Pisot property for Arnoux-Rauzy-Poincaré Multidimensional Continued Fraction Algorithm

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Based on joint work with Valérie Berthé and Pierre Arnoux

Outline

1. Sturmian words and their generalizations
2. MCF algorithms and S-adic sequences
3. Result : Factor complexity
4. Result : Pisot Property
5. Result : Ergodicity and density function
6. Open question

Plan

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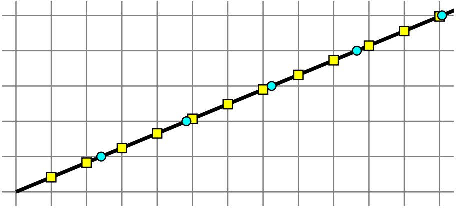
Sturmian sequences



The characteristic (starting at origin) Sturmian word of slope $1/\sqrt{2}$ is :

$$0010010001001000100100100010010001001001 \cdots$$

Factor complexity

Let $w \in \mathcal{A}^{\mathbb{N}}$. The factor complexity is a function $p_w(n) : \mathbb{N} \to \mathbb{N}$ counting the number of factors of length n, noted $\mathrm{L}_w(n)$, in the sequence w.

$$w = 000100\boxed{0100}01001000100010001001000010001001$$

$$\mathrm{L}_w(4) = \{0001, 0010, 0100, 1000, 1001\} \implies p_w(4) = 5$$

Definition

A sturmian word is an infinite word having exactly $p(n) = n + 1$ factors of length n.

Balanced sequences

Notation : If $u \in \{0,1\}^*$, then $\vec{u} = (|u|_0, |u|_1)$ is the abelian vector of u.

Example : $\overrightarrow{00100} = (4,1)$.

Definition

An infinite word $\mathbf{w} \in \mathcal{A}^{\mathbb{N}}$ is said to be finitely balanced or C-balanced or balanced if there exists a constant $C \in \mathbb{N}$ such that

for all pairs of factors u, v of $\mathbf{w}$ of the same length,

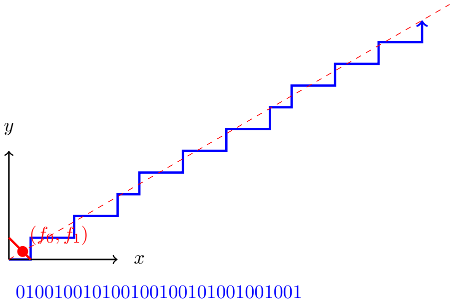
$$||\vec{u} - \vec{v}||_\infty \leq C.$$

Base 2 development of $(\pi)_2 = 11.001001000011111101101010100 0\cdots$ is not 1-balanced because

$$||\overrightarrow{0000} - \overrightarrow{1111}||_\infty = ||(4,0)-(0,4)||_\infty = ||(4,-4)||_\infty = 4 > 1.$$

Fact (Morse, Hedlund, 1940)

Sturmian words are exactly the aperiodic 1-balanced sequences.



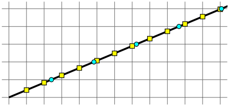
Question

*Given a vector $\vec{f} \in \mathbb{R}_+^d$ with $\|\vec{f}\|_1 = 1$ can we construct an infinite word **w** on the alphabet $\mathcal{A} = \{1, 2, \cdots, d\}$ satisfying each of below conditions?*

- *frequency of letters in **w** exists and is equal to $\vec{f}$,*
- ***w** stays at bounded distance from $\mathbb{R}_+\vec{f}$ (**w** is balanced),*
- ***w** has a linear factor complexity.*

When $d = 3$: Cutting and billiard sequences

Sturmian word are obtained
from the cutting sequence of a line :



This can be generalized as billiard sequences :

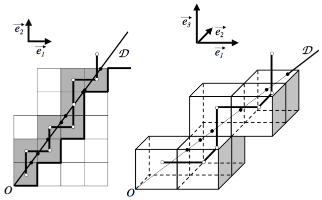


Image credit : Borel (2006)

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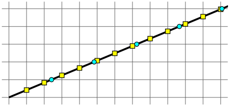

