WORDS AND TILINGS

Talk #3 At most two

March ~~April~~ 29, 2011
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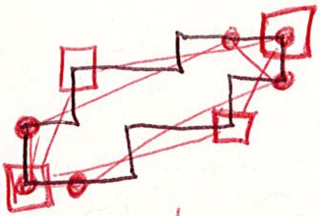
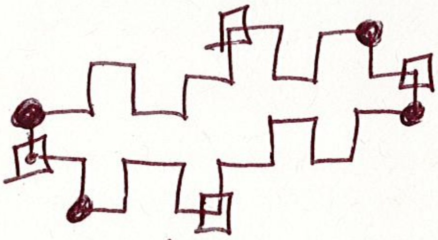
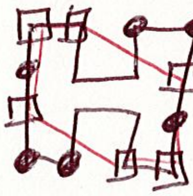


Square tile + Hexagone tile



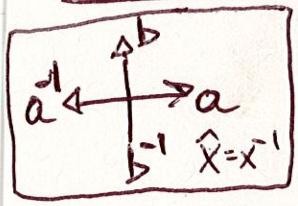
Double Square



Double Hexagon

DEF Two BN factorisations alternate if • □ the factorization point alternate on the boundary word.

Question Can we find a polyomino with three independant regular tilings? i.e. a boundary word w

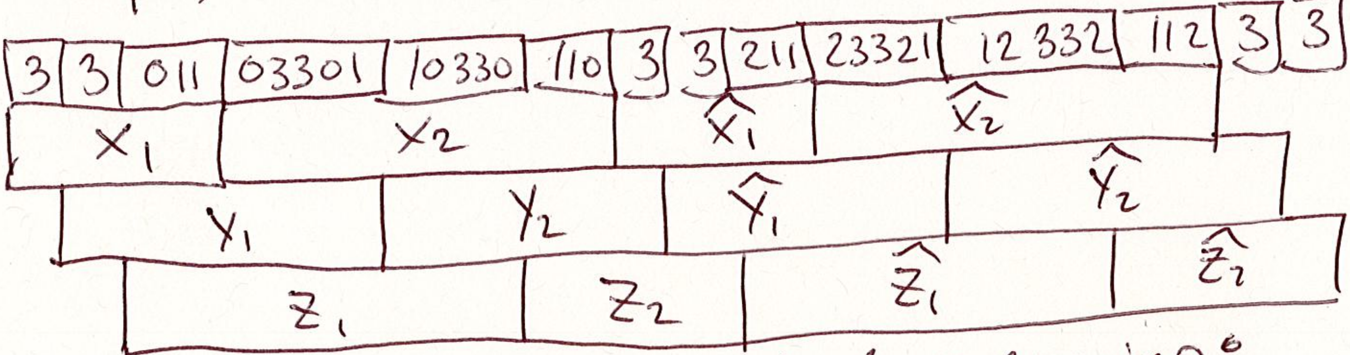


s.t. $w = x_1 x_2 x_3 \hat{x}_1 \hat{x}_2 \hat{x}_3 \equiv y_1 y_2 y_3 \hat{y}_1 \hat{y}_2 \hat{y}_3 \equiv z_1 z_2 z_3 \hat{z}_1 \hat{z}_2 \hat{z}_3$

Tarski Question 1945 asked whether for free groups on two or more generators, the first order theory is decidable.

$\boxed{\text{YES}}$ Makanin 1983: 1 equation; Rasborov 1985: syst. of equations; Kharlampovich and Myasnikov (2006)

For example, $x_1 x_2 \hat{x}_1 \hat{x}_2 \equiv y_1 y_2 \hat{y}_1 \hat{y}_2 \equiv z_1 z_2 \hat{z}_1 \hat{z}_2$ has a solution:



but this is not the boundary word of a polyomino:

