On Jeandel-Rao aperiodic tilings

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LOMA Theory Day
31 mai 2018
Laboratoire Ondes et Matière d'Aquitaine
Bordeaux, France

Outline

1. Wang tiles, aperiodicity and quasicrystals
2. The search for small aperiodic set of Wang tiles
3. Tilings coded by a $\mathbb{Z}^2$-action on a torus

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1. **Wang tiles, aperiodicity and quasicrystals**
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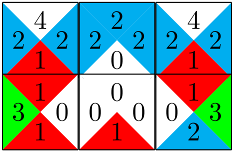
Wang tiles

A **Wang tile** is a square tile with a color on each border

Tile set T : a finite collection of such tiles.
A tiling of the plane : an assignment

$$\mathbb{Z}^2 \to T$$

of tiles on infinite square lattice so that the contiguous edges of adjacent tiles have the same color.



Note : rotation not allowed.

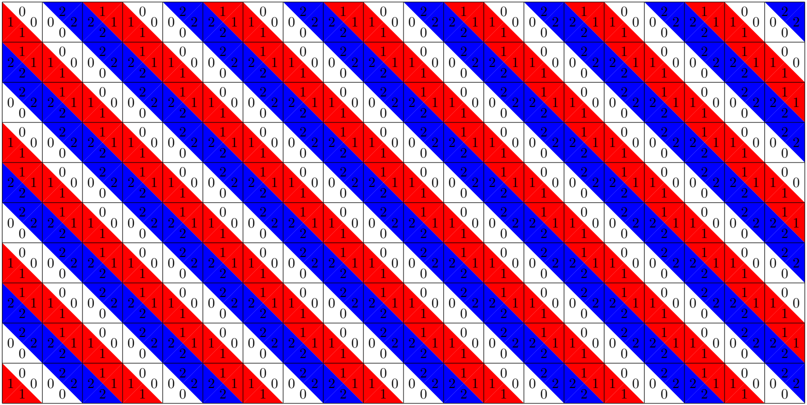
Eternity II puzzle (2007)

- A puzzle which involves **placing 256 square puzzle** pieces into a 16 by 16 grid constrained by the requirement to **match adjacent edges**
- A **2 million prize** was offered for the first complete solution
- **No solution found** before the competition ended on 31 Dec 2010
- At most $256! \times 4^{256} \approx 1.15 \times 10^{661}$ possibilities to check.

Source : `https://en.wikipedia.org/wiki/Eternity_II_puzzle`

Periods

A tiling is called **periodic** if it is invariant under some non-zero translation of the plane.



A Wang tile set that admits a periodic tiling also admits a **doubly periodic** tiling : a tiling with a horizontal and a vertical period.

Aperiodicity

A tile set is **finite** if there is no tiling of the plane with this set.
A tile set is **aperiodic** if it tiles the plane, but no tiling is periodic

Conjecture (Wang 1961)

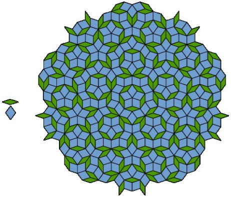
Every set of Wang tiles is either **finite or periodic**.

- 1966 (Berger) : There **exists an aperiodic** set of Wang tiles
- 1976 (Penrose) : discovered an aperiodic set of **two tiles**
- 1982 (Shechtman) : observed that certain aluminium-manganese alloys produced a **quasicrystals structure**
- 2011 : Dan Shechtman receives **Nobel Prize** in Chemistry

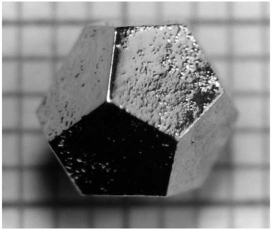
"*His discovery of quasicrystals revealed a **new principle for packing of atoms and molecules***", stated the Nobel Committee that "*led to a **paradigm shift** within chemistry*".