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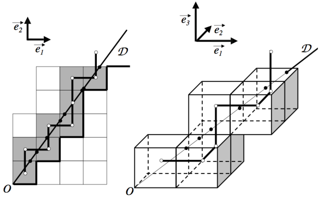


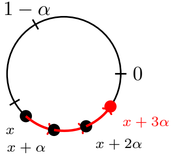
Image credit : Borel (2006)

Theorem (Arnoux et al. 1994 ; Baryshnikov, 1995 ; Bédaride, 2003)

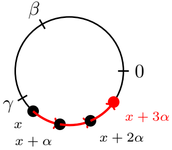
If directions $(\alpha_1, \alpha_2, \alpha_3)$ and $(\alpha_1^{-1}, \alpha_2^{-1}, \alpha_3^{-1})$ are both $\mathbb{Q}$ independent, the number of factors appearing in the Billiard word in a cube is exactly $p(n) = n^2 + n + 1$.

When $d = 3$: Coding of rotations and IET

Sturmian word are obtained
from coding of rotations :



This can be generalized to larger alphabet with coding of rotations on more intervals and more generally to interval exchange transformations (IET) :



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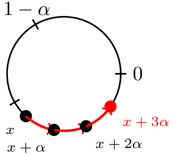

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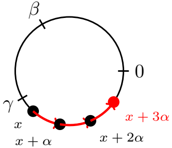

Such sequences have linear factor complexity but are not balanced.

Anton Zorich. Deviation for interval exchange transformations. *Ergodic Theory Dynam. Systems*, 17(6) :1477–1499, 1997.

When $d \geq 2$: Arnoux-Rauzy sequences

An infinite word $\mathbf{w} \in \{1, 2, \ldots, d\}^{\mathbb{N}}$ is an Arnoux-Rauzy word if all its factors occur infinitely often, and if $p(n) = (d-1)n + 1$ for all n, with exactly one left special and one right special factor of length n.

Theorem (Delecroix, Hejda, Steiner, WORDS 2013)

For μ-almost every $\mathbf{f}$ in the Rauzy gasket, the Arnoux-Rauzy word $w_{AR}(\mathbf{f})$ is finitely balanced.

When $d = 3$, three substitutions :

$$\alpha_1 = 1 \mapsto 1, 2 \mapsto 21, 3 \mapsto 31$$
$$\alpha_2 = 1 \mapsto 12, 2 \mapsto 2, 3 \mapsto 32$$
$$\alpha_3 = 1 \mapsto 13, 2 \mapsto 23, 3 \mapsto 3$$

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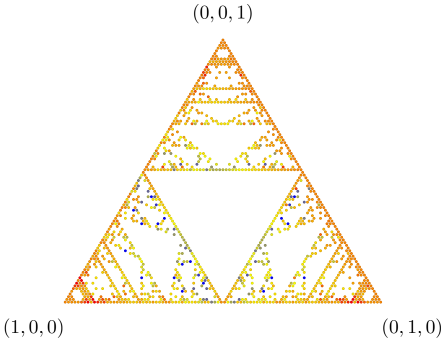
Pierre Arnoux and Štěpán Starosta. The Rauzy Gasket. 2013.



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Multidimensional Continued fraction algorithms

A Multidimensional Continued Fraction (MCF) algorithm is a function

$$\begin{array}{rccc} T: & \Lambda & \to & \Lambda \\ & \mathbf{x} & \mapsto & M(\mathbf{x})^{-1}\cdot\mathbf{x}. \end{array}$$

where $\Lambda \subset \mathbb{R}^d$ is a cone and M is

- an homogeneous function of degree 0 : $M(\alpha\mathbf{x}) = \alpha M(\mathbf{x})$,
- piecewise constant on subcones,
- that associates to each $\mathbf{x} \in \Lambda$ an invertible matrix $M(\mathbf{x})$,
- (most of the time) the entries of $M(\mathbf{x})$ are nonnegative integers.

Classical references : Schweiger 2000, Brentjes 1981.