$\overrightarrow{110332} = (0,0)$

$A=\{0,1,2,3\}\cong\mathbb{Z}/4\mathbb{Z}$

Def $w\in A^*$ is closed if $\vec{w}=(0,0)$

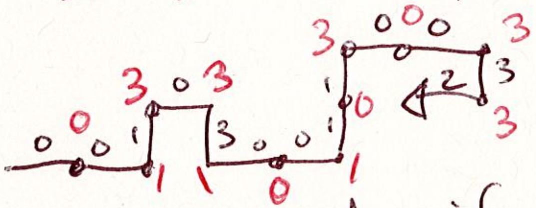
Def $w\in A^*$ is simple if no nonempty proper factor of w is closed, i.e. $w=pus \Rightarrow \vec{u}\neq(0,0)$
$w\neq u$
$u\neq\varepsilon$

simple — closed and simple — closed notsimple

Def ~~Remark~~ Let $w=w_0w_1\cdots w_{n-1}\in A^*$. We define the sequence of turns:

$$\Delta(w)=(w_1-w_0)(w_2-w_1)\cdots(w_{n-1}-w_{n-2})$$
$$\mathring{\Delta}(w)=(w_1-w_0)(w_2-w_1)\cdots(w_{n-1}-w_{n-2})(w_0-w_{n-1})$$

$\boxed{\text{EX}}$



$w=0010300110032$
$\Delta(w)=013310103033$

Def w is reduced if $w\in A^*\setminus A^*\{02,20,13,31\}A^*$

If w is reduced, then $\Delta(w)\in\{0,1,3\}$
$0\to$ going straight
$1\to$ left turn
$3\to$ right turn

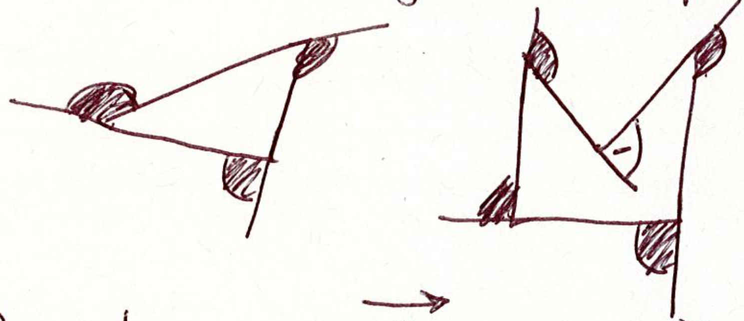
Def the turning number of a reduced word $w\in A^*$ is $T(w)=\dfrac{|\Delta(w)|_1-|\Delta(w)|_3}{4}$

or $\mathring{T}(w)=\dfrac{|\mathring{\Delta}(w)|_1-|\mathring{\Delta}(w)|_3}{4}$ for circular words w

Prop The turning number of a simple closed reduced word $w\in A^*$ is $\mathring{T}(w)=\pm1$.

Proof follows from the Polygon exterior angle sum theorem

theorem The sum of the measures of exterior angles of a polygon is 2π.



Remarks

- $\overrightarrow{\hat{w}} = -\overrightarrow{w}$
- w, w' conjugates $\Rightarrow \overrightarrow{w} = \overrightarrow{w'}$
- $T(w) = -T(\hat{w})$, $w \in A^*$ reduced word
- $T(ab) = -T(\bar{b}\bar{a})$, $a, b \in A$

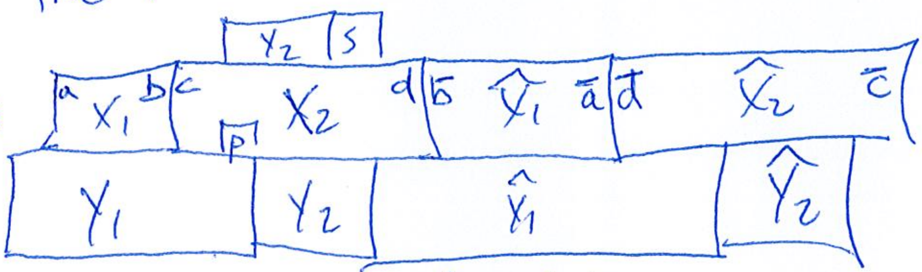
Alternating factorisation

Lemma If the conjugates of a simple closed reduced word $w \in W$ have two distinct square factorisation

$$w \equiv X_1 X_2 \hat{X}_1 \hat{X}_2 \quad \text{et} \quad w \equiv Y_1 Y_2 \hat{Y}_1 \hat{Y}_2$$

then the two fact. must alternate.

