①

Cassaigne invariant measure

9 nov 2016
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Goal (Δ, f, μ) dyn. system $\mu \equiv$ Lebesgue

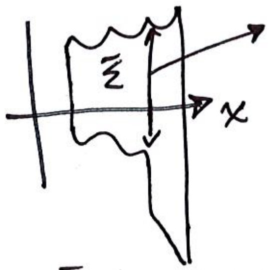
Find density function of μ, i.e. $\delta(x)$ s.t. $\mu(A) = \int_A \delta(x)\,dx$

Strategies 1: Guess it like Gauss did

$$\delta(x) = \frac{1}{1+x} \qquad \text{for } f : [0,1[\to [0,1[,\quad x \mapsto \begin{cases} \{\frac{1}{x}\} & \text{si } x \neq 0 \\ 0 & \text{si } x = 0 \end{cases}$$

Strategy 2: Show that f is conjugate to g for which you know the invariant measure.

Strategy 3: Using the natural extension $\tilde{f}$ of f for $(\Sigma, \tilde{f}, \text{Leb})$ which Lebesgue measure is invariant



$\delta(x)$ s.t. $\tilde{f}(x,a) = (f(x), \cdot)$

$$\delta(x) = \int_{\{a : (x,a) \in \Sigma\}} 1\, da$$

Rmk: Σ has to be of positive measure for this method to work (not always true)

Ref (Arnoux, Nogueira, 93) Farey, Brun, Selmer, Gauss, add, mult, dim d, sorted

(Arnoux, Labbé, 2015) arXiv:1508:07814
Farey, Brun, Reverse, Cassaigne, ARP, sorted

method does not work

~~Algo Cassaigne~~

~~Calcule de la densité δ(x)~~

(2)

~~Cassaigne invariant measure~~

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~~Réponse courte:~~

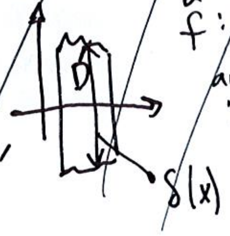
- ~~Guess it like Gauss did: $\frac{dx}{1+x}$ for $T(x) = \left\{\frac{1}{x}\right\}$~~
- ~~Cassaigne is conjugate to Selmer thus Cassaigne inv. measure can be computed from Selmer one~~

~~$f : \Delta \to \Delta$~~
~~Dyn. system (Δ, f, μ)~~

~~Idea → Compute $\delta(x)$ s.t. $\mu(f^{-1}(A)) = \mu(A)$ $\mu(A) = \int_A \delta(x)\,dx$~~
~~density function~~

~~using some natural extension $\tilde{f}$ of f s.t. $\lambda(\tilde{f}^{-1}(A)) = \lambda(A)$~~
~~Lebesgue (λ) is inv. measure of $\tilde{f}$.~~

~~$\tilde{f} : D \to D$~~
~~and $\tilde{f}|_{(x,y)} = (f(x), \cdot)$~~



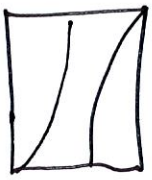
~~Ref (Arnoux, Nogueira, 93) Farey, Gauss, Brun, Selmer additif, multi, dim d, sorted~~
~~(Arnoux, L, 2015 arxiv: 1508.07814) Farey, Brun, Reveuse, Cassaigne, unsorted, d=3~~

Algo Farey

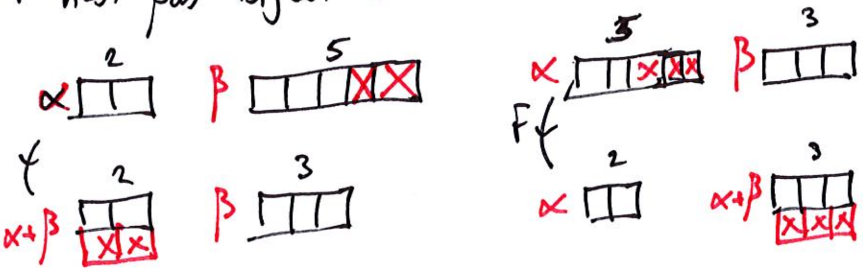
cône $\Lambda = \{(x,y) \in \mathbb{R}^2 \mid 0 < x,\ 0 < y\}$

$$F : (x,y) \mapsto \begin{cases} (x, y-x) & \text{si } y > x \\ (x-y, y) & \text{sinon} \end{cases}$$

$$f(x,y) = \frac{F(x,y)}{\|F(x,y)\|_1} \quad ; \quad f : [0,1] \to [0,1], \quad x \mapsto \begin{cases} \frac{x}{1-x} & \text{si } x < \frac{1}{2} \\ 2 - \frac{1}{x} & \text{si } x > \frac{1}{2} \end{cases}$$



