Suites équilibrées de faible complexité et algorithmes de fractions continues multidimensionnelles

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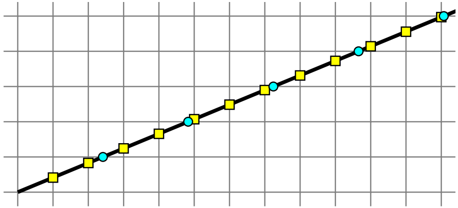
Caen, 9 février 2016

Based on joint work with Valérie Berthé and Pierre Arnoux

Outline

Plan

Sturmian sequences



The characteristic (starting at origin) Sturmian word of slope $1/\sqrt{2}$ is :

$$0010010001001000100100010010001001001\cdots$$

Factor complexity

Let $w \in \mathcal{A}^{\mathbb{N}}$. The factor complexity is a function $p_w(n) : \mathbb{N} \to \mathbb{N}$ counting the number of factors of length n, noted $\mathrm{L}_w(n)$, in the sequence w.

$$w = 000100\boxed{0100}0100100010001000100100010001001$$

$$\mathrm{L}_w(4) = \{0001, 0010, 0100, 1000, 1001\} \quad \Longrightarrow \quad p_w(4) = 5$$

Definition

A sturmian word is an infinite word having exactly $p(n) = n + 1$ factors of length n.

Balanced sequences

Notation : If $u \in \{0,1\}^*$, then $\vec{u} = (|u|_0, |u|_1)$ is the abelian vector of u.

Example : $\overrightarrow{00100} = (4,1)$.

Definition

An infinite word $\mathbf{w} \in \mathcal{A}^{\mathbb{N}}$ is said to be finitely balanced or C-balanced or balanced if there exists a constant $C \in \mathbb{N}$ such that
for all pairs of factors u, v of $\mathbf{w}$ of the same length,

$$||\vec{u} - \vec{v}||_\infty \leq C.$$

Base 2 development of $(\pi)_2 = 11.001001000011111101101010100 0\cdots$
is not 1-balanced because

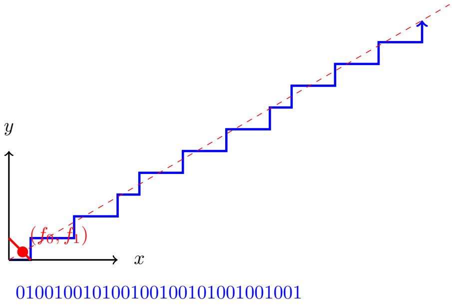
$$||\overrightarrow{0000} - \overrightarrow{1111}||_\infty = ||(4,0) - (0,4)||_\infty = ||(4,-4)||_\infty = 4 > 1.$$

Fact (Morse, Hedlund, 1940)

Sturmian words are exactly the aperiodic 1-balanced sequences.

"À la mort d'un vieux chamelier, un notaire est en charge de partager les chameaux de celui-ci entre ses trois fils : Arthas, Balnazarr et César. Lors de l'écriture de son testament, le vieux chamelier avait 18 chameaux et avait demandé que 9 soient donnés à Arthas, 6 à Balnazarr et 3 à César. Malheureusement, depuis l'écriture du testament, certains chameaux sont morts, d'autres ont été volés et d'autres encore ont été prêtés à des cousins nomades. Finalement le notaire ne sait pas combien de chameaux doivent partagés entre les trois fils (car certains chameaux qui ont été prêtés sont susceptibles de revenir dans les prochains mois). Pour contourner le problème, le notaire décide que les chameaux seront donnés un par un et au fur et à mesure qu'on les retrouvera, selon un ordre préétabli, codé par un mot sur $\{a, b, c\}$*. Par exemple, le mot* $aba\ldots$ *signifie que le premier chameau doit être donné à Arthas, le deuxième à Balnazarr, le troisième à Arthas..."*

Vincent Jost, Ordonnancement chromatique : polyèdres, complexité et classification, thèse de doctorat, 2006, p. 265.



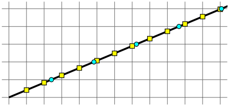
Question

Given a vector $\vec{f} \in \mathbb{R}^d_+$ with $\|\vec{f}\|_1 = 1$ can we construct an infinite word **w** *on the alphabet $\mathcal{A} = \{1, 2, \cdots, d\}$ satisfying each of below conditions ?*

- *frequency of letters in* **w** *exists and is equal to $\vec{f}$,*
- **w** *stays at bounded distance from $\mathbb{R}_+\vec{f}$ (***w** *is balanced),*
- **w** *has a linear factor complexity.*

When $d = 3$: Cutting and billiard sequences

Sturmian word are obtained
from the cutting sequence of a line :



This can be generalized as billiard sequences :

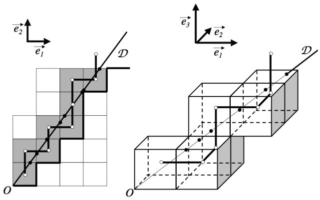


Image credit : Borel (2006)

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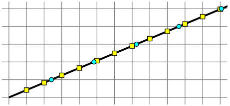

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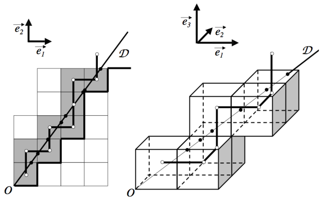
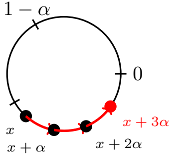


Image credit : Borel (2006)

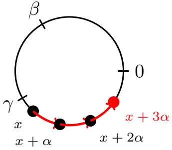
Theorem (Arnoux et al. 1994 ; Baryshnikov, 1995 ; Bédaride, 2003)

If directions $(\alpha_1, \alpha_2, \alpha_3)$ and $(\alpha_1^{-1}, \alpha_2^{-1}, \alpha_3^{-1})$ are both $\mathbb{Q}$ independent, the number of factors appearing in the Billiard word in a cube is exactly $p(n) = n^2 + n + 1$.