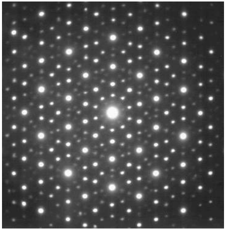
Quasicrystals

Penrose tiles and tiling :



A Ho-Mg-Zn **icosahedral quasicrystal** formed as a pentagonal dodecahedron and its **electron diffraction** pattern :





Source : `https://en.wikipedia.org/wiki/Quasicrystal`

Some notions and results (< 2000)

Definition

A discrete set X in $\mathbb{R}^d$ is a **Delone set** if it is uniformly discrete and relatively dense. It is called a **Meyer set** if the self-difference set $X - X$ a Delone set.

"The notion of Delone sets as fundamental objects of study in crystallography was introduced by the Russian school in the 1930's ; in particular, by Boris Delone [...]. One can think about a Delone set as an idealized model of an ***atomic structure*** *of a material [...]"*

Source : Boris Solomyak, `arxiv:1802.02370`

Theorem (Lagarias, Meyer)

Let X be a Meyer set in $\mathbb{R}^d$ such that $\eta X \subseteq X$ for a real number $\eta > 1$, then η is a **Pisot number** or a **Salem number**.

Other important results relates them to **cut-and-project set**, **regular tetrahedron packing**, **meteorites**, etc.

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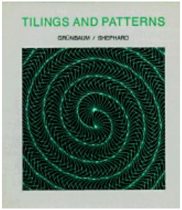
Discoveries of aperiodic Wang tile sets (< 2000)

Image credit : http://chippewa.canalblog.com/archives/2010/06/04/18115718.html

- 1966 (Berger) : 20426 tiles (lowered down later to 104)
- 1968 (Knuth) : 92 tiles
- 1971 (Robinson) : 56 tiles
- 1971 (Ammann) : 16 tiles
- 1987 (Grunbaum) : 24 tiles
- 1996 (Kari) : 14 tiles
- 1996 (Culik) : (same method) 13 tiles

Grünbaum, Shephard, Tilings and patterns, 1987



The reduction in the number of Wang tiles in an aperiodic set from over 20,000 to 16 has been a notable achievement. Perhaps the minimum possible number has now been reached. If, however, further reductions are possible then it seems certain that new ideas and methods will be required. The discovery of such remains one of the outstanding challenges in this field of mathematics. One can, of course, look at the problem from the opposite point of view. Is it possible to prove that, for example, 15 tiles are not enough? It is difficult to see how any such proof could be constructed, and the only result we know in this direction is an unpublished theorem of Robinson that no aperiodic set of *four* Wang tiles can exist.

Source : Grünbaum, Shephard, Tilings and patterns, 1987, p. 596.

Ammann set of 16 Wang tiles

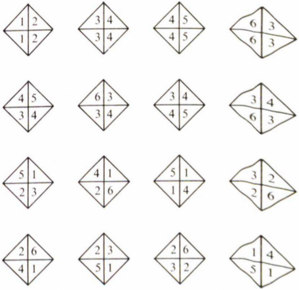


Figure 11.1.13
The 16 Wang tiles that correspond to the tiles of Figure 11.1.12.
These form the smallest known aperiodic set.

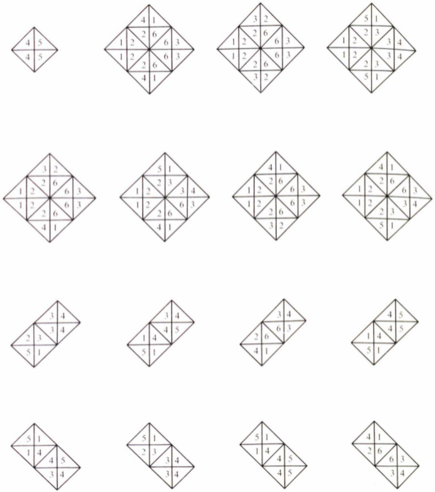


Figure 11.1.16
This diagram shows how the Wang tiles of Figure 11.1.13 can be "decomposed".

Grünbaum, Shephard, Tilings and patterns, 1987, p. 595-597.

