WORDS AND TILINGS

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Talk #4 Generation of doublesquare tiles

with
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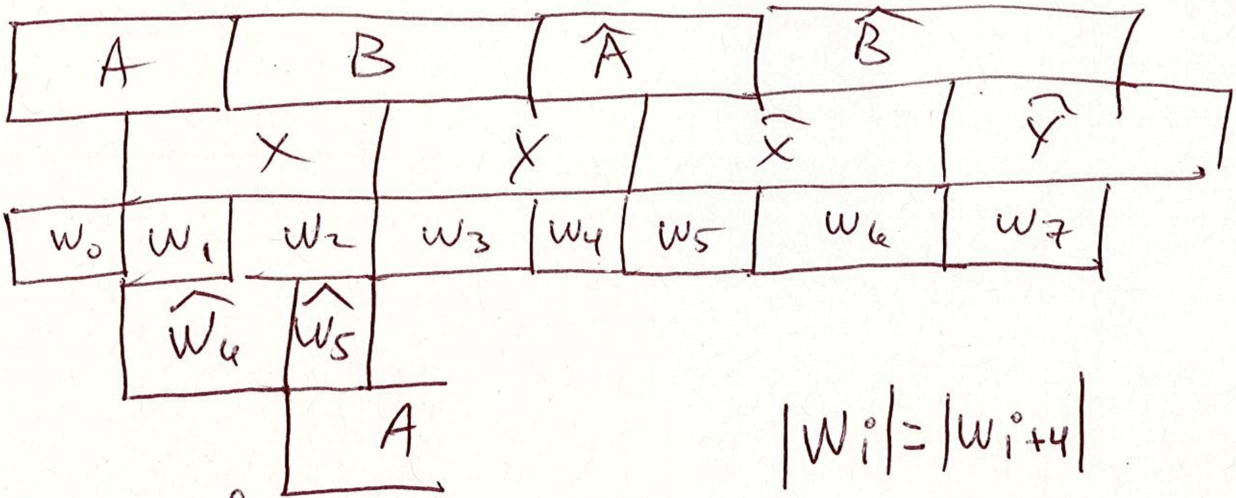
Let $A B \hat{A} \hat{B} \equiv X Y \hat{X} \hat{Y}$ be the factorizations of a double square tile.



$|w_i| = |w_{i+4}|$

In general,

$d_i = |w_{i-1}| + |w_{i+1}|$ is a period of $w_{i-1} w_i w_{i+1}$

Hence we write

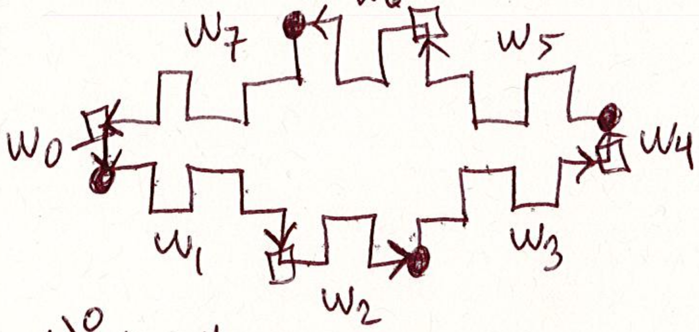
$w_i = (u_i v_i)^{n_i} u_i$ where $|u_i v_i| = |w_{i-1}| + |w_{i+1}|$

and $0 \leq |u_i| < |u_i v_i|$

Also $u_i v_i = \widehat{w_{i-3}} w_0$

$v_i u_i =$

EXAMPLE

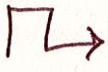


$w_3 = (u_3 v_3)^1 u_3$

$d_3 = |w_2| + |w_4| = 6$

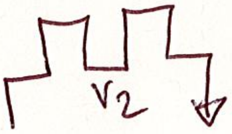
$\Rightarrow |u_3| = 8 - 6 = 2$

$\Rightarrow u_3 =$ 

$\Rightarrow v_3 =$ 

$w_2 = (u_2 v_2)^0 u_2 = u_2$

$d_2 = |w_1| + |w_3| = |u_2 v_2| = 16$

$\Rightarrow |v_2| = 11$, $v_2 =$ 

- **Proposition (Lothaire)** Let x, ~~y~~, z be nonempty words and y be a word s.t. $xy = yz$. Then there exists unique word u, v, $v \neq \varepsilon$, and integer $n \geq 0$ s.t. $x = uv$, $y = (uv)^n u$ and $z = vu$.