Farey's algorithm

The positive cone $\Lambda = \mathbb{R}^2_+$ is partionned as $\Lambda = \Lambda_1 \cup \Lambda_2$ where

$$\Lambda_1 = \{(x_1, x_2) \in \Lambda \mid x_1 < x_2\},$$
$$\Lambda_2 = \{(x_1, x_2) \in \Lambda \mid x_2 < x_1\}.$$

The matrices are

$$M(\mathbf{x}) = M_i \qquad \text{if and only if} \qquad \mathbf{x} \in \Lambda_i.$$

with

$$M_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \qquad M_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

More simply,

$$F : (x, y) \mapsto \begin{cases} (x, y - x) & \text{if } x < y, \\ (x - y, y) & \text{if } x > y. \end{cases}$$

Continued fractions

The convergents p_n/q_n of $\alpha = \frac{\sqrt{3}-1}{2} = [0; 2, 1, 2, 1, 2, 1, \cdots] = 0.36602540\cdots$ are

$$0, \frac{1}{2}, \frac{1}{3}, \frac{3}{8}, \frac{4}{11}, \frac{11}{30}, \frac{15}{41}, \frac{41}{112}, \frac{56}{153}, \frac{153}{418}, \frac{209}{571}, \frac{571}{1560}, \frac{780}{2131}, \cdots$$

With

$$L = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

Convergents :

$$\begin{pmatrix} q_0 \\ p_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = R^0 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} q_1 \\ p_1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} = R^0 L^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} q_2 \\ p_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} = R^0 L^2 R^1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} q_3 \\ p_3 \end{pmatrix} = \begin{pmatrix} 8 \\ 3 \end{pmatrix} = R^0 L^2 R^1 L^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} q_4 \\ p_4 \end{pmatrix} = \begin{pmatrix} 11 \\ 4 \end{pmatrix} = R^0 L^2 R^1 L^2 R^1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} q_5 \\ p_5 \end{pmatrix} = \begin{pmatrix} 30 \\ 11 \end{pmatrix} = R^0 L^2 R^1 L^2 R^1 L^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Execution of Farey (or Euclid) algorithm :

$$\begin{pmatrix} 30 \\ 11 \end{pmatrix} = L^2 R^1 L^2 R^1 L^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 8 \\ 11 \end{pmatrix} = R^1 L^2 R^1 L^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 8 \\ 3 \end{pmatrix} = L^2 R^1 L^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix} = R^1 L^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} = L^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Continued fractions : substitutions from matrices

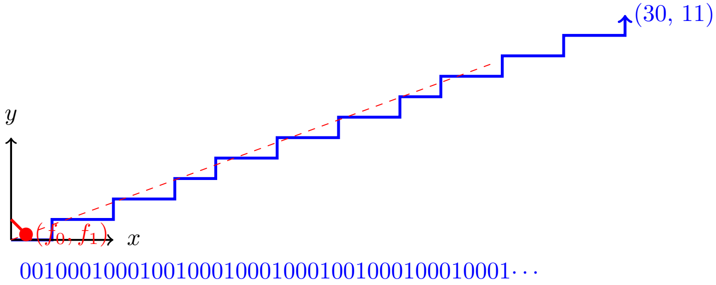
With

$$L = \begin{array}{l} 0 \mapsto 0 \\ 1 \mapsto 01 \end{array} \qquad \text{and} \qquad R = \begin{array}{l} 0 \mapsto 10 \\ 1 \mapsto 1 \end{array}$$

the convergents can be transformed into finite sequences over $\mathcal{A}$:

$$L^2R^1L^2R^1L^2(1) = 00100010001001000100010001001000100010001$$

which again corresponds to sturmian sequences.