Unique composition property in $\mathbb{R}^d$

Informally, two conditions that imply aperiodicity :

Ammann, Grünbaum, Shephard, 1992

Let $\mathcal{T}$ be a tile set. If

(a) in every tiling admitted by $\mathcal{T}$ there is a **unique way** in which the tiles can be grouped into patches which lead to a tiling by **supertiles** ; and

(b) the markings on the supertiles, inherited from the original tiles, imply a matching condition for the supertiles which is **exactly equivalent** to that originally specified for the tiles,

then $\mathcal{T}$ is **aperiodic**.

Mossé 1992 (on $\mathbb{Z}$) ; Solomyak 1998 (in $\mathbb{R}^d$)

A self-similar tiling has the **unique composition property** if and only if it is **nonperiodic**.

Computer assisted proofs

arXiv.org > math > arXiv:1708.00274

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Mathematics > Combinatorics

Exhaustive search of convex pentagons which tile the plane

Michael Rao

(Submitted on 1 Aug 2017)

We present an exhaustive search of all families of convex pentagons which tile the plane. This research shows that there are no more than the already 15 known families. In particular, this implies that there is no convex polygon which allows only non-periodic tilings.

ACTUALITÉ DE L'ENS DE LYON

PAR JOUR PAR SEMAINE PAR MOIS

PRIX LA RECHERCHE 2017 DANS LA CATÉGORIE "MATHÉMATIQUE" ATTRIBUÉ À MICHAËL RAO

Like 0

Date de publication

14/02/2018

Prix et distinction

Divers prix et récompenses 2017

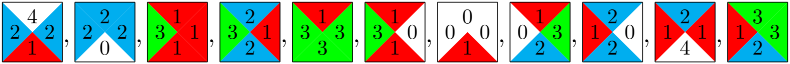
Discoveries of aperiodic Wang tile sets (2015)

- Jeandel, Rao : every set of ≤ 10 tiles is **finite or periodic**
- Jeandel, Rao : an aperiodic set of **11** Wang tiles

Their algorithm is pictured below :

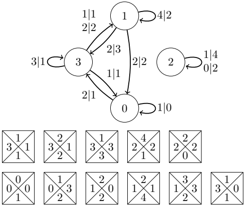
Image credit : Le Bagger 288, `http://i.imgur.com/YH9xX.jpg`

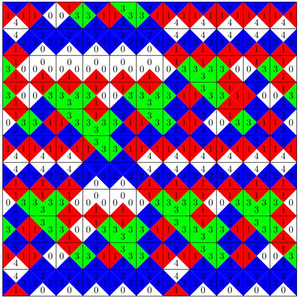
Jeandel-Rao 11 tiles set, `arxiv:1506.06492`

$\mathcal{T} = \{$  $\}.$

4 An aperiodic Wang set of 11 tiles - Proof Sketch

Using the same method presented in the last section, we were able to enumerate and test sets of 11 tiles, and found a few potential candidates. Of these few candidates, two of them were extremely promising and we will indeed prove that they are aperiodic sets.



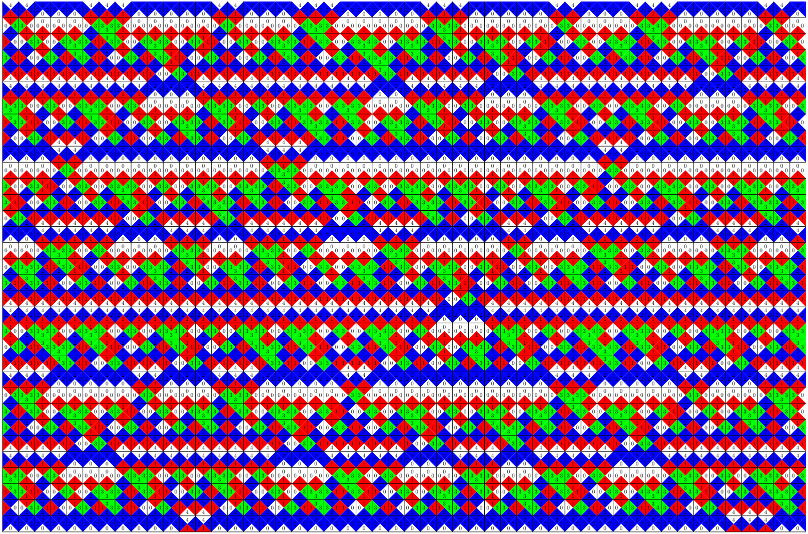


Theorem (Jeandel, Rao, 2015)

The 11 Wang tile set $\mathcal{T}$ is **aperiodic**.

