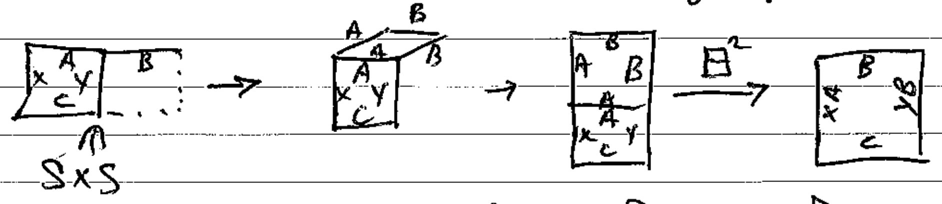
$$\boxed{\Omega_U \xleftarrow{\alpha} \Omega_V \xleftarrow{\beta} \Omega_W \overset{\gamma}{\underset{\sim}{\leftrightarrows}} \Omega_U}$$

1st paper

Ω_U is selfsimilar, minimal, aperiodic

Sheering top. conjugacy $\eta : \Omega_6 \to \Omega_5$

- No markers in Ω_5 because markers are on lines of slope 1
- Idea: for each domino



$\in T$

- leads to a top. conjugacy on tilings $\Omega_5 \leftrightarrow \Omega_T$

Results

$\boxed{\text{THM}}$ Let $X_i = \overline{\omega_i(X_{i+1})}^{\sigma}$ for $i = 0, \dots, 11$ $[\pi = \omega_4, \eta = \omega_5]$

- $X_{12} = \Omega_{12}$ is s.s. aperiodic minimal
- $X_i = \Omega_i$ is aperiodic, minimal $\forall i$, $5 \le i \le 11$
- $X_i \subsetneq \Omega_i$ is ——"—— $\forall i$, $0 \le i \le 4$
- Frequencies of tiles/patches in X_i $\forall i$

Conjecture: $\Omega_i \setminus X_i$ has measure 0, $\forall i$, $0 \le i \le 4$

$\Rightarrow X_0$ can be decomposed into self/similar patches of size 45, 70, 72, 112.