Sturmian word are obtained from coding of rotations :



This can be generalized to larger alphabet with coding of rotations on more intervals and more generally to interval exchange transformations (IET) :



When $d = 3$: Coding of rotations and IET

Sturmian word are obtained
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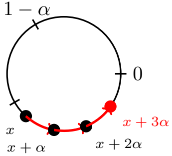

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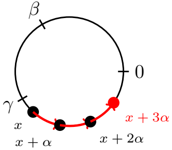

Such sequences have linear factor complexity but are not balanced.

Anton Zorich. Deviation for interval exchange transformations. *Ergodic Theory Dynam. Systems*, 17(6) :1477–1499, 1997.

When $d \geq 2$: Arnoux-Rauzy sequences

An infinite word $\mathbf{w} \in \{1, 2, \ldots, d\}^{\mathbb{N}}$ is an Arnoux-Rauzy word if all its factors occur infinitely often, and if $p(n) = (d-1)n + 1$ for all n, with exactly one left special and one right special factor of length n.

Theorem (Delecroix, Hejda, Steiner, WORDS 2013)

For μ-almost every $\mathbf{f}$ in the Rauzy gasket, the Arnoux-Rauzy word $w_{AR}(\mathbf{f})$ is finitely balanced.

When $d = 3$, three substitutions :

$$\alpha_1 = 1 \mapsto 1, 2 \mapsto 21, 3 \mapsto 31$$
$$\alpha_2 = 1 \mapsto 12, 2 \mapsto 2, 3 \mapsto 32$$
$$\alpha_3 = 1 \mapsto 13, 2 \mapsto 23, 3 \mapsto 3$$

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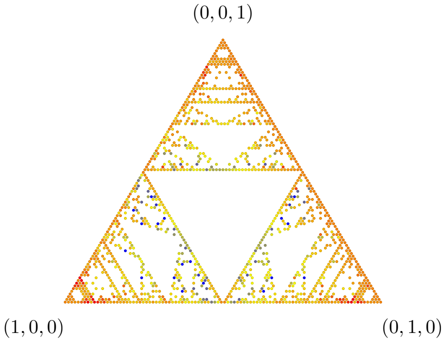


Pierre Arnoux and Štěpán Starosta. The Rauzy Gasket. 2013.

Plan

Multidimensional Continued fraction algorithms

A Multidimensional Continued Fraction (MCF) algorithm is a function

$$\begin{array}{rccc} T: & \Lambda & \to & \Lambda \\ & \mathbf{x} & \mapsto & M(\mathbf{x})^{-1} \cdot \mathbf{x}. \end{array}$$

where $\Lambda \subset \mathbb{R}^d$ is a cone and M is

- an homogeneous function of degree 0 : $M(\alpha\mathbf{x}) = M(\mathbf{x})$,
- piecewise constant on subcones,
- that associates to each $\mathbf{x} \in \Lambda$ an invertible matrix $M(\mathbf{x})$,
- (most of the time) the entries of $M(\mathbf{x})$ are nonnegative integers.

Classical references : Schweiger 2000, Brentjes 1981.

Farey's algorithm

The positive cone $\Lambda = \mathbb{R}^2_+$ is partionned as $\Lambda = \Lambda_1 \cup \Lambda_2$ where

$$\Lambda_1 = \{(x_1, x_2) \in \Lambda \mid x_1 < x_2\},$$
$$\Lambda_2 = \{(x_1, x_2) \in \Lambda \mid x_2 < x_1\}.$$

The matrices are

$$M(\mathbf{x}) = M_i \qquad \text{if and only if} \qquad \mathbf{x} \in \Lambda_i.$$

with

$$M_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \qquad M_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

More simply,

$$F : (x, y) \mapsto \begin{cases} (x, y - x) & \text{if } x < y, \\ (x - y, y) & \text{if } x > y. \end{cases}$$

Continued fractions

The convergents p_n/q_n of $\alpha = \frac{\sqrt{3}-1}{2} = [0;2,1,2,1,2,1,\cdots] = 0.36602540\cdots$ are

$$0, \frac{1}{2}, \frac{1}{3}, \frac{3}{8}, \frac{4}{11}, \frac{11}{30}, \frac{15}{41}, \frac{41}{112}, \frac{56}{153}, \frac{153}{418}, \frac{209}{571}, \frac{571}{1560}, \frac{780}{2131}, \cdots$$

With

$$L = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad R = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

Convergents :

$$\begin{pmatrix} q_0 \\ p_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = R^0 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} q_1 \\ p_1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} = R^0 L^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} q_2 \\ p_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} = R^0 L^2 R^1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} q_3 \\ p_3 \end{pmatrix} = \begin{pmatrix} 8 \\ 3 \end{pmatrix} = R^0 L^2 R^1 L^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} q_4 \\ p_4 \end{pmatrix} = \begin{pmatrix} 11 \\ 4 \end{pmatrix} = R^0 L^2 R^1 L^2 R^1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} q_5 \\ p_5 \end{pmatrix} = \begin{pmatrix} 30 \\ 11 \end{pmatrix} = R^0 L^2 R^1 L^2 R^1 L^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Execution of Farey (or Euclid) algorithm :

$$\begin{pmatrix} 30 \\ 11 \end{pmatrix} = L^2 R^1 L^2 R^1 L^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 8 \\ 11 \end{pmatrix} = R^1 L^2 R^1 L^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 8 \\ 3 \end{pmatrix} = L^2 R^1 L^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix} = R^1 L^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} = L^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Continued fractions : substitutions from matrices

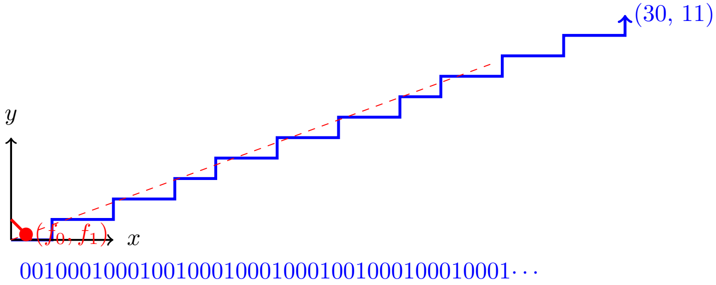
With

$$L = \begin{array}{l} 0 \mapsto 0 \\ 1 \mapsto 01 \end{array} \qquad \text{and} \qquad R = \begin{array}{l} 0 \mapsto 10 \\ 1 \mapsto 1 \end{array}$$

the convergents can be transformed into finite sequences over $\mathcal{A}$:

$$L^2 R^1 L^2 R^1 L^2(1) = 001000100010010001000100010010001000100010001$$

which again corresponds to sturmian sequences.