Question

What is the **structure** of Jeandel-Rao aperiodic tiling ?



Outline

Decimal expansion

- Let $A = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ be digits
- Consider the **multiplication by 10** on the circle $\mathbb{R}/\mathbb{Z} = [0, 1[$

$$\begin{array}{rcl} T_{10} : [0,1[& \to & [0,1[\\ x & \mapsto & 10x - \lfloor 10x \rfloor \end{array}$$

- Consider the shift map σ on $A^{\mathbb{N}}$:

$$\sigma : (a_0 a_1 a_2 \dots) \mapsto (a_1 a_2 a_3 \dots)$$

- We have a **measurable isomorphism** :

$$\begin{array}{ccc} A^{\mathbb{N}} & \xrightarrow{\sigma} & A^{\mathbb{N}} \\ \downarrow & & \downarrow \\ [0,1[& \xrightarrow{T_{10}} & [0,1[\end{array}$$

- Exponential complexity, positive entropy.

SYMBOLIC DYNAMICS II. STURMIAN TRAJECTORIES.*

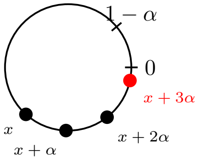
By MARSTON MORSE and GUSTAV A. HEDLUND. (1940)

- Let $S_\alpha \subset \{a,b\}^{\mathbb{Z}}$ be the set (subshift) of **Sturmian words** of slope $\alpha \in \mathbb{R} \setminus \mathbb{Q}$
- Consider the **rotation** on the circle $\mathbb{R}/\mathbb{Z} = [0,1[$

$$\begin{array}{rcll} R_\alpha : [0,1[& \to & [0,1[& \\ x & \mapsto & (x+\alpha) & \mod 1 \end{array}$$

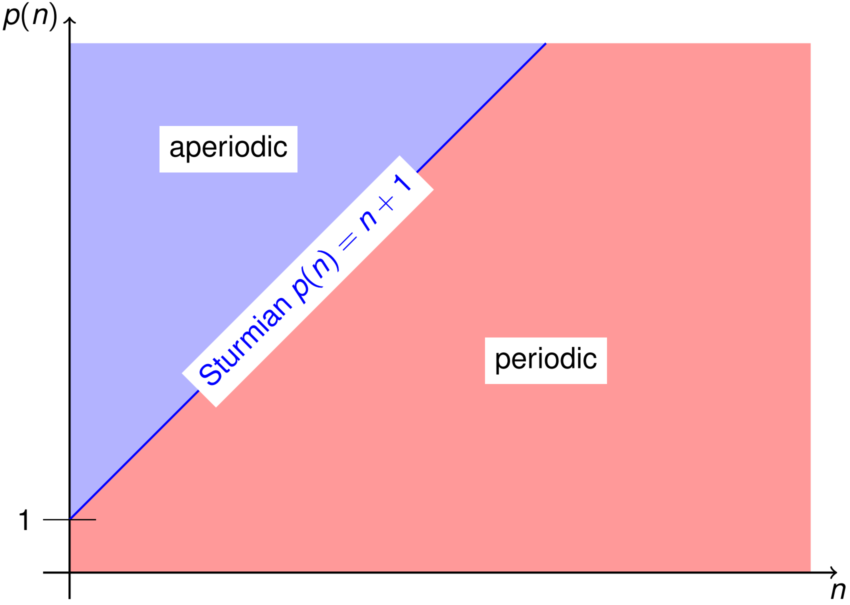
- We have a **measurable isomorphism** :

$$\begin{array}{ccc} S_\alpha & \xrightarrow{\sigma} & S_\alpha \\ \downarrow & & \downarrow \\ [0,1[& \xrightarrow{R_\alpha} & [0,1[\end{array}$$



- Sturmian words can be **desubtituted** by $0 \mapsto 0, 1 \mapsto 01$ or $0 \mapsto 1, 1 \mapsto 01$.