Note F n'est pas bijective



Extension naturelle de F

$$D = \{(x,y,\alpha,\beta) \mid x,y,\alpha,\beta > 0\}$$

$$\tilde{F} : D \to D$$

$$\begin{pmatrix} x \\ y \\ \alpha \\ \beta \end{pmatrix} \mapsto \begin{cases} \begin{pmatrix} 1&0&0&0 \\ -1&1&0&0 \\ 0&0&1&0 \\ 0&0&1&1 \end{pmatrix} \begin{pmatrix} x \\ y \\ \alpha \\ \beta \end{pmatrix} & \text{si } y > x, \\ \begin{pmatrix} 1&-1&0&0 \\ 0&1&0&0 \\ 0&0&1&1 \\ 0&0&0&1 \end{pmatrix} \begin{pmatrix} x \\ y \\ \alpha \\ \beta \end{pmatrix} & \text{si } y < x \end{cases}$$

Les matrices sont de la forme $\begin{pmatrix} A^{-1} & 0 \\ 0 & A^T \end{pmatrix}$

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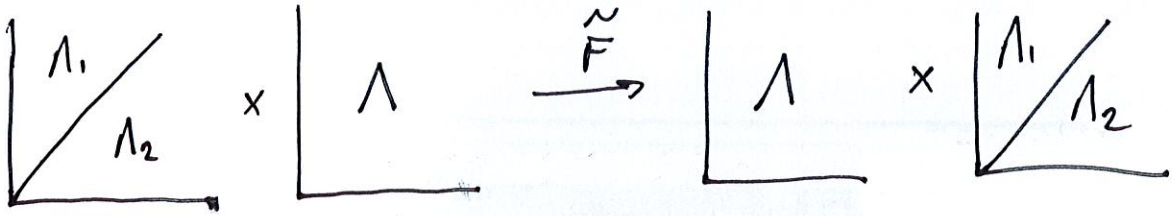

$$D = \underline{(\Lambda_1 \times \Lambda)} \cup \underline{(\Lambda_2 \times \Lambda)} = \underline{(\Lambda \times \Lambda_1)} \cup \underline{(\Lambda \times \Lambda_2)}$$

$\tilde{F}$ is 1-to-1

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- Thus $\tilde{F}$ is a bijection on D.
- $\tilde{F}$ preserves Lebesgue measure (jacobian is 1)
- $\tilde{F}$ preserves the forme $x\alpha + y\beta$ (sum of area of rectangles)

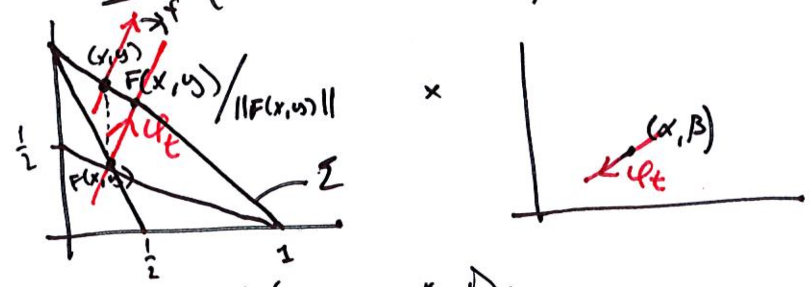
Flot

$$\varphi_t : \begin{pmatrix} x \\ y \\ \alpha \\ \beta \end{pmatrix} \mapsto \begin{pmatrix} e^{+t} & & & \\ & e^{t} & & \\ & & e^{-t} & \\ & & & e^{-t} \end{pmatrix} \begin{pmatrix} x \\ y \\ \alpha \\ \beta \end{pmatrix}$$

- φ_t bien défini sur D
- φ_t a un jacobien 1, donc préserve la mesure Lebesgue sur D
- φ_t préserve la forme $x\alpha + y\beta$
- $\varphi_t \circ \tilde{F} = \tilde{F} \circ \varphi_t$

Natural extension of ζ

Define $D_1 = \{ (x, y, \alpha, \beta) \in D \mid x\alpha + y\beta = 1 \}$



$\tilde{F}$ and φ_t preserve Lebesgue of D_1

Define $\Omega = D_1 / \tilde{F}$ set of orbits of the map $\tilde{F}$. φ_t acts on Ω.