Cocyles and S-adic words

Definition

The algorithm T defines a cocycle $M_n : \Lambda \to SL(d, \mathbb{Z})$

$$M_0(\mathbf{x}) = I \quad \text{and} \quad M_n(\mathbf{x}) = M(\mathbf{x})M(T\mathbf{x})M(T^2\mathbf{x}) \cdots M(T^{n-1}\mathbf{x}).$$

If, for all $\mathbf{x} \in \Lambda$, $\sigma(\mathbf{x}) : \mathcal{A}^* \to \mathcal{A}^*$ is a morphism of monoid such that its incidence matrix is $M(\mathbf{x})$, i.e., $\overrightarrow{\sigma(w)} = M(\mathbf{x})(\overrightarrow{w})$.

Definition

The algorithm T defines the function $\sigma_n : \Lambda \to (\mathcal{A}^*)^{\mathcal{A}}$

$$\sigma_n(\mathbf{x}) = \sigma(\mathbf{x})\sigma(T\mathbf{x})\sigma(T^2\mathbf{x}) \cdots \sigma(T^{n-1}\mathbf{x})$$

and an S-adic word

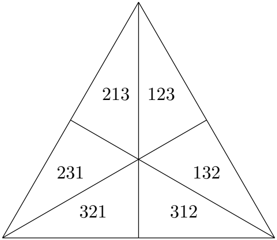
$$W(\mathbf{x}, a) = \lim_{n \to \infty} \sigma_n(\mathbf{x})(a).$$

Poincaré's algorithm (1884)

The positive cone $\Lambda = \mathbb{R}^3_+$ is partionned as $\Lambda = \cup_{\pi \in \mathcal{S}_3} \Lambda_\pi$ where

$$\Lambda_\pi = \{(x_1, x_2, x_3) \in \Lambda \mid x_{\pi 1} < x_{\pi 2} < x_{\pi 3}\}.$$

Matrices are $M(\mathbf{x}) = M_\pi$ and substitutions are $\sigma(\mathbf{x}) = \sigma_\pi$ if $\mathbf{x} \in \Lambda_\pi$.



$$M_{123} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \quad M_{132} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \quad M_{213} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

$$M_{231} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad M_{312} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad M_{321} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\sigma_{ijk} = i \mapsto ijk, j \mapsto jk, k \mapsto k \text{ for all } ijk \in \mathcal{S}_3$$

For example, if $(x_1, x_2, x_3) \in \Lambda_{123}$, then

$$F(x_1, x_2, x_3) = (x_1, x_2 - x_1, x_3 - x_2).$$

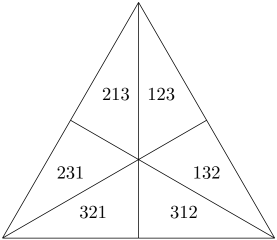
It subtracts the min to the median and the median to the max.

Brun's algorithm (1958)

The positive cone $\Lambda = \mathbb{R}^3_+$ is partionned as $\Lambda = \cup_{\pi \in \mathcal{S}_3} \Lambda_\pi$ where

$$\Lambda_\pi = \{(x_1, x_2, x_3) \in \Lambda \mid x_{\pi 1} < x_{\pi 2} < x_{\pi 3}\}.$$

Matrices are $M(\mathbf{x}) = M_\pi$ and substitutions are $\sigma(\mathbf{x}) = \sigma_\pi$ if $\mathbf{x} \in \Lambda_\pi$.



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$$\sigma_{ijk} = i \mapsto i, j \mapsto jk, k \mapsto k \text{ for all } ijk \in \mathcal{S}_3$$

For example, if $(x_1, x_2, x_3) \in \Lambda_{123}$, then

$$F(x_1, x_2, x_3) = (x_1, x_2, x_3 - x_2).$$

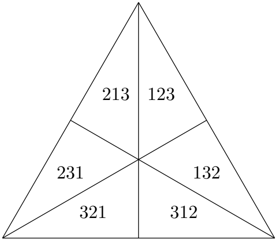
In other words, it subtracts the second largest entry to the largest.

Selmer's algorithm (1961)

The positive cone $\Lambda = \mathbb{R}^3_+$ is partionned as $\Lambda = \cup_{\pi \in \mathcal{S}_3} \Lambda_\pi$ where

$$\Lambda_\pi = \{(x_1, x_2, x_3) \in \Lambda \mid x_{\pi 1} < x_{\pi 2} < x_{\pi 3}\}.$$

Matrices are $M(\mathbf{x}) = M_\pi$ and substitutions are $\sigma(\mathbf{x}) = \sigma_\pi$ if $\mathbf{x} \in \Lambda_\pi$.



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$$\sigma_{ijk} = i \mapsto ik, j \mapsto j, k \mapsto k \text{ for all } ijk \in \mathcal{S}_3$$

For example, if $(x_1, x_2, x_3) \in \Lambda_{123}$, then

$$F(x_1, x_2, x_3) = (x_1, x_2, x_3 - x_1).$$

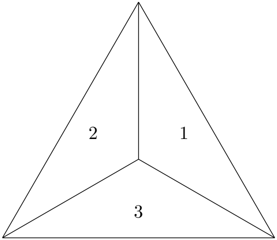
In other words, it subtracts the smallest entry to the largest.