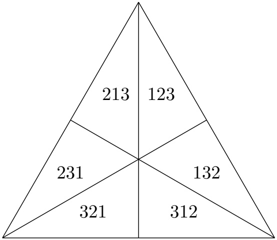


Poincaré's algorithm (1884)

The positive cone $\Lambda = \mathbb{R}^3_+$ is partionned as $\Lambda = \cup_{\pi \in \mathcal{S}_3} \Lambda_\pi$ where

$$\Lambda_\pi = \{(x_1, x_2, x_3) \in \Lambda \mid x_{\pi 1} < x_{\pi 2} < x_{\pi 3}\}.$$

Matrices are $M(\mathbf{x}) = M_\pi$ and substitutions are $\sigma(\mathbf{x}) = \sigma_\pi$ if $\mathbf{x} \in \Lambda_\pi$.



$$M_{123} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \quad M_{132} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \quad M_{213} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

$$M_{231} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad M_{312} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad M_{321} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\sigma_{ijk} = i \mapsto ijk, j \mapsto jk, k \mapsto k \text{ for all } ijk \in \mathcal{S}_3$$

For example, if $(x_1, x_2, x_3) \in \Lambda_{123}$, then

$$F(x_1, x_2, x_3) = (x_1, x_2 - x_1, x_3 - x_2).$$

It subtracts the min to the median and the median to the max.

Brun's algorithm (1958)

The positive cone $\Lambda = \mathbb{R}^3_+$ is partionned as $\Lambda = \cup_{\pi \in \mathcal{S}_3} \Lambda_\pi$ where

$$\Lambda_\pi = \{(x_1, x_2, x_3) \in \Lambda \mid x_{\pi 1} < x_{\pi 2} < x_{\pi 3}\}.$$

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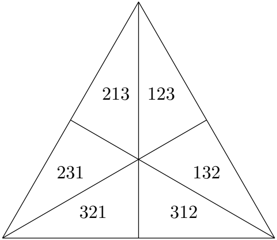

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$$\sigma_{ijk} = i \mapsto i, j \mapsto jk, k \mapsto k \text{ for all } ijk \in \mathcal{S}_3$$

For example, if $(x_1, x_2, x_3) \in \Lambda_{123}$, then

$$F(x_1, x_2, x_3) = (x_1, x_2, x_3 - x_2).$$

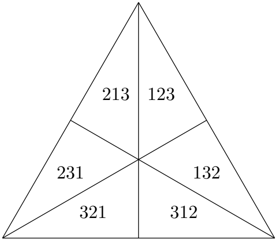
In other words, it subtracts the second largest entry to the largest.

Selmer's algorithm (1961)

The positive cone $\Lambda = \mathbb{R}^3_+$ is partionned as $\Lambda = \cup_{\pi \in \mathcal{S}_3} \Lambda_\pi$ where

$$\Lambda_\pi = \{(x_1, x_2, x_3) \in \Lambda \mid x_{\pi 1} < x_{\pi 2} < x_{\pi 3}\}.$$

Matrices are $M(\mathbf{x}) = M_\pi$ and substitutions are $\sigma(\mathbf{x}) = \sigma_\pi$ if $\mathbf{x} \in \Lambda_\pi$.



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$$\sigma_{ijk} = i \mapsto ik, j \mapsto j, k \mapsto k \text{ for all } ijk \in \mathcal{S}_3$$

For example, if $(x_1, x_2, x_3) \in \Lambda_{123}$, then

$$F(x_1, x_2, x_3) = (x_1, x_2, x_3 - x_1).$$

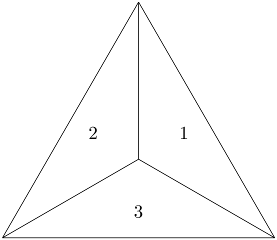
In other words, it subtracts the smallest entry to the largest.

Fully subtractive algorithm (≈ 1980)

The positive cone $\Lambda = \mathbb{R}^3_+$ is partionned as $\Lambda = \cup_{i\in\{1,2,3\}}\Lambda_i$ where

$$\Lambda_i = \{(x_1, x_2, x_3) \in \Lambda \mid x_i = \min\{x_1, x_2, x_3\}\}.$$

Matrices are $M(\mathbf{x}) = M_i$ and substitutions are $\sigma(\mathbf{x}) = \sigma_i$ if $\mathbf{x} \in \Lambda_i$.



$$M_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad M_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad M_3 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\sigma_1 = 1 \mapsto 123, 2 \mapsto 2, 3 \mapsto 3$$
$$\sigma_2 = 1 \mapsto 1, 2 \mapsto 231, 3 \mapsto 3$$
$$\sigma_3 = 1 \mapsto 1, 2 \mapsto 2, 3 \mapsto 312$$

For example, if $(x_1, x_2, x_3) \in \Lambda_1$, then

$$F(x_1, x_2, x_3) = (x_1, x_2 - x_1, x_3 - x_1).$$

It subtracts the smallest entry to the other two.