Proof $w \equiv$



case 1 y_2 not centered factor of X_2

$\Rightarrow \exists u = y_2 s = p y_2$ with s, and p non empty

$\Rightarrow$ s and p are conjugates

s suffix of Y_2, $\hat{p}$ prefix of $\hat{Y}_1$

$\Rightarrow s\hat{p}$ proper factor of w, because $|X_1| > 0$

$s\hat{p}$ is a closed factor.

$$\overrightarrow{s\hat{p}} = \vec{s} + \vec{\hat{p}} = \vec{s} - \vec{p} = \vec{0}$$

Case 2 y_2 centered factor of X_2.

Let a, b be the fst a last letter of X_1

c, d ——— // — X_2.

$$\mathring{T}(w) = \mathring{T}(X_1) + T(X_2) + T(\hat{X}_1) + T(\hat{X}_2) + T(bc) + T(d\bar{b}) + T(\bar{a}\bar{d}) + T(\bar{c}a)$$

Centered factor $\Rightarrow c = \bar{d}$

$$= T(bc) + T(\bar{c}\bar{b}) + T(\bar{a}c) + T(\bar{c}a) = 0 \;\lightning$$

$$= 0$$

$\lightning$

Thm (BMBL, 2012) Every polyomino yields at most two distinct square tilings.

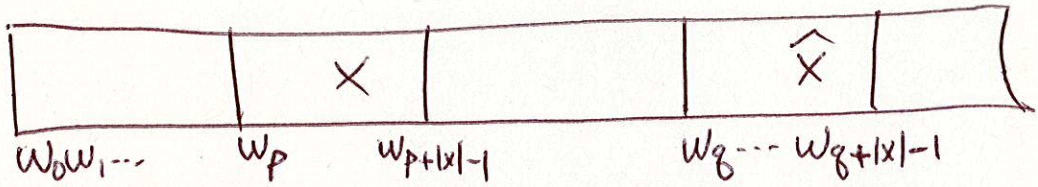
Idea of Proof

Let $w = w_0 w_1 \cdots w_{n-1} \in A^*$ boundary word s.t.

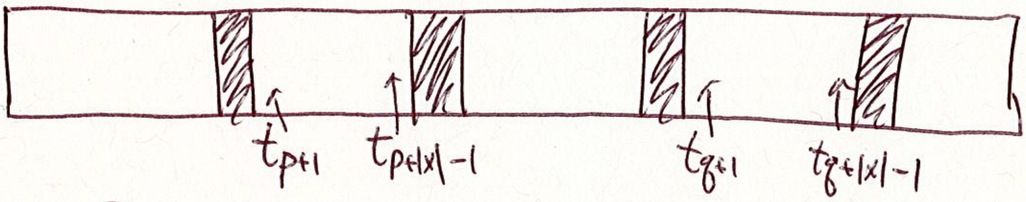
$$w \equiv X_1 X_2 \hat{X}_1 \hat{X}_2 \equiv Y_1 Y_2 \hat{Y}_1 \hat{Y}_2 \equiv Z_1 Z_2 \hat{Z}_1 \hat{Z}_2$$

Let $t = \mathring{\Delta}(w) = t_1 \cdots t_{n-1} t_n$

① $w =$



$t = \Delta(w) =$



$$R_{x,p,q} = \{(p+i,\, q+|x|-i) : 1 \le i \le |x|-1\}$$

② These relations are subsets of edges of a bigger graph $\Gamma = (V,E)$ with $V = \{1, \ldots, n\}$

$$\text{edges} = E = \left\{ (i,j) : \begin{array}{l} t_i = 1 \text{ and } t_j = 3 \text{ or} \\ t_i = 3 \text{ and } t_j = 1 \end{array} \right\}$$

This is a complete bipartite graph

③ We show the existence of an odd-length (5 in fact) path from a left turn to some other left turn in that graph Γ. ↯ ∎