Fully subtractive algorithm ($\approx$ 1980)

The positive cone $\Lambda = \mathbb{R}^3_+$ is partionned as $\Lambda = \cup_{i\in\{1,2,3\}}\Lambda_i$ where

$$\Lambda_i = \{(x_1, x_2, x_3) \in \Lambda \mid x_i = \min\{x_1, x_2, x_3\}\}.$$

Matrices are $M(\mathbf{x}) = M_i$ and substitutions are $\sigma(\mathbf{x}) = \sigma_i$ if $\mathbf{x} \in \Lambda_i$.



$$M_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad M_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad M_3 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\sigma_1 = 1 \mapsto 123, 2 \mapsto 2, 3 \mapsto 3$$
$$\sigma_2 = 1 \mapsto 1, 2 \mapsto 231, 3 \mapsto 3$$
$$\sigma_3 = 1 \mapsto 1, 2 \mapsto 2, 3 \mapsto 312$$

For example, if $(x_1, x_2, x_3) \in \Lambda_1$, then

$$F(x_1, x_2, x_3) = (x_1, x_2 - x_1, x_3 - x_1).$$

It subtracts the smallest entry to the other two.

Example

Cocycle for Brun's algorithm on $\mathbf{x} = (7, 4, 6)$:

$$\begin{aligned} M_6(\mathbf{x}) &= M(\mathbf{x})M(T\mathbf{x})M(T^2\mathbf{x})M(T^3\mathbf{x})M(T^4\mathbf{x})M(T^5\mathbf{x}) \\ &= M_{231}M_{123}M_{132}M_{123}M_{312}M_{312} = \begin{pmatrix} 7 & 3 & 2 \\ 4 & 2 & 1 \\ 6 & 3 & 2 \end{pmatrix} \end{aligned}$$

and the associated S-adic word

$$\begin{aligned} \sigma_6(\mathbf{x})(1) &= \left[\sigma(\mathbf{x})\sigma(T\mathbf{x})\sigma(T^2\mathbf{x})\sigma(T^3\mathbf{x})\sigma(T^4\mathbf{x})\sigma(T^5\mathbf{x})\right](1) \\ &= \left[\sigma_{231}\sigma_{123}\sigma_{132}\sigma_{123}\sigma_{312}\sigma_{312}\right](1) \\ &= 1231312312313123 1 \end{aligned}$$

Convergence

An MCF algorithm is weakly convergent at $\mathbf{x} \in \Lambda$ with $\|\mathbf{x}\| = 1$ if for all i with $1 \leq i \leq d$, we have

$$\lim_{n\to\infty} \frac{M_n(\mathbf{x})\mathbf{e}_i}{\|M_n(\mathbf{x})\mathbf{e}_i\|} = \mathbf{x}.$$

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$$\lim_{n\to\infty} M_n(\mathbf{x})\mathbf{e}_i - \|M_n(\mathbf{x})\mathbf{e}_i\|\mathbf{x} = 0.$$

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$$\lim_{n\to\infty} M_n(\mathbf{x})\mathbf{e}_i - \|M_n(\mathbf{x})\mathbf{e}_i\|\mathbf{x} = 0.$$

The discrepancy of an infinite word $\mathbf{w} \in \mathcal{A}^{\mathbb{N}}$ having frequency vector $\mathbf{x}$ for letters $a \in \mathcal{A}$ with $\|\mathbf{x}\| = 1$ is defined as

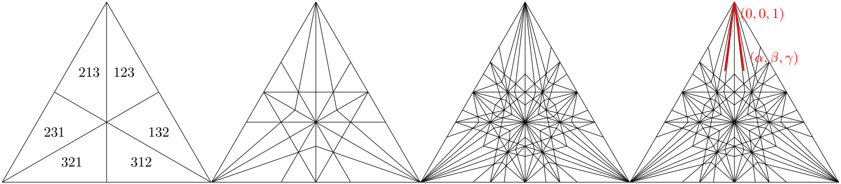
$$\sup_{n\in\mathbb{N}} \left\| \overrightarrow{\mathbf{w}_{[0,n)}} - n \cdot \mathbf{x} \right\|_\infty .$$

$\mathbf{w}$ is balanced $\iff$ $\mathbf{w}$ has finite discrepancy $\iff$ $\mathbf{w}$ stays at bounded distance from the euclidean line of direction $\mathbf{x}$

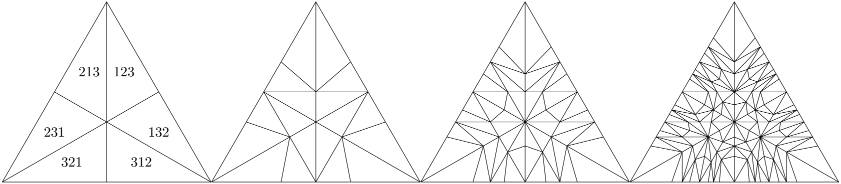
Boris Adamczewski. Balances for fixed points of primitive substitutions. *Theoretical Computer Science*, 307(1) :47 – 75, 2003.

Convergence

Poincaré's algorithm when $d = 3$ is not everywhere weakly convergent since the points on the segment $(\alpha, \beta, \gamma) = (-2+\sqrt{5}, \frac{7-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2})$ to $(0,0,1)$ have the same cocycle $M_n(\mathbf{x})$ structure.

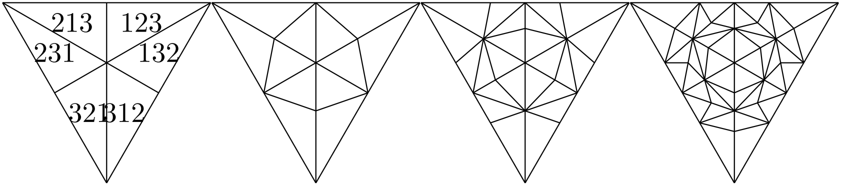


Brun's algorithm when $d = 3$ is almost everywhere strongly convergent :

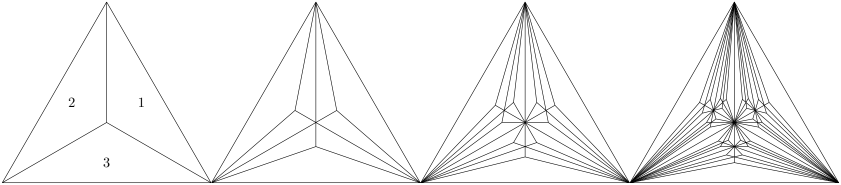


Convergence

Selmer's algorithm when $d = 3$ is almost everywhere strongly convergent :



Fully subtractive algorithm when $d = 3$ is not weakly convergent :



Lemma (Schweiger, 2000, Chap. 14)

Strong convergence $\implies$ *detection of rational dependencies.*

Experimentations on discrepancy

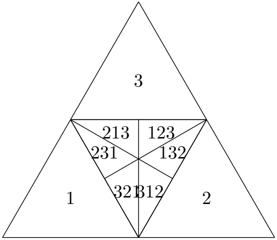
Mean and Maximum values for the discrepancy of S-adic words for strictly positive integer vectors (a_1, a_2, a_3) such that $a_1 + a_2 + a_3 = 100$.