Sequences of complexity $p(n) = n + 1$



Wang tiles from codings of $\mathbb{Z}^2$-actions

- Let D be a **set** ,
- $u, v : D \to D$, two **invertible transformations** s.t. $u \circ v = v \circ u$,
- I and J : two finite **sets of colors**,
- $D = \cup_{i \in I} X_i$ and $D = \cup_{j \in J} Y_j$ be two **partitions** of D.

This gives the **left and bottom colors** :

$$\begin{array}{rccl} \ell : & D & \to & I \\ & \mathbf{x} & \mapsto & i \text{ if } \mathbf{x} \in X_i, \end{array} \qquad \begin{array}{rccl} b : & D & \to & J \\ & \mathbf{x} & \mapsto & j \text{ if } \mathbf{x} \in Y_j. \end{array}$$

and the **right and top colors** $r : D \to I$, $t : D \to J$ as :

$$r = \ell \circ u \quad \text{and} \quad t = b \circ v,$$

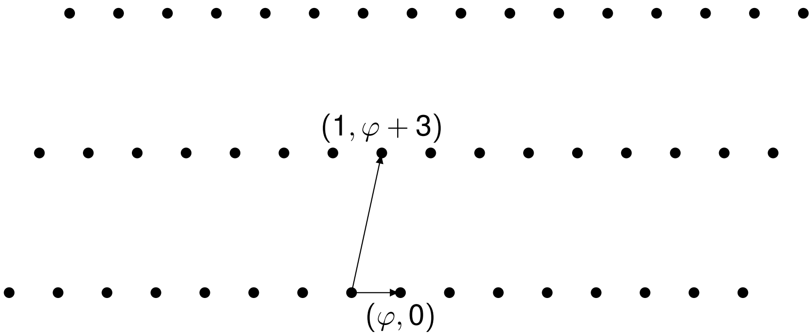
The **Wang tile coding** :

$$\begin{array}{rccl} c : & D & \to & I \times J \times I \times J \\ & \mathbf{x} & \mapsto & (r(\mathbf{x}), t(\mathbf{x}), \ell(\mathbf{x}), b(\mathbf{x})). \end{array}$$

Then $\mathcal{T} = c(D)$ is a **Wang tile set**.
For all $\mathbf{x} \in D$, $f_{\mathbf{x}} : (m, n) \mapsto c(u^m v^n \mathbf{x})$ is a **Wang tiling of the plane**.

Codings of $\mathbb{Z}^2$-actions : Example 3

Let $\varphi = \frac{1+\sqrt{5}}{2}$. Consider the **lattice** $\Gamma = \langle(\varphi, 0), (1, \varphi + 3)\rangle_{\mathbb{Z}}$.



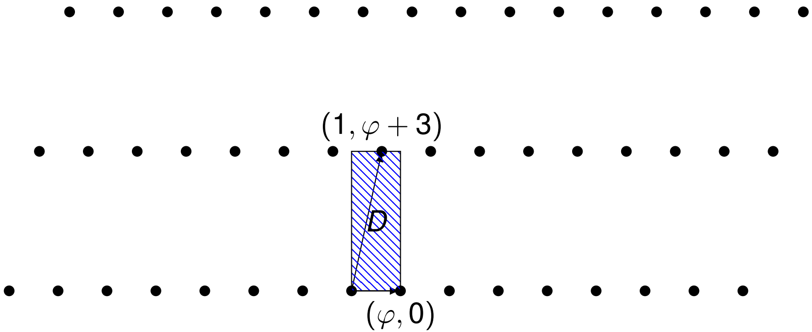
On the **torus** $\mathbb{R}^2/\Gamma$, we consider the **translations**

$$u : \begin{array}{ccc} \mathbb{R}^2/\Gamma & \to & \mathbb{R}^2/\Gamma \\ (x, y) & \mapsto & (x + 1, y) \end{array} \quad \text{and} \quad v : \begin{array}{ccc} \mathbb{R}^2/\Gamma & \to & \mathbb{R}^2/\Gamma \\ (x, y) & \mapsto & (x, y + 1). \end{array}$$