We have

$$\widehat{w_6}\, w_0 w_1 = \widehat{w_6}\, \widehat{w_5}\, \widehat{w_4} = w_1 w_2 \widehat{w_4}$$

$\Rightarrow \exists!\ u_1, v_1$, $v_1 \neq \varepsilon$, integer n_1 s.t.

$$\widehat{w_6}\, w_0 = u_1 v_1 \quad \text{and} \quad w_1 = (u_1 v_1)^{n_1} u_1$$
$$w_2 \widehat{w_4} = v_1 u_1$$

and similarly for all $i \in \mathbb{Z}_8$

Lemma
- $u_i v_i w_i = w_i v_i u_i$ (easy)
- $w_4 u_5 v_5 = \hat{u}_1 \hat{v}_1 w_4$ (13)

Proof $w_4 u_5 v_5 = w_4 \widehat{w_2}\, w_4 = \widehat{w_2 \widehat{w_4}}\, w_4 = \widehat{v_1 u_1}\, w_4 = \hat{u}_1 \hat{v}_1 w_4$

Then
- $\widehat{w_4}\, v_1 u_1 = \hat{v}_5 \hat{u}_5 \widehat{w_4}$

Lemma
- $\widehat{w_4}\, v_1 = \hat{v}_5 w_0$ (see the paper for the proof)

Lemma SWAP is well defined:

$$w_0' w_1' = \widehat{w_4}\, (v_1 u_1)^{n_1} v_1 = (\hat{v}_5 \hat{u}_5)^{n_1} \widehat{w_4}\, v_1 = (\hat{v}_5 \hat{u}_5)^{n_5} \hat{v}_5 w_0$$
$$= \widehat{w_5'}\, \widehat{w_4'} = \widehat{w_4' w_5'}$$

$w_2 \hat{u}_1 \hat{v}_1 w_4 = w_2 w_2 w_4 \cdots$ ✓

Operations on D.S $S=(w_0, w_1, \ldots, w_7)$ a fact. of a D

~~SHRIN~~K$_0(S) = \big((u_0v_0)^{n_0-1}u_0, w_1, w_2, w_3, (u_4v_4)^{n_4-1}u_4, w_5, w_6, w_7\big)$
TRIM

if $n_0 (= n_4) \geqslant 1$.

$\text{EXTEND}_0(S) = \big((u_0v_0)^{n_0+1}u_0, w_1, w_2, w_3, (u_4v_4)^{n_4+1}u_4, w_5, w_6, w_7\big)$

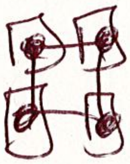
$$\text{SWAP}_0(S) = \big(\widehat{w}_4, (v_1u_1)^{n_1}v_1, \widehat{w}_6, (v_3u_3)^{n_3}v_3, \widehat{w}_0, (v_5u_5)^{n_5}v_5, \widehat{w}_2, (v_7u_7)^{n_7}v_7\big)$$

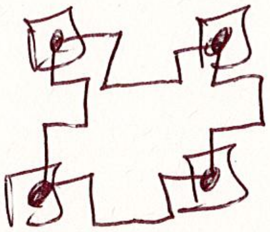
All credits given to Ariane Garon for SWAP idea.

$\boxed{\text{EXERCISE}}$ Prove that TRIM, EXTEND, SWAP PRESERVES Double squares.