	Mean	Max	$1 - \theta_2/\theta_1$ (std)
Brun	1.100	2.000	1.36834 (0.000065)
Selmer	2.151	12.75	1.38710 (0.000072)
Fully subtractive	6.047	14.21	$\nexists$
Poincaré	2.476	11.13	$\nexists$
Arnoux-Rauzy (AR)	0.8922	1.200	
AR-Fully subtractive	1.154	4.000	
AR-Selmer	0.9991	1.600	
AR-Brun	0.9169	1.520	
AR-Poincaré	0.9066	1.320	1.38881 (0.000050)

Arnoux-Rauzy-Poincaré's algorithm

$$\Lambda_i = \{(x_1, x_2, x_3) \in \Lambda \mid 2x_i > x_1 + x_2 + x_3\}, \qquad i \in \{1,2,3\},$$
$$\Lambda_\pi = \{(x_1, x_2, x_3) \in \Lambda \mid x_{\pi 1} < x_{\pi 2} < x_{\pi 3}\}, \qquad \pi \in \mathcal{S}_3.$$

$$M(\mathbf{x}) = \begin{cases} M_i & \text{if } \mathbf{x} \in \Lambda_i, \\ M_\pi & \text{else if } \mathbf{x} \in \Lambda_\pi \end{cases} \quad \text{and} \quad \sigma(\mathbf{x}) = \begin{cases} \alpha_i & \text{if } \mathbf{x} \in \Lambda_i, \\ \sigma_\pi & \text{else if } \mathbf{x} \in \Lambda_\pi \end{cases}$$



$$M_1 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad M_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

$$M_{123} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \quad M_{132} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \quad M_{213} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

$$M_{231} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad M_{312} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad M_{321} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\alpha_1 = 1 \mapsto 1, 2 \mapsto 21, 3 \mapsto 31, \qquad \sigma_{123} = 1 \mapsto 123, 2 \mapsto 23, 3 \mapsto 3$$

Plan

Factor complexity of ARP S-adic sequences

Let T be the Arnoux-Rauzy-Poincaré algorithm and σ be its substitution function. Recall that

$$W(\mathbf{x}, a) = \lim_{n\to\infty} \left[\sigma(\mathbf{x})\sigma(T\mathbf{x})\sigma(T^2\mathbf{x})\cdots\sigma(T^{n-1}\mathbf{x})\right](a)$$

Theorem (Berthé, L., 2015)

For every totally irrational vector $\mathbf{x} \in \Lambda = \mathbb{R}^3_+$ *and* $a \in \{1, 2, 3\}$, *the factor complexity* $p(n)$ *of the* S*-adic word* $W(\mathbf{x}, a)$ *satisfies :*

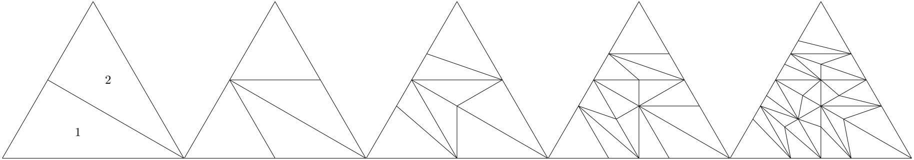
- $2n + 1 \leq p(n) \leq 3n + 1$ *for all* $n \geq 0$ *;*
- $p(n + 1) - p(n) \in \{2, 3\}$ *for all* $n \geq 0$ *;*
- $\limsup_{n\to\infty} \frac{p(n)}{n} \leq \frac{5}{2} < 3$ *(not sharp).*

Cassaigne's algorithm (2015)

On $\Lambda = \mathbb{R}^3_+$, Julien Cassaigne recently proposed the algorithm

$$F(x_1, x_2, x_3) = \begin{cases} (x_1 - x_3, x_3, x_2) & \text{if } x_1 > x_3 \\ (x_2, x_1, x_3 - x_1) & \text{if } x_1 < x_3. \end{cases}$$

$$M_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad M_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad \sigma_1 = \begin{cases} 1 \mapsto 1 \\ 2 \mapsto 13 \\ 3 \mapsto 2 \end{cases} \quad \sigma_2 = \begin{cases} 1 \mapsto 2 \\ 2 \mapsto 13 \\ 3 \mapsto 3 \end{cases}$$



Theorem (Cassaigne, 2015)

The map F is convergent and for every totally irrational vector $\mathbf{x} \in \Lambda = \mathbb{R}^3_+$ and $a \in \{1, 2, 3\}$, the factor complexity of $W(\mathbf{x}, a)$ is $p(n) = 2n + 1$.

Plan

A square matrix with nonnegative entries is Pisot if its dominant eigenvalue is simple and all other eigenvalue have modulus less than one.

Theorem (Avila, Delecroix, 2015)

Let T be the Brun algorithm on $\Lambda = \mathbb{R}^3_+$ and M be its matrix function. If the matrix

$$M_n(\mathbf{x}) = M(\mathbf{x})M(T\mathbf{x})M(T^2\mathbf{x}) \cdots M(T^{n-1}\mathbf{x})$$

is primitive, then it is Pisot.
The same holds for Fully subtractive algorithm on $\Lambda = \mathbb{R}^d_+$.