Codings of $\mathbb{Z}^2$-actions : Example 3

A **fundamental domain** of $\mathbb{R}^2/\Gamma$ is

$$D = [0, \varphi[\times [0, \varphi + 3[.$$

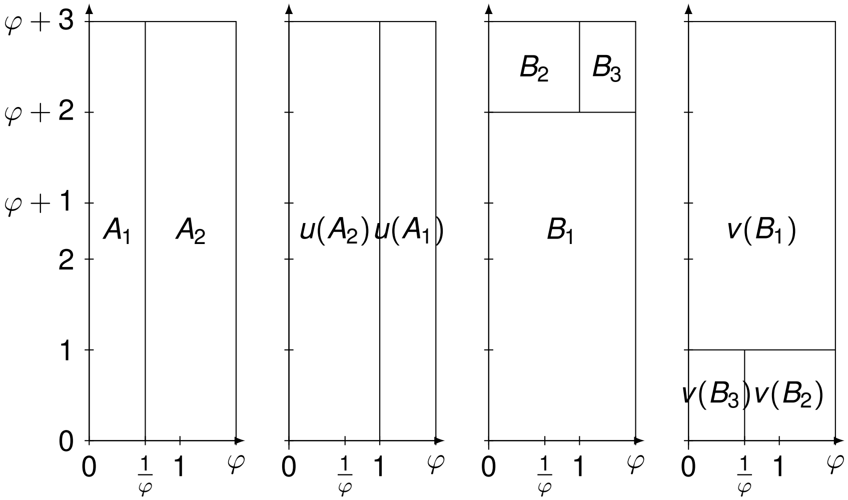


After renormalisation of transformations u and v on the torus $\mathbb{R}^2/\mathbb{Z}^2$, we observe that each translation vect. is not rationally independent :

$$\begin{pmatrix} \phi & 1 \\ 0 & \phi + 3 \end{pmatrix}^{-1} = \begin{pmatrix} \phi - 1 & -\frac{4}{11}\phi + \frac{5}{11} \\ 0 & -\frac{1}{11}\phi + \frac{4}{11} \end{pmatrix}.$$

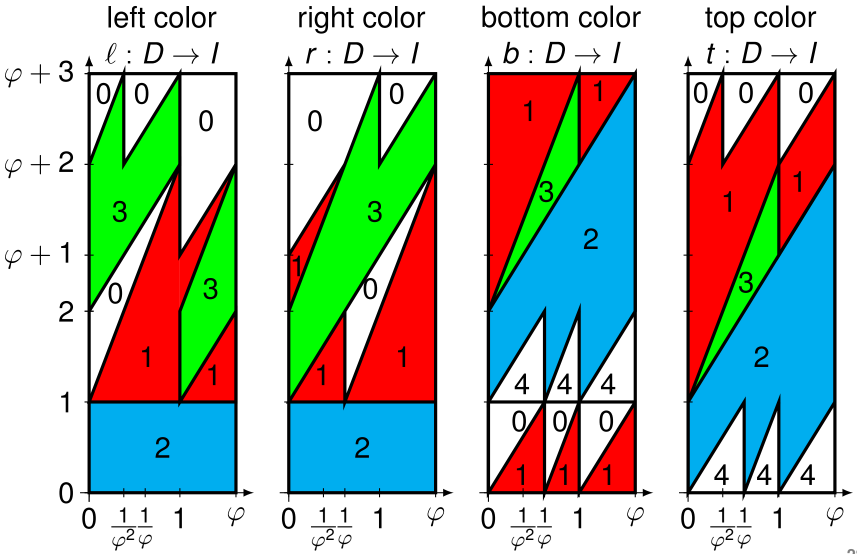
Codings of $\mathbb{Z}^2$-actions : Example 3

Transformations u and v are one-to-one **piecewise translations** of pieces on the fundamental domain D.



