WORDS and TILINGS
Talk #1: Convexity

~~April~~ March 27, 2017
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1. Preliminaries

Let $x = (x_1, x_2, \dots, x_d) \in \mathbb{Z}^d$.

$$\|x\|_1 = |x_1| + |x_2| + \dots + |x_d|$$

$$\|x\|_\infty = \max\{|x_i| : 1 \le i \le d\}$$

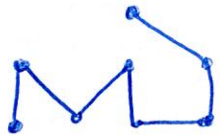
$\boxed{\text{Ex}}$ $d=2$ $\|x\|_1 = 1 \iff x \in \{(1,0), (0,1), (0,-1), (-1,0)\}$

$\|x\|_\infty = 1 \iff x \in \{(1,0), (0,1), (0,-1), (-1,0), (1,1), (-1,-1), (1,-1), (-1,1)\}$

Def A sequence $p_0, p_1, \dots, p_n \in \mathbb{Z}^d$ is a <u>k-connected path</u> if $\|p_i - p_{i-1}\|_\infty \le 1$ and $\|p_i - p_{i-1}\|_1 \le k$ $\forall i$ s.t. $1 \le i \le n$. $\quad 1 \le k \le d$

$\boxed{\text{Ex}}$ $d=2$

is 1-connected path

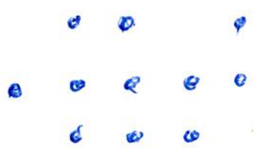


2-connected path

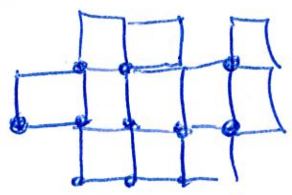
Def $P \subseteq \mathbb{Z}^d$ is <u>k-connected</u> if $\forall x, y \in P$ $\exists$ k-connected path $(p_i)_{0 \le i \le n}$ s.t. $x = p_0$, $y = p_n$ and $p_i \in P$ $\forall i$.

Def $P \subseteq \mathbb{Z}^d$ is a <u>polyomino</u> if
1) P is finite
2) P is 1-connected
3) $\mathbb{Z}^d \setminus P$ is 1-connected

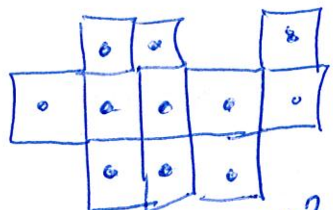
<u>Representations of a polyomino:</u>



$P \subseteq \mathbb{Z}^2$

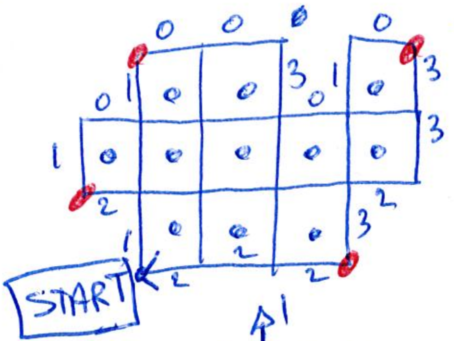


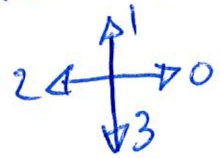
$P + [0,1[^2 \subseteq \mathbb{R}^2$



$P + [-\frac{1}{2}, \frac{1}{2}[^2 \subseteq \mathbb{R}^2$

Def The <u>boundary word</u> of a polyomino $P \subseteq \mathbb{Z}^2$ is a word $w \in \{0,1,2,3\}^*$ ~~describing~~ coding the 1-connected path of the boundary of $P + [-\frac{1}{2}, \frac{1}{2}[^2$.





NW-part NE SE SO

$$w = 12 \,|\, 101 \,|\, 0030\,1\,0 \,|\, 3323 \,|\, 222$$

Def Two words w and $w' \in A^*$ are <u>conjugate</u> if $\exists\, u, v \in A^*$ s.t. $w = uv$ and $w' = vu$.

Conjugacy of words is an equivalence relation (exercise): $w \equiv w'$.

<u>Exercise</u>: $w \equiv w' \iff \exists z$ s.t. $wz = zw'$.

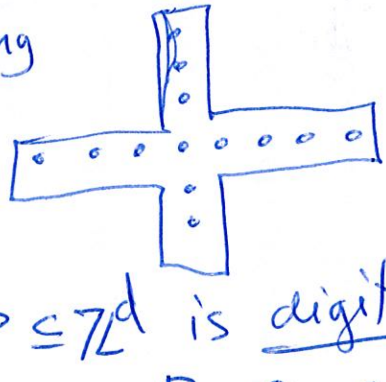
2. Digital geometry

DG's goal is to adapt Euclidean Geometry on $\mathbb{Z}^d$.
straight lines, circles, plane, hyperplanes, rotations, etc

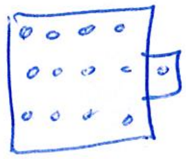
<u>Question</u>: When is a polyomino convex?