Avila, Delecroix, 2015

Recall that to a non-zero vector $v \in \mathbb{R}^d$, we associate its dual hyperplane

$$H_v = \{z \in \mathbb{R}^d \mid \langle v, z \rangle = 0\}.$$

Given a norm $\|.\|$ on $\mathbb{R}^d$ and a non-zero vector v in $\mathbb{R}^d$ we define the following semi-norm on $d \times d$ matrices

$$\|B\|_v = \sup_{z \in H_v \setminus \{0\}} \frac{\|Bz\|}{\|z\|}.$$

If $\Lambda \subset \mathbb{R}^d$ is a cone

$$\|B\|_\Lambda = \sup_{v \in \Lambda} \|B\|_v.$$

Avila, Delecroix, 2015 : Pisot property for Brun algo

Let $v \in \Lambda_{123}$. Then for some $k \in \mathbb{R}$ and $\mu_1, \mu_2, \mu_3 \geq 0$ such that $\mu_1 + \mu_2 + \mu_3 = 1$ we have

$$\begin{aligned} kv &= \mu_1(1,1,1) + \mu_2(0,1,1) + \mu_3(0,0,1) \\ &= (\mu_1, \mu_1 + \mu_2, 1) = (0,1,1) - (-\mu_1, \mu_3, 0). \end{aligned}$$

If $\langle v, z \rangle = 0$, then $(0,1,1) \cdot z = (-\mu_1, \mu_3, 0) \cdot z$ and

$$(M_{123})^T z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} z = \begin{pmatrix} 1 & 0 & 0 \\ -\mu_1 & \mu_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} z =: M' z$$

Therefore

$$\|(M_{123})^T\|_v = \sup_{z \in H_v \setminus \{0\}} \frac{\|(M_{123})^T z\|}{\|z\|} = \sup_{z \in H_v \setminus \{0\}} \frac{\|M' z\|}{\|z\|} \leq \|M'\|_\infty = 1.$$

Pisot property for ARP algorithm

Theorem (Arnoux, Berthé, L.)

Let T be the Arnoux-Rauzy-Poincaré algorithm on $\Lambda = \mathbb{R}^3_+$ and M be its matrix function. If the matrix

$$M_n(\mathbf{x}) = M(\mathbf{x})M(T\mathbf{x})M(T^2\mathbf{x}) \cdots M(T^{n-1}\mathbf{x})$$

is primitive, then it is Pisot.

Pisot property for ARP algorithm

Let $v \in \Lambda_{321} \setminus \Lambda_1$. Then for some $k \in \mathbb{R}$ and $\mu_1, \mu_2, \mu_3 \geq 0$ such that $\mu_1 + \mu_2 + \mu_3 = 1$ we have

$$\begin{aligned} kv &= \mu_1(1,1,0) + \mu_2(1, \frac{1}{2}, \frac{1}{2}) + \mu_3(1,1,1) \\ &= (1,1,0) - \left(0, \frac{1}{2}\mu_2, -\frac{1}{2}\mu_2 - \mu_3\right) = (1,1,1) - \left(0, \frac{1}{2}\mu_2, \mu_1 + \frac{1}{2}\mu_2\right). \end{aligned}$$

If $\langle v, z \rangle = 0$ then

$$(1,1,0)\cdot z = (0, \frac{1}{2}\mu_2, -\frac{1}{2}\mu_2 - \mu_3)\cdot z \quad \text{and} \quad (1,1,1)\cdot z = (0, \frac{1}{2}\mu_2, \mu_1 + \frac{1}{2}\mu_2)\cdot z.$$

$$(M_{321})^T z = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2}\mu_2 & -\frac{1}{2}\mu_2 - \mu_3 \\ 0 & \frac{1}{2}\mu_2 & \mu_1 + \frac{1}{2}\mu_2 \end{pmatrix} z =: M' z.$$

We obtain

$$\|(M_{321})^T\|_v = \sup_{z \in H_v \setminus \{0\}} \frac{\|(M_{321})^T z\|}{\|z\|} = \sup_{z \in H_v \setminus \{0\}} \frac{\|M' z\|}{\|z\|} \leq \|M'\|_1 = 1.$$

$\Theta_2^\mu < 0$

Lemma (Avila, Delecroix, 2015)

Let $(A^{(i)})_{i\in\Sigma}$ be a finite or countable set of nonnegative matrices in $SL(d,\mathbb{Z})$. Let (Δ, T, A) be the associated full shift with its cocycle. Let $D \subset \mathbb{P}(\mathbb{R}^d)$ be adapted, i.e., it is nonempty, it is the closure of its interior and for all $i \in \Sigma$ we have $A_i D \subset D$. Assume that

$$\text{for all } i \in \Sigma, \qquad \left\| {A^{(i)}}^T \right\|_{A^{(i)}D} \leq 1$$

Let μ be a T-invariant and ergodic measure on D so that

- *the cocycle A_n is log-integrable,*
- *there exists a cylinder $[w]$ such that $\mu([w]) > 0$, $A_n(w)$ is positive and $\left\| A_n(w)^T \right\|_{A_n(w)D} < 1$.*

Then the two first Lyapunov exponents of the cocycle A_n for the measure μ satisfies $\Theta_1^\mu > 0 > \Theta_2^\mu$.

$\Theta_2^\mu < 0$ implies balanced

Theorem (Berthé, Delecroix (Survey Thm. 6.4), 2013)

Let $T : \Delta \to \Delta$ be a MCF algorithm and μ be a T-invariant and ergodic measure on Δ so that

- *the cocycle M_n is log-integrable,*
- *there exists a cylinder $[w]$ such that $\mu([w]) > 0$ and $M_n(w)$ is positive.*

Let Θ_1^μ and Θ_2^μ denote the two first Lyapunov exponents of M_n with respect to μ. If $\Theta_2^\mu < 0$, then for μ-almost all frequency vector $\mathbf{x} \in \Delta$ and letter $a \in \mathcal{A}$, the S-adic word $W(\mathbf{x}, a)$ generated by the algorithm T is finitely balanced.

Summary

Generalizations of Sturmian words :

	$\forall v \in \mathbb{R}^3_+$	$p(n)$ linear	Balanced
Arnoux Rauzy words	No	Yes	Almost always
Billiard, Andres discrete line	Yes	No	Yes
Coding of rotations and of IET	Yes	Yes	No
Brun S-adic sequences	Yes	?	Almost always
ARP S-adic sequences	Yes	Yes : $\frac{5}{2}n$	Almost always
Cassaigne S-adic sequences	Yes	Yes : $2n+1$	?