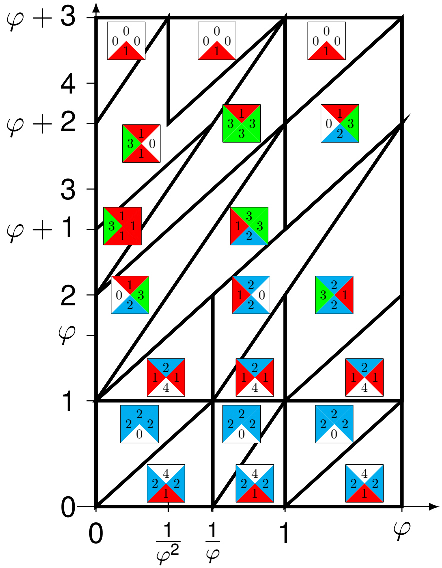
Codings of $\mathbb{Z}^2$-actions : Example 3

The **left, right bottom and top color codings** satisfying $r = \ell \circ u$ and $t = b \circ v$.



Codings of $\mathbb{Z}^2$-actions : Example 3

We deduce the **tile coding** $c : D \to \mathcal{T}$.



Theorem

$c(\mathbb{R}^2/\Gamma) = \mathcal{T}$ where $\mathcal{T}$ is the **Jeandel-Rao tile set**.

Theorem

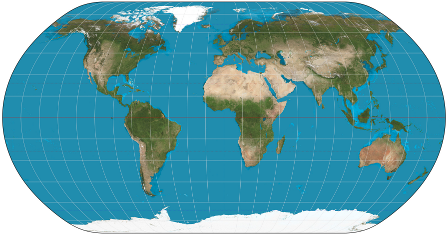
For every $\mathbf{x} \in \mathbb{R}^2/\Gamma$, $f_{\mathbf{x}} : \mathbb{Z}^2 \to \mathcal{T}$ is a **Jeandel-Rao Wang tiling of the plane**.

Conjecture

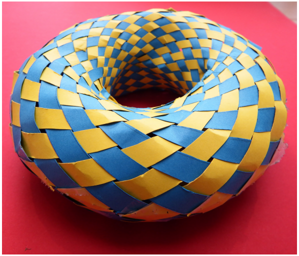
$(\Omega_{\mathcal{T}}, \sigma)$ is **measurably conjugate** to a $\mathbb{Z}^2$-rotation on the torus $\mathbb{R}^2/\mathbb{Z}^2$.

Summary of the contribution

Walking on the Earth is like walking on a **sphere** :



Walking on Jeandel-Rao tiling is like walking on a **torus** :



The Open Questions

- Can we generalize Jeandel-Rao tilings to **other Pisot numbers** ?
- For which $\mathbb{Z}^2$-translations on the torus does there exists a **SFT that computes** exactly their orbits ? Is it quadratic Pisot units ? Computable translations ?
- Find the **10 line proof** for the aperiodicity of Jeandel-Rao tilings.
- I think the fundamental domain can be identified with the **window** of a cut-and-project set with dimension $2 + 2$.
- What are the structure of the **other aperiodic tile sets** of cardinality 11 found by Jeandel-Rao ?
- Does there **exists an aperiodic** Wang shift with less than 13 tiles with positive entropy ?
- Does there **exists an aperiodic self-similar** Wang tile set of cardinality less than 16 ?

Sage code

Some of my code is **open-source** :

```
https://github.com/seblabbe/slabbe
```

It is part of the optional Sage package `slabbe-0.4.1` which can be **installed** with :

```
sage -pip install slabbe
```

I wrote new **Python modules** while working on this project :

```
sage: from slabbe import WangTileSet, WangTileSolver
sage: from slabbe import Substitution2d
```