Cocyles and S-adic words

Definition

The algorithm T defines a cocycle $M_n : \Lambda \to SL(d, \mathbb{Z})$

$$M_0(\mathbf{x}) = I \quad \text{and} \quad M_n(\mathbf{x}) = M(\mathbf{x})M(T\mathbf{x})M(T^2\mathbf{x}) \cdots M(T^{n-1}\mathbf{x}).$$

If, for all $\mathbf{x} \in \Lambda$, $\sigma(\mathbf{x}) : \mathcal{A}^* \to \mathcal{A}^*$ is a morphism of monoid such that its incidence matrix is $M(\mathbf{x})$, i.e., $\overrightarrow{\sigma(w)} = M(\mathbf{x})(\overrightarrow{w})$.

Definition

The algorithm T defines the function $\sigma_n : \Lambda \to (\mathcal{A}^*)^{\mathcal{A}}$

$$\sigma_n(\mathbf{x}) = \sigma(\mathbf{x})\sigma(T\mathbf{x})\sigma(T^2\mathbf{x}) \cdots \sigma(T^{n-1}\mathbf{x})$$

and an S-adic word

$$W(\mathbf{x}, a) = \lim_{n \to \infty} \sigma_n(\mathbf{x})(a).$$

Example

Cocycle for Brun's algorithm on $\mathbf{x} = (7, 4, 6)$:

$$\begin{aligned} M_6(\mathbf{x}) &= M(\mathbf{x})M(T\mathbf{x})M(T^2\mathbf{x})M(T^3\mathbf{x})M(T^4\mathbf{x})M(T^5\mathbf{x}) \\ &= M_{231}M_{123}M_{132}M_{123}M_{312}M_{312} = \begin{pmatrix} 7 & 3 & 2 \\ 4 & 2 & 1 \\ 6 & 3 & 2 \end{pmatrix} \end{aligned}$$

and the associated S-adic word

$$\begin{aligned} \sigma_6(\mathbf{x})(1) &= \left[\sigma(\mathbf{x})\sigma(T\mathbf{x})\sigma(T^2\mathbf{x})\sigma(T^3\mathbf{x})\sigma(T^4\mathbf{x})\sigma(T^5\mathbf{x})\right](1) \\ &= [\sigma_{231}\sigma_{123}\sigma_{132}\sigma_{123}\sigma_{312}\sigma_{312}](1) \\ &= 1231312312313123 1 \end{aligned}$$

Convergence

An MCF algorithm is weakly convergent at $\mathbf{x} \in \Lambda$ with $\|\mathbf{x}\| = 1$ if for all i with $1 \leq i \leq d$, we have

$$\lim_{n\to\infty} \frac{M_n(\mathbf{x})\mathbf{e}_i}{\|M_n(\mathbf{x})\mathbf{e}_i\|} = \mathbf{x}.$$

An MCF algorithm is strongly convergent at $\mathbf{x} \in \Lambda$ with $\|\mathbf{x}\| = 1$ if for all i with $1 \leq i \leq d$, we have

$$\lim_{n\to\infty} M_n(\mathbf{x})\mathbf{e}_i - \|M_n(\mathbf{x})\mathbf{e}_i\|\mathbf{x} = 0.$$

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$$\lim_{n\to\infty} \frac{M_n(\mathbf{x})\mathbf{e}_i}{\|M_n(\mathbf{x})\mathbf{e}_i\|} = \mathbf{x}.$$