Define $\Sigma = \{ (x, y, \alpha, \beta) \in D_1 \mid |x| + |y| = 1 \}$, and $\bar{\Sigma}$ its projection on Ω

Consider $\zeta : \bar{\Sigma} \to \bar{\Sigma}$ first return map of flow φ_t on Ω.

$$\tilde{\zeta}(x, y, \alpha, \beta) = \left(f(x,y),\ M(x,y)^T \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \cdot \|F(x,y)\| \right)$$

④ Calcul de la densité $\delta(x)$

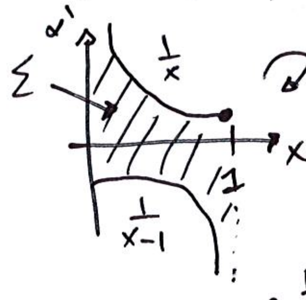
$$\Sigma = \{ (x,y,\alpha,\beta) \mid x\alpha + y\beta = 1,\ x+y=1,\ x,y,\alpha,\beta > 0 \}$$

$\Big\downarrow$ $\alpha' = \alpha - \beta$, $\tau = -\log(x+y)$, $e = x\alpha + y\beta$, jacobian is 1, so Lebesgue measure is $dx\,d\alpha'\,d\tau\,de$

$$= \{ (x,\alpha',\tau,e) \mid \tau = 0,\ e = 1,\ \alpha'+\beta>0,\ \beta>0,\ x\alpha'+\beta = 1 \}$$

$$= \{ (x,\alpha',\tau,e) \mid \tau=0,\ e=1,\ \alpha'+1-x\alpha'>0,\ 1-x\alpha'>0 \}$$

$$= \{ (x,\alpha',\tau,e) \mid \tau=0,\ e=1,\ \tfrac{1}{x-1} < \alpha' < \tfrac{1}{x} \}$$



$\tilde f$ leaves invariant measure $dx\,d\alpha'$

$$\tilde f : \Sigma \longrightarrow \Sigma$$
$$(x,\alpha') \longmapsto \begin{cases} \left(\frac{x}{1-x},\ 1-x+\alpha'(1-x)^2 \right) & \text{if } x < \frac{1}{2} \\ \left(2-\frac{1}{x},\ \alpha' x^2 - x \right) & \text{if } x > \frac{1}{2} \end{cases}$$

$$\delta(x) = \int_{\frac{1}{x-1}}^{\frac{1}{x}} 1\, d\alpha' = \frac{1}{x} - \frac{1}{x-1} = \frac{1}{x(1-x)}$$

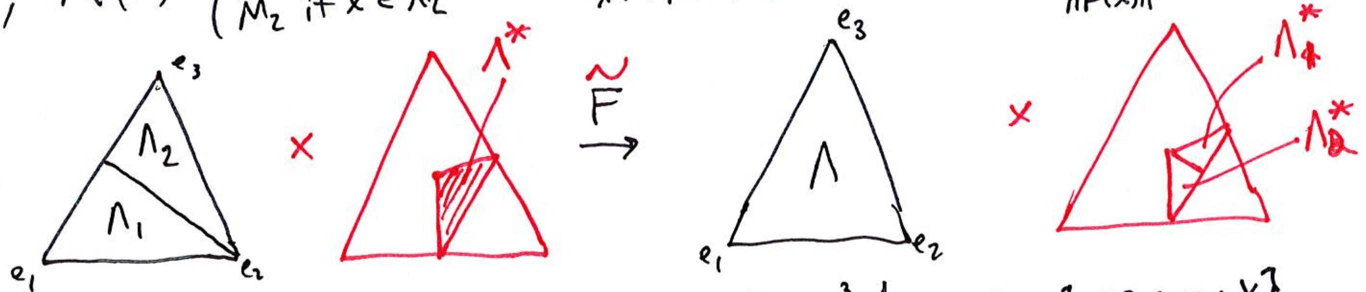
Cassaigne Algorithm

$\Lambda = \mathbb{R}_+^3$ $\quad \Lambda_1 = \{ (x_1,x_2,x_3) \in \Lambda \mid x_1 > x_3 \}$ $\quad M_1 = \begin{pmatrix} 1&1&0 \\ 0&0&1 \\ 0&1&0 \end{pmatrix}$