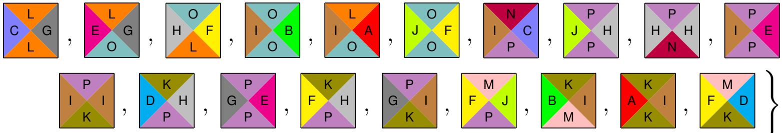
arXiv.org > math > arXiv:1802.03265

Mathematics > Dynamical Systems

A self-similar aperiodic set of 19 Wang tiles

Sébastien Labbé

(Submitted on 9 Feb 2018)

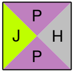
$\mathcal{U} = \{$  $\}$

Theorem

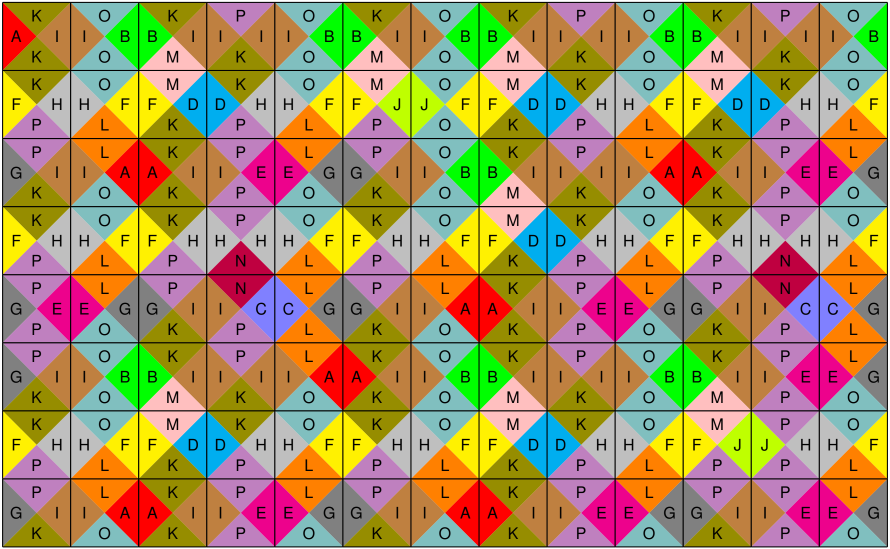
The Wang shift $\Omega_\mathcal{U}$ is **self-similar**, **aperiodic** and **minimal**.

$$\Omega_\mathcal{U} \xleftarrow{\alpha:\square\mapsto\square,\square\mapsto\boxminus} \Omega_\mathcal{V} \xleftarrow{\beta:\square\mapsto\square,\square\mapsto\square\square} \Omega_\mathcal{W} \xleftarrow{\gamma:\square\mapsto\square} \Omega_\mathcal{U}$$

Some patch in $\Omega_{\mathcal{U}}$

Let $\omega = \alpha\beta\gamma$ and $u_7 =$ 

, then $\omega^5(u_7) =$



Regular tetrahedron packing arrangement (2009)

Letter

Disordered, quasicrystalline and crystalline phases of densely packed tetrahedra

Amir Haji-Akbari, Michael Engel, Aaron S. Keys, Xiaoyu Zheng, Rolfe G. Petschek, Peter Palffy-Muhoray & Sharon C. Glotzer

"One of the simplest shapes for which the densest packing arrangement remains unresolved is the ***regular tetrahedron*** *[...]. Using a novel approach involving thermodynamic computer simulations that allow the system to evolve naturally* ***towards high-density states****, Sharon Glotzer and colleagues have worked out the* ***densest ordered packing yet for tetrahedra****, a configuration with a packing fraction of 0.8324. Unexpectedly, the structure is a* ***dodecagonal quasicrystal****, [...] "*

Source (2009) : `doi:10.1038/nature08641` ,
`https://www.quantamagazine.org/`
`digital-alchemist-sharon-glotzer-seeks-rules-of-emergence-20170`

The Russian meteorite (2016)

*"They **loaded the minerals** found in the rock into a chamber, and then, using a four-meter-long propellant gun, **fired a projectile** into the stack of ingredients. [...] The findings [...] indicate that the **quasicrystals** in the Russian meteorite did indeed **form during a shock event**."*

Source (2016) : `https://www.quantamagazine.org/a-quasicrystals-shocking-origin-20160708/`