Plan

Bounded variation for ARP

The projective version of F on a codimension 1 compact domain

$$\Delta = \{\mathbf{x} \in \Lambda \mid \|\mathbf{x}\| = 1\}$$

for some norm $\|\cdot\|$ is explicitly given by :

$$\begin{array}{rccc} f: & \Delta & \to & \Delta \\ & \mathbf{x} & \mapsto & \frac{F(\mathbf{x})}{\|F(\mathbf{x})\|}. \end{array}$$

Theorem (Arnoux, Berthé, L.)

The function f for Arnoux-Rauzy-Poincaré algorithm admits a unique ergodic absolutely continuous invariant probability measure.

Proof : Bounded variation on n-cylinders of an accelerated version of ARP algorithm.

Natural extension of MCF algorithms

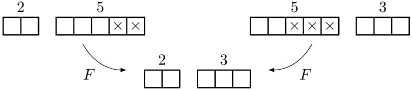
Explicit formulas for the invariant density of an abs. cont. invariant measure of f can be computed from a geometric model for the natural extension of f with good properties.
A potential geometric model for the *natural extension* $\widetilde{F}$ of F is

$$\begin{array}{rccc} \widetilde{F}: & \Lambda \times \mathbb{R}_+^d & \to & \Lambda \times \mathbb{R}_+^d \\ & \begin{pmatrix} \mathbf{x} \\ \mathbf{a} \end{pmatrix} & \mapsto & \begin{pmatrix} M(\mathbf{x})^{-1} & 0 \\ 0 & M(\mathbf{x})^\top \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{a} \end{pmatrix} \end{array}$$

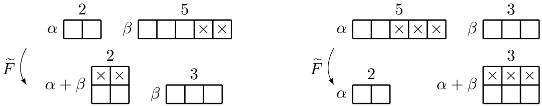
which has jacobian 1.
Main Problem :

find a conical domain $D \subset \Lambda \times \mathbb{R}_+^d$ on which $\widetilde{F}$ is a bijection.

The Farey map F is not a bijection :



One way to transform F into a bijection is to stack the removed part :



We thus constructed $\widetilde{F} : (x, y, \alpha, \beta) \mapsto \begin{cases} (x, y - x, \alpha + \beta, \beta) & \text{if } x < y, \\ (x - y, y, \alpha, \alpha + \beta) & \text{if } x > y. \end{cases}$

It is a bijection on the domain

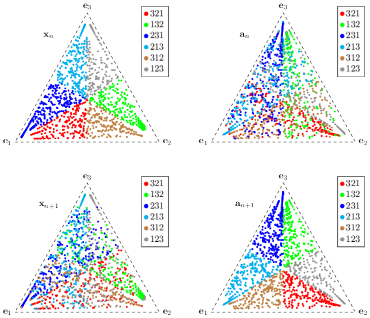
$$D = \{(x, y, \alpha, \beta) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid x > 0, y > 0, \alpha > 0, \beta > 0\}.$$

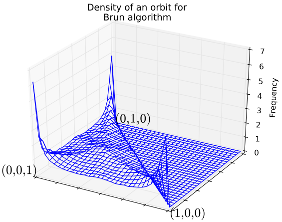
Skiping some details, we get the invariant density of $f = \frac{F(x)}{\|F(x)\|}$:

$$\delta(x) = \frac{1}{x(1-x)}.$$

Natural extension of Brun's algorithm

Sequences $\mathbf{x}_{n+1} = M(\mathbf{x}_n)^{-1}\mathbf{x}_n$ and $\mathbf{a}_{n+1} = M(\mathbf{x}_n)^{\top}\mathbf{a}_n$ helps to find D :





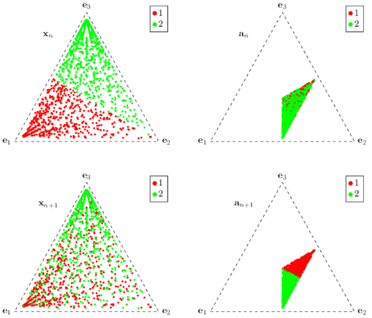
The density function of the invariant measure of $f : \Delta \to \Delta$ for the Brun algorithm is

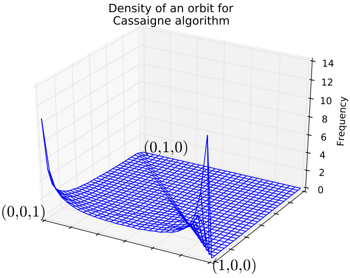
$$\frac{1}{2\, x_{\pi(2)}(1 - x_{\pi(2)})(1 - x_{\pi(1)} - x_{\pi(2)})}$$

on the part $\mathbf{x} = (x_1, x_2, x_3) \in \Lambda_\pi \cap \Delta$.

Natural extension of Cassaigne's algorithm

Sequences $\mathbf{x}_{n+1} = M(\mathbf{x}_n)^{-1}\mathbf{x}_n$ and $\mathbf{a}_{n+1} = M(\mathbf{x}_n)^{\top}\mathbf{a}_n$ helps to find D :





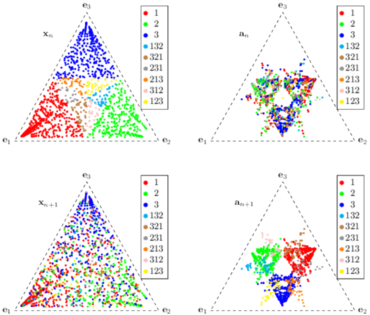
Theorem (Arnoux, L. 2015)

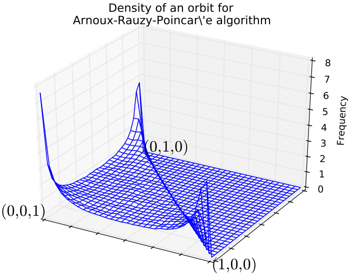
The density function of the invariant measure of $f : \Delta \to \Delta$ for the Cassaigne algorithm is

$$\frac{1}{(1-x)(1-z)}.$$

Natural extension of ARP algorithm

Sequences $\mathbf{x}_{n+1} = M(\mathbf{x}_n)^{-1}\mathbf{x}_n$ and $\mathbf{a}_{n+1} = M(\mathbf{x}_n)^{\top}\mathbf{a}_n$ helps to find D :