$\Lambda_2 = \{ (x_1, x_2, x_3) \in \Lambda \mid x_1 < x_3 \}$ $\quad M_2 = \begin{pmatrix} 0&1&0 \\ 1&0&0 \\ 0&1&1 \end{pmatrix}$

$f : \Delta \to \Delta = \{ x \in \Lambda \mid \|x\| = 1 \}$

$x \in \Lambda$, $M(x) = \begin{cases} M_1 & \text{if } x \in \Lambda_1 \\ M_2 & \text{if } x \in \Lambda_2 \end{cases}$, $F : \Lambda \to \Lambda$, $x \mapsto M(x)^{-1} x$, $x \mapsto \frac{F(x)}{\|F(x)\|}$



$$\tilde F \begin{pmatrix} x \\ a \end{pmatrix} = \begin{pmatrix} M(x)^{-1} & 0 \\ 0 & M(x)^T \end{pmatrix} \begin{pmatrix} x \\ a \end{pmatrix}$$

$$\Lambda^* = \{ (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}_+^3 \mid \max\{\alpha,\gamma\} < \beta < \alpha + \gamma \}$$
$$\Lambda_1^* = \{ (\alpha,\beta,\gamma) \in \Lambda^* \mid \alpha < \gamma \}$$
$$\Lambda_2^* = \{ (\alpha,\beta,\gamma) \in \Lambda^* \mid \alpha > \gamma \}$$

$$D = \Lambda \times \Lambda^* = \Lambda_1 \times \Lambda^* \cup \Lambda_2 \times \Lambda^* = \Lambda \times \Lambda_1^* \cup \Lambda \times \Lambda_2^*$$

$\tilde F$ is 1-to-1 onto

$\tilde F$ 1-to-1 and onto

$\tilde F$ bijection on D , D of positive measure

Thus the methods will work.

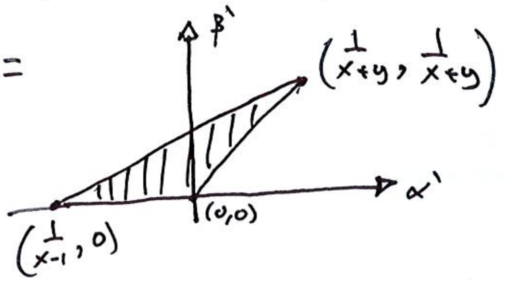
(5) Density of inv. meas. of Cassaigne

$$\Sigma = \{(x,y,z,\alpha,\beta,\gamma)\in\mathbb{R}_+^6 \mid x\alpha+y\beta+z\gamma=1,\ x+y+z=1,\ \alpha<\beta,\ \gamma<\beta,\ \beta<\alpha+\gamma\}$$

$\downarrow$ $\alpha'=\alpha-\gamma$, $\beta'=\beta-\gamma$, $e=x\alpha+y\beta+z\gamma$, $\tau=-\log(x+y+z)$

$$=\{(x,y,\alpha',\beta',\tau,e)\in\mathbb{R}_+^6 \mid \tau=0,\ e=1,\ \alpha'<\beta',\ 0<\beta',\ \beta'<\alpha'+\gamma\}$$

$=$



whose area is

$$\delta(x) = \frac{1}{2}\times\left(0-\frac{1}{x-1}\right)\left(\frac{1}{x+y}\right) = \frac{1}{2(1-x)(x+y)}$$

$\frac{\varepsilon}{2} < x$ is $1-2x =$ $\frac{\varepsilon}{1}\left(\frac{\varepsilon}{2}+x\right)2$

$\frac{\varepsilon}{2} > x$ is $\frac{x}{\varepsilon}$ $\Big\} = f(x)$