Def $S=(w_0, \ldots, w_7)$ is singular if $\exists i$ s.t. $|w_i| + |w_{i+1}| =$

$\boxed{\text{EX}}$





Algorithm: Reduction of DS tiles

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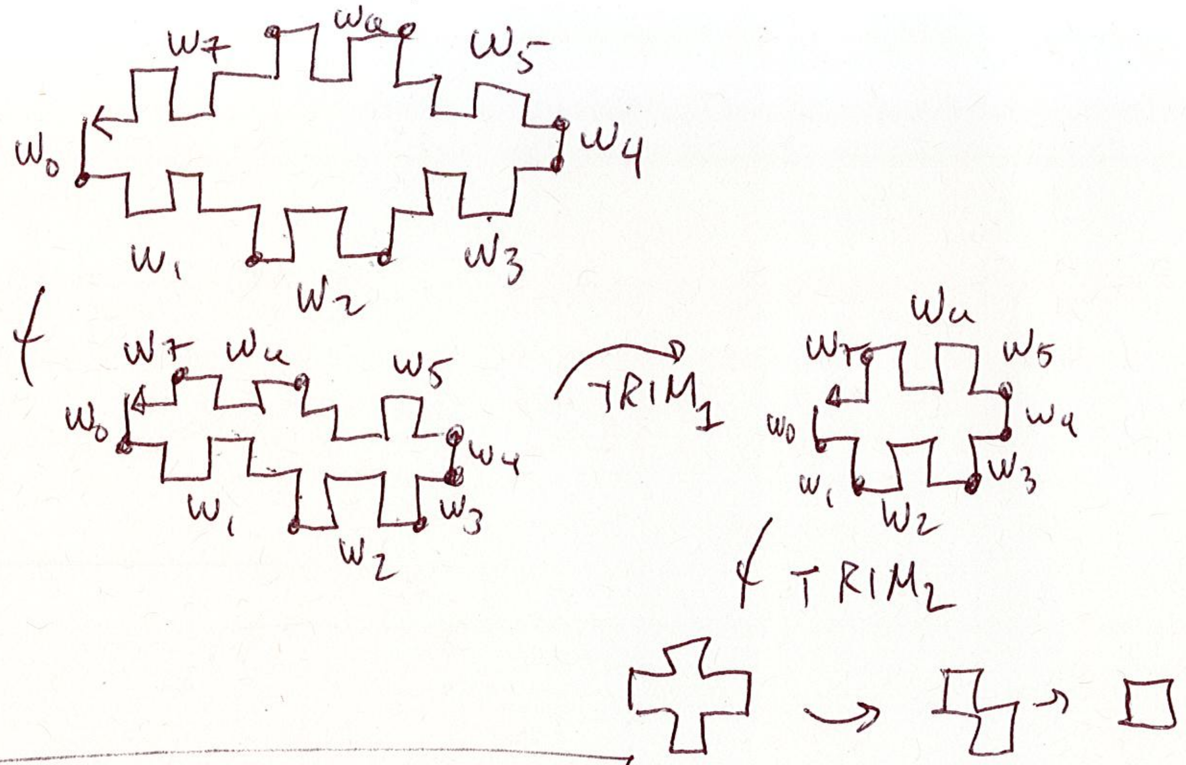
INPUT: $S=(w_0, \ldots, w_7)$   s.t. $\mathring{T}(S) = \pm 1$
OUTPUT: sequence of opererators {SHRINK$_i$, SWAP$_i$} + singular tile
while S not singular
   if $\exists i$ s.t. $d_i \leq |w_i|$ then
      S = ~~SHRIN~~K$_i$(S)
           TRIM
   else [Thm $\exists i$ s.t. |SWAP$_i$(S)| < |S|]
      S = SWAP$_i$(S)
   endif
endwhile.
```

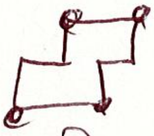
EX

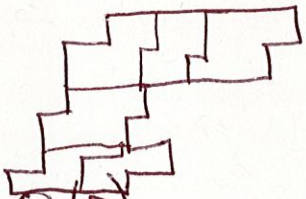


EXAMPLES WITH SAGE

Theorem Every DS-factorization of double square reduces to a singular DS-factorization.


P prime


Q square tile


Q(P) not prime

Theorem D double square tile. If D is prime, then D reduces to the unit square.

Theorem If D is a prime double square tile, then D is invariant under rotation of π.

Proof EXTEND and SWAP preserve palindromes.

Open questions

- Show that $S \mapsto (|w_0|, |w_1|, |w_2|, |w_3|)$ is injective
- Find algorithm decides whether a polyomino is prime
- If $\alpha\alpha$ factor of boundary word of double square D, $\alpha \in A$, then D is not prime
- Extends to ~~continuous~~ results to any shapes a polygons
- Prove that there are no triple hexagons tiles
- Converse of this
- Describe the set of all double hexagons
- // — double octogons (hyperbolic plane)
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- Describe 2-isohedral tiles
- Find a direct proof of that + generalize to double hexagons