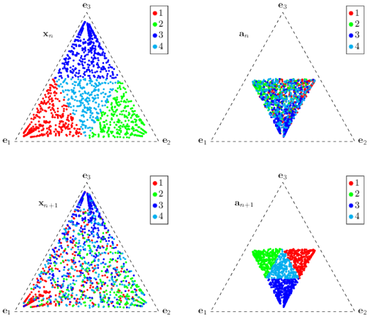


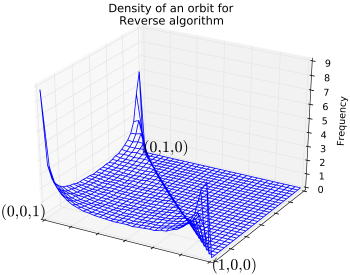


Question : Is this fractal domain D of positive mesure ?

Natural extension of Reverse algorithm

Sequences $\mathbf{x}_{n+1} = M(\mathbf{x}_n)^{-1}\mathbf{x}_n$ and $\mathbf{a}_{n+1} = M(\mathbf{x}_n)^{\top}\mathbf{a}_n$ helps to find D :





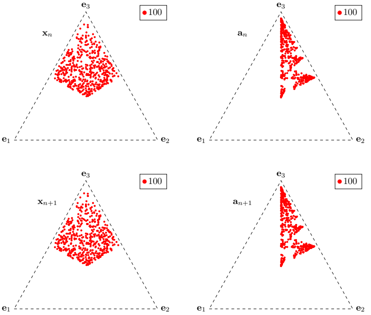
Theorem (Arnoux, L. 2015)

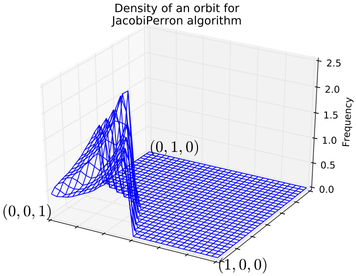
The density function of the invariant measure of $f : \Delta \to \Delta$ for the Reverse algorithm is

$$\frac{1}{(1-x)(1-y)(1-z)}.$$

Natural extension of Jacobi-Perron

Sequences $\mathbf{x}_{n+1} = M(\mathbf{x}_n)^{-1}\mathbf{x}_n$ and $\mathbf{a}_{n+1} = M(\mathbf{x}_n)^{\top}\mathbf{a}_n$ helps to find D :





Open Question : Find the density function of the invariant measure of Jacobi Perron.

Plan

These are good MCF algorithms, i.e.,

- weakly convergent,
- have a finite invariant measure, thus conservative,
- $F^k(\mathbf{x}) \to 0$ for every rationaly independent $\mathbf{x} \in \Lambda$.

Farey	Brun	Selmer	ARP
$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix}$

They are of the good form

$$A = \begin{pmatrix} I_{d-1} & 0 \\ \mathbf{a} & 1 \end{pmatrix}$$

where I_{d-1} is a $(d-1) \times (d-1)$ identity matrix and $\mathbf{a} = (a_1, a_2, \ldots, a_{d-1}) \in \mathbb{R}^{d-1}$ is some vector taking only negative values with at least one of them being non zero.

Not good MCF algorithms and not good form :

Poincaré ($d=3$)

$$\left(\begin{array}{cc|c} 1 & 0 & 0 \\ -1 & 1 & 0 \\ \hline 0 & -1 & 1 \end{array}\right)$$

Fully subtractive

$$\left(\begin{array}{cc|c} 1 & 0 & 0 \\ -1 & 1 & 0 \\ \hline -1 & 0 & 1 \end{array}\right)$$

Good by blocks :

Double Farey

$$\left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{array}\right)$$

Poincaré ($d=4$)

$$\left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ \hline 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{array}\right)$$

Brun-Selmer

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{array}\right)$$

Let $\Lambda = \{(x_1, x_2, \ldots, x_d) \in \mathbb{R}^d \mid 0 < x_1 < x_2 < \ldots < x_d\}$.

Conjecture

A sorted MCF algorithm $F : \Lambda \to \Lambda : \mathbf{x} \mapsto M(\mathbf{x})^{-1}\mathbf{x}$ is good if and only if its matrix $M(\mathbf{e}_d)$ is decomposed by blocks into a lower triangular matrix such that the diagonal block B_i are of the good form :

$$M(\mathbf{e}_d) = \begin{pmatrix} B_1 & 0 & 0 & 0 \\ * & B_2 & 0 & 0 \\ * & * & \cdots & 0 \\ * & * & * & B_k \end{pmatrix} \quad \text{and} \quad B_i = \begin{pmatrix} I & 0 \\ \mathbf{a} & 1 \end{pmatrix}$$

Conjecture (Nogueira 1995)

Let F be the Poincaré algorithm on $\Lambda \subset \mathbb{R}^d_+$ and $f(\mathbf{x}) = \frac{F(\mathbf{x})}{\|F(\mathbf{x})\|}$.

- *if d is even : F is ergodic and f is ergodic and conservative.*
- *if d is odd : F is nonergodic and f is ergodic, although nonconservative, and f has an attractor, $f^k(\mathbf{x}) \to (0, \ldots 0, 1)$ as $k \to \infty$, for a.e. $\mathbf{x}$.*

Proved for $d = 3$ (Nogueira 1995) and $d = 4, 5$ (Nogueira 2001).

Plan

Results on the factor complexity and invariant measures are available here :

V. Berthé and S. Labbé. Factor complexity of S-adic words generated by the Arnoux-Rauzy-Poincaré algorithm. *Adv. in Appl. Math.*, 63 :90–130, 2015.

Pierre Arnoux and Sébastien Labbé. On some symmetric multidimensional continued fraction algorithms. *arXiv:1508.07814*, August 2015.

Images (and code to reproduce them) are available in those Cheat Sheets :

Sébastien Labbé. 3-dimensional Continued Fraction Algorithms Cheat Sheets. *arXiv:1511.08399*, November 2015.