1. Preliminaries

Let \( x = (x_1, x_2, \ldots, x_d) \in \mathbb{Z}^d \).

\[ \|x\|_1 = |x_1| + |x_2| + \ldots + |x_d| \]

\[ \|x\|_\infty = \max \{ |x_i| : 1 \leq i \leq d \} \]

**Ex**

\( d=2 \)

\[ \|x\|_1 = 1 \iff x \in \{ (1,0), (0,1), (0,-1), (-1,0) \} \]

\[ \|x\|_\infty = 1 \iff x \in \{ (1,0), (0,1), (0,-1), (-1,0), (1,1), (-1,-1), (-1,1), (1,-1) \} \]

**Def** A sequence \( P_0, P_1, \ldots, P_n \in \mathbb{Z}^d \) is \( 1 \leq k \leq d \)

A **\( k \)-connected path** if \( \|P_i - P_{i-1}\|_\infty \leq 1 \)

and \( \|P_i - P_{i-1}\|_1 \leq k \ \forall i \) s.t. \( 1 \leq i \leq n \).

**Ex**

\( d=2 \)

\[ \]

is \( 1 \)-connected path

\[ \]

2-connected path

**Def** \( P \subseteq \mathbb{Z}^d \) is **\( k \)**-connected if \( \forall x, y \in P \)

\( \exists \ \text{\( k \)-connected path} \ \text{(} P_i \text{)}_{i \in \mathbb{N}} \text{ s.t. } x = P_0, \ y = P_n \)

and \( P \subseteq P \ \forall i \).

**Def** \( P \subseteq \mathbb{Z}^d \) is a **polyomino** if

1) \( P \) is finite

2) \( P \) is \( 1 \)-connected

3) \( \mathbb{Z}^d \setminus P \) is \( 1 \)-connected

**Representation of a polyomino:**

\[ \dot{\bullet} \quad \bullet \quad \bullet \quad \bullet \quad \bullet \]

\( P \subseteq \mathbb{Z}^2 \)

\[ P \cup [0,1]^2 \subseteq 1R^2 \]

\[ P + [-\frac{1}{2}, \frac{1}{2}]^2 \subseteq 1R^2 \]
Def The boundary word of a polyomino \( P \subseteq \mathbb{Z}^d \) is a word \( w = \{0, 1, 2, 3\}^* \) coding the 1-connected path of \( P \) along the boundary of \( P + \{ -\frac{1}{2}, \frac{1}{2} \} \mathbb{Z}^2 \).

\[ w = 12101000301003323222 \]

Def Two words \( w \) and \( w' \in A^* \) are conjugate if \( \exists \; u, v \in A^* \) so that \( w = uvv' \).

 Conjugaity of words is an equivalence relation \( : w \equiv w' \).

Conjugacy of words is an equivalence relation \( : w \equiv w' \).

Exercise: \( w \equiv w' \iff \exists \; z \; s.t. \; wz = zw' \).

2. Digital geometry

DG's goal is to adapt Euclidean Geometry on \( \mathbb{Z}^d \).

Straight lines, circles, planes, hyperplanes, rotations, etc.

Question: When is a polyomino convex?

(Appleby, combinatorics, Lothaire) suggest when Perimeter = 2H + 2L allowing

Def \( P \subseteq \mathbb{Z}^d \) is digitally-convex if

\[ P \subseteq \text{CONVEXHULL}_{\mathbb{Z}^d}(P) \cap \mathbb{Z}^d \subseteq P \]

Def \( S \subseteq \mathbb{Z}^d \)

\[ \text{CONVEXHULL}(S) = \left\{ \sum_{i=1}^{n} \alpha_i P_i \mid \begin{array}{l} P_1, \ldots, P_n \in S \\ 0 \leq \alpha_i \leq 1 \\ \alpha_1 + \alpha_2 + \ldots + \alpha_n = 1 \end{array} \right\} \]
3. Christoffel Words

- Jean Benoulli (1771), first apparition
- Jean Berstel (1960), first to call them Christoffel words
- Elwin B. Christoffel (1829–1900) mathematician/physicist

Let \( p, b \in \mathbb{Z} \) coprime. The Christoffel word of parameter \((p, b)\) is the word \( w \in \{0, 1\}^* \) coding the \( L \)-connected path of unitary steps \((1, b)\) and \((0, 1)\) from the origin \((0, 0)\) to \((p, b)\) strictly below the segment \((0, 0) \rightarrow (p, b)\) and the nearest possible to this segment.

If \((p, b) \neq (0, 1)\), then \( w \) is of the form \( w=0m1 \) with middle word \( m \in \{0, 1\}^* \).

Lemma \( m \) is a palindrome.

Proof: translation \( (-1, 1) \) keeps \( m \) invariant, rotation \( 180^\circ \) preserves the figure. \( \square \)

see Berstel (2007): 14 characterization of C. words.

Exercise 2: Write an algorithm that produces/recognizes Christoffel words.

4. Lyndon Words

introduced by Roger Lyndon (1917–1988) in 1954 under the name of "standard lexicographic sequences" used to construct bases of certain free abelian groups.

Def A Lyndon word is a word \( e^A \) which is strictly less than any of its conjugates (circular permutation).

\( \exists x \in \{0, 1\}^* \) such that \( x \preceq \text{atoire} \), then \( \text{atoire} \) is not Lyndon.

Lyndon words on \( \{0, 1\} \) are:

- \( 0, 1, 01, 011, 001, 0011, 0111, 0001, \ldots \)
Exercise Prove that a C. word is a L. word.

Prop (Frederickson, Maiorana, 1978) The concatenation in increasing order of the Lyndon words over alphabet \( \mathcal{A} \) of length dividing \( n \), \( \forall n \geq 1 \), is a de Bruijn sequence, i.e., a word where every word of \( \mathcal{A}^n \) appears once and only once.

\[ A = \{01^n \} \quad 0 \quad 0001 \quad 0011 \quad 01 \quad 0111 \quad 1 \]

Can be computed in linear time and logarithmic space (Berstel, Perrin, 2007).

Thm (Lyndon, 1954) Every nonempty word \( W \) admits a unique factorization as a lexicographically decreasing sequence of Lyndon words

\[ W = l_1^n l_2^{n_2} \cdots l_k^{n_k}, \quad l_1 > l_2 > \cdots > l_k, \quad n_i > 1 \] where \( n_i > 1 \) and \( l_i \) are Lyndon words, \( \forall 1 \leq i \leq k \).

Exercise Enumerate the Lyndon words, factorization into Lyndon words.

S. Convexity

Theorem (Bri..K, Lachaud, Provençal, Percinman, 2009)

A word \( W \in \{0,1\}^* \) codes the NW part of a digitally convex polyomino iff its unique factorization

\[ W = l_1^n l_2^{n_2} \cdots l_k^{n_k} \]

is such that every \( l_i \) is a Christoffel word.

\[ w = 1(01)(01)^2(0001)(0) \]