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$$\lim_{n\to\infty} M_n(\mathbf{x})\mathbf{e}_i - \|M_n(\mathbf{x})\mathbf{e}_i\|\mathbf{x} = 0.$$

The discrepancy of an infinite word $\mathbf{w} \in \mathcal{A}^{\mathbb{N}}$ having frequency vector $\mathbf{x}$ for letters $a \in \mathcal{A}$ with $\|\mathbf{x}\| = 1$ is defined as

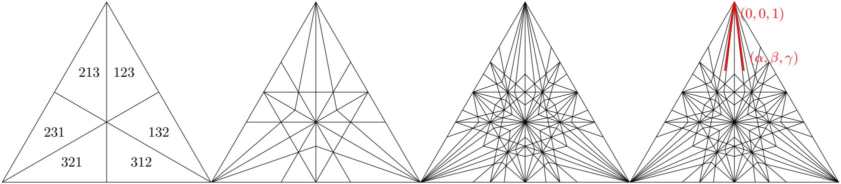
$$\sup_{n\in\mathbb{N}} \left\| \overrightarrow{\mathbf{w}_{[0,n)}} - n \cdot \mathbf{x} \right\|_\infty .$$

$\mathbf{w}$ is balanced $\iff$ $\mathbf{w}$ has finite discrepancy $\iff$ $\mathbf{w}$ stays at bounded distance from the euclidean line of direction $\mathbf{x}$

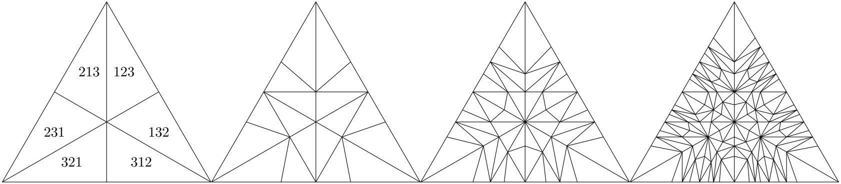
Boris Adamczewski. Balances for fixed points of primitive substitutions. *Theoretical Computer Science*, 307(1) :47 – 75, 2003.

Convergence

Poincaré's algorithm when $d = 3$ is not everywhere weakly convergent since the points on the segment $(\alpha, \beta, \gamma) = (-2 + \sqrt{5}, \frac{7-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2})$ to $(0, 0, 1)$ have the same cocycle $M_n(\mathbf{x})$ structure.

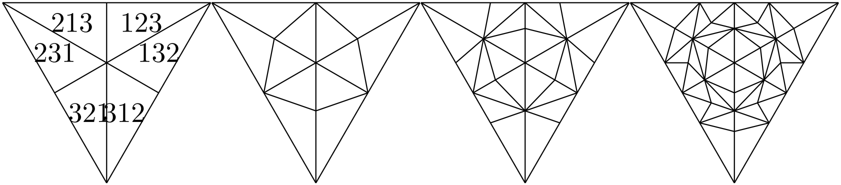


Brun's algorithm when $d = 3$ is almost everywhere strongly convergent :

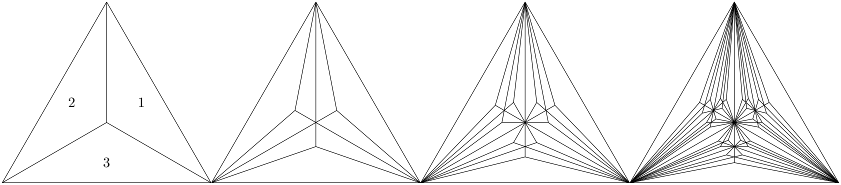


Convergence

Selmer's algorithm when $d = 3$ is almost everywhere strongly convergent :



Fully subtractive algorithm when $d = 3$ is not weakly convergent :



Lemma (Schweiger, 2000, Chap. 14)

Strong convergence $\Longrightarrow$ detection of rational dependencies.

Experimentations on discrepancy

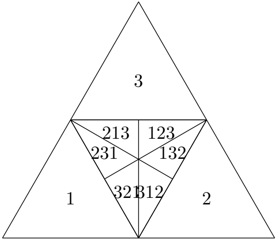
Mean and Maximum values for the discrepancy of S-adic words for strictly positive integer vectors (a_1, a_2, a_3) such that $a_1 + a_2 + a_3 = 100$.