(Applied comb. on Words, Lothaire) suggest when Perimeter $= 2H + 2L$

allowing



and



but not

Def $P \subseteq \mathbb{Z}^d$ is <u>digitally-convex</u> if $\mathrm{CONVEXHULL}_{\mathbb{R}^d}(P) \cap \mathbb{Z}^d \subseteq P$

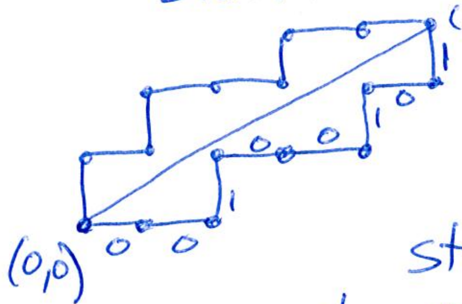
$$\underbrace{P \subseteq}_{\text{always true}} \mathrm{CONVEXHULL}_{\mathbb{R}^d}(P) \cap \mathbb{Z}^d$$

Def $S \subseteq \mathbb{R}^d$

$$\mathrm{CONVEXHULL}(S) = \left\{ \sum_{i=1}^{n} \alpha_i p_i \;\middle|\; \begin{array}{l} p_1, \dots, p_n \in S \\ 0 \le \alpha_i \le 1 \\ \alpha_1 + \alpha_2 + \dots + \alpha_n = 1 \end{array} \right\}$$

3. Christoffel Words

- Jean Bernouilli (1771), first apparition
- Jean Berstel (1990), first to call them Christoffel words
- Elwin B. Christoffel (1829–1900), mathematician, physicist



Let $p,q \in \mathbb{Z}$ coprime. The Christoffel word of parameter (p,q) is the word $w \in \{0,1\}^*$ coding the 1-connected path of unitary steps $(1,0)$ and $(0,1)$ from the origin $(0,0)$ to (p,q) strictly below the segment $(0,0)$–(p,q) and the nearest possible of this segment.

$w = 0\underbrace{010010}_{m}1$

If $(p,q) \neq (0,1)$, then w is of the form $w = 0m1$ with middle word $m \in \{0,1\}^*$.

Lemma m is a palindrome

Proof translation $\nwarrow (-1,1)$ keeps m invariant
rotation $180°$ preserve the figure. $\square$

see Berstel (2007): 14 characterizations of C. words.

Exercise: Write an algorithm that produces/recognize Christoffel words.

4. Lyndon Words

introduced by Roger Lyndon (1917–1988) ~~in~~ 1954 under the name of "standard lexicographic sequences"
used to construct bases of certain free abelian groups

Def A Lyndon word is a word $\in A^+$ which is strictly less than any of its conjugates (circular permutation).

$\boxed{\text{EX}}$ atoire $= \min_{lex} \{$atoire, toirea, oireat, ireato, reatoi, eatoir$\}$

atoireatoire is not Lyndon

$\boxed{\text{EX}}$ Lyndon words on $\{0,1\}$ are

0, 1, 01, 001, 011, 0001, 0011, 0111, 00001, ...

$\boxed{\text{Exercise}}$ Prove that a C. word is a L. word.

<u>Prop</u> (Fredericksen, Maiorana, 1978) The concatenation in increasing order of the Lyndon words over alphabet A of length dividing n, $\forall n \geq 1$, is a de Bruijn sequence, i.e. a word where every word of A^n appears once and only once.

$\boxed{\text{EX}}$ $A = \{0,1\}$ 0 0001 0011 01 0111 1

Can be computed in linear time and logarithmic space (Berstel, Perrin, 2007).

<u>Thm</u> (Lyndon, 1954) Every nonempty word w admits a unique factorisation as a ~~sequence~~ lexicographically decreasing sequence of Lyndon words

$$w = l_1^{n_1} l_2^{n_2} \cdots l_k^{n_k}, \quad l_1 \geq_{lex} l_2 \geq_{lex} \cdots \geq_{lex} l_k$$

where $n_i \geq 1$ and l_i are Lyndon word, $\forall 1 \leq i \leq k$.

$\boxed{\text{EX}}$ c o m | b i n | a t o i r e

2 | 1 | 0 1 2 | 0 1 | 0 1

$\boxed{\text{Exercise}}$ Find algo for Enumeration of Lyndon words, factorization into Lyndon words.

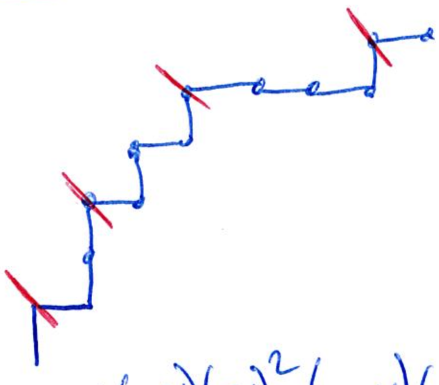
<u>S. convexity</u>

<u>Theorem</u> (Brlek, Lachaud, Provençal, Reutenauer, 2009)

A word $w \in \{0,1\}^*$ codes the NW part of a digitally convex polyomino iff its unique Lyndon factorisation

$$w = l_1^{n_1} l_2^{n_2} \cdots l_k^{n_k}$$

is s.t. every l_i is a Christoffel word.



$w = 1(011)(01)^2(0001)(0)$