WORDS AND TILING

Talk #2 Double Squares Tiles

~~April~~ March 28, 2017
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1 Tiling by translations

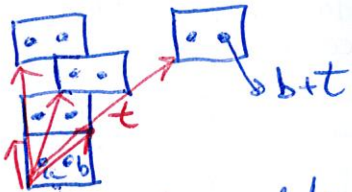
G abelian group

$A, B, C \subseteq G$, $A+B = \{a+b \mid a\in A, b\in B\}$

$A \oplus B = C$ if $\forall c\in C\ \exists!\, a\in A, b\in B$ s.t. $c = a+b$

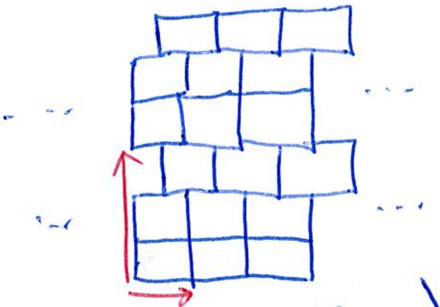
Def $d \geq 1$, $F \subseteq \mathbb{Z}^d$ finite set. A tiling of $\mathbb{Z}^d$ by translations of F is a subset $T \subseteq \mathbb{Z}^d$ s.t. $F \oplus T = \mathbb{Z}^d$

$\boxed{\text{EX}}$



Def A tiling by translation T is fully periodic if $\exists$ finite index subgroup $\Lambda \subseteq \mathbb{Z}^d$ s.t. $T + \Lambda = T$.

$\boxed{\text{EX}}$



$\Lambda = \langle (2,0), (0,3) \rangle_{\mathbb{Z}}$

$T = \{(0,0), (0,1), (1,2)\} + \Lambda$

Rmk fully periodic $\not\Rightarrow$ isohedral
$\Rightarrow$ k-isohedral for some k

Def A tiling T is regular if $T = \Lambda$ for some finite index subgroup Λ of $\mathbb{Z}^d$.

Thm (Bhattacharya, 2016) If a finite set $F \subset \mathbb{Z}^2$ tiles $\mathbb{Z}^2$ by translations, then it also admits a periodic tiling.
~~Open~~ proof uses ergodic theory. Still open for $d \geq 3$.

Cor The problem whether a finite set F tiles $\mathbb{Z}^2$ by translation is decidable.

Exercise Find the algorithm!

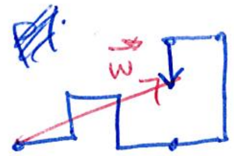
2. Boundary criterion

$A = \{0,1,2,3\} = \mathbb{Z}/4\mathbb{Z}$

Rotation $\rho : A^* \to A^*$ morphism defined by $a \mapsto (a+1) \bmod 4 \ \forall a \in A$

$\rho(uv) = \rho(u)\rho(v)$

$\boxed{EX}$ $w = 0103001122$ $\quad \rho(w) = 1210112230$

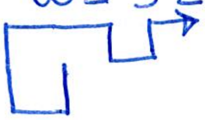


rotation of angle $\frac{\pi}{2}$

Reversal $\sim : A^* \to A^*$ anti morphism $a \mapsto a, \ \forall a \in A$

$\widetilde{uv} = \tilde{v}\tilde{u}$

$\boxed{EX}$ $\tilde{w} = 3211003010$



Backward $\hat{w} := \rho^2(\tilde{w}) = \widetilde{\rho^2(w)}$

$\hat{w} = 1033221232$

Vector $w \in A^*$, $\vec{w} = (|w|_0 - |w|_2, |w|_1 - |w|_3) \in \mathbb{Z}^2$

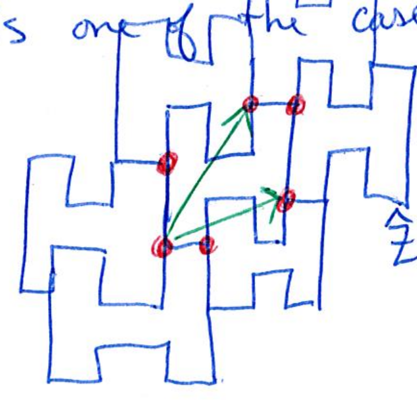
Theorem (Beauquier, Nivat, 1991) If a polyomino P tiles $\mathbb{Z}^2$ by translation, then P admits a regular tiling T and the boundary word w of P can be factorized as

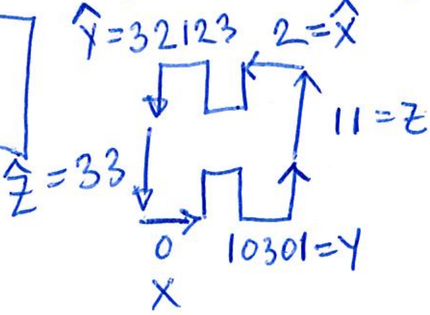
$$w \equiv XYZ\hat{X}\hat{Y}\hat{Z} \quad (1)$$

with $X, Z \in A^+$ and $Y \in A^*$ and $T = \langle \overrightarrow{XY}, \overrightarrow{YZ} \rangle_{\mathbb{Z}}$.

Remark (1) is one of the cases of isohedral tilings classified by Heesch and Kienzle (1963)

$\boxed{EX}$





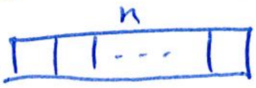
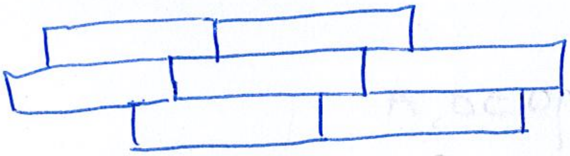
$\overrightarrow{XY} = (3,1)$

$\overrightarrow{YZ} = (2,3)$

3. Number of regular tilings of a polyomino

Can a polyomino P have more then one regular tilings?

Yes, a lot:

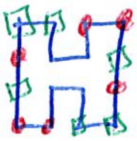
$P = $  

$$T_i = \langle (n,0), (i,1) \rangle \quad , \quad 0 \leq i < n$$

P has n distinct tilings.

<u>DEF</u> Tilings T and T' of P are <u>independant</u> if

$$T_1 \cap T_2 = \{\vec{0}\}$$

$\boxed{EX}$  has two independant tilings

$T = \langle (3,1), (2,3) \rangle_{\mathbb{Z}}$ and $T' = \langle (1,2), (3,-1) \rangle$

~~Question Can a tile P have more than two mutually independant regular tilings?~~

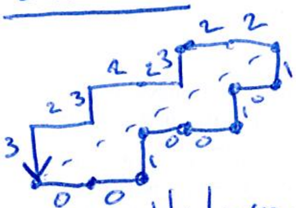
4. Double Squares and hexagons

<u>DEF</u> A tile P is a <u>double hexagon</u> if it has two independant tilings.

and <u>double square</u> if both tiling are of the form $X Z \hat{X} \hat{Z}$.

<u>Thm (Pirillo, 2001)</u> $w = 0m1$ is Christoffel $\Longleftrightarrow$ $w \equiv 1m0$

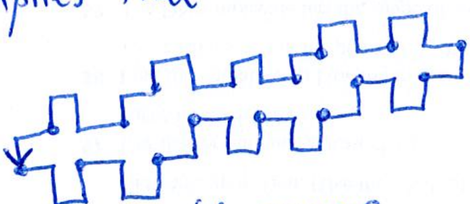
<u>Exercise</u>: Show that this implies that Christoffel snakes are double hexagons tiles



double hexagon

$$\xrightarrow{\lambda}$$

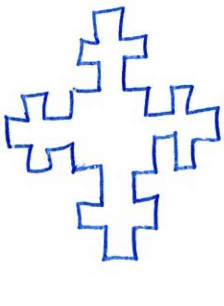
$0 \mapsto 0301$
$1 \mapsto 01$
$2 \mapsto 2123$
$3 \mapsto 23$



double square

<u>Thm (BBMGL)</u> $w \equiv 0m1 \in \{0,1\}^*$ is a Christoffel word iff $\lambda(w\, e^2(w))$ is the boundary word of a double square tile.

<u>Proof</u> EXERCISE OR DONE LATER

<u>Exercise</u> Complete the sequence: , , , ...?