	Mean	Max	$1 - \theta_2/\theta_1$ (std)
Brun	1.100	2.000	1.36834 (0.000065)
Selmer	2.151	12.75	1.38710 (0.000072)
Fully subtractive	6.047	14.21	$\nexists$
Poincaré	2.476	11.13	$\nexists$
Arnoux-Rauzy (AR)	0.8922	1.200	
AR-Fully subtractive	1.154	4.000	
AR-Selmer	0.9991	1.600	
AR-Brun	0.9169	1.520	
AR-Poincaré	0.9066	1.320	1.38881 (0.000050)

Arnoux-Rauzy-Poincaré's algorithm

$$\Lambda_i = \{(x_1, x_2, x_3) \in \Lambda \mid 2x_i > x_1 + x_2 + x_3\}, \qquad i \in \{1,2,3\},$$
$$\Lambda_\pi = \{(x_1, x_2, x_3) \in \Lambda \mid x_{\pi 1} < x_{\pi 2} < x_{\pi 3}\}, \qquad \pi \in \mathcal{S}_3.$$

$$M(\mathbf{x}) = \begin{cases} M_i & \text{if } \mathbf{x} \in \Lambda_i, \\ M_\pi & \text{else if } \mathbf{x} \in \Lambda_\pi \end{cases} \quad \text{and} \quad \sigma(\mathbf{x}) = \begin{cases} \alpha_i & \text{if } \mathbf{x} \in \Lambda_i, \\ \sigma_\pi & \text{else if } \mathbf{x} \in \Lambda_\pi \end{cases}$$



$$M_1 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad M_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$
$$M_{123} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \quad M_{132} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \quad M_{213} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$
$$M_{231} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad M_{312} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad M_{321} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\alpha_1 = 1 \mapsto 1, 2 \mapsto 21, 3 \mapsto 31, \qquad \sigma_{123} = 1 \mapsto 123, 2 \mapsto 23, 3 \mapsto 3$$

Plan

1 Sturmian words and their generalizations

2 MCF algorithms and S-adic sequences

3 Result : Factor complexity

4 Result : Pisot Property

5 Result : Ergodicity and density function

6 Open question

Factor complexity of ARP S-adic sequences

Let T be the Arnoux-Rauzy-Poincaré algorithm and σ be its substitution function. Recall that

$$W(\mathbf{x}, a) = \lim_{n\to\infty} \left[\sigma(\mathbf{x})\sigma(T\mathbf{x})\sigma(T^2\mathbf{x})\cdots\sigma(T^{n-1}\mathbf{x})\right](a)$$

Theorem (Berthé, L., 2015)

For every totally irrational vector $\mathbf{x} \in \Lambda = \mathbb{R}^3_+$ and $a \in \{1, 2, 3\}$, the factor complexity $p(n)$ of the S-adic word $W(\mathbf{x}, a)$ satisfies :

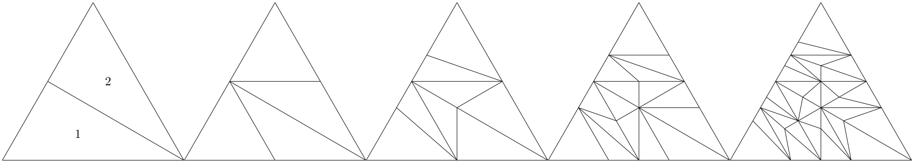
- $2n + 1 \leq p(n) \leq 3n + 1$ *for all* $n \geq 0$ *;*
- $p(n + 1) - p(n) \in \{2, 3\}$ *for all* $n \geq 0$ *;*
- $\limsup_{n\to\infty} \frac{p(n)}{n} \leq \frac{5}{2} < 3$ *(not sharp).*

Cassaigne's algorithm (2015)

On $\Lambda = \mathbb{R}^3_+$, Julien Cassaigne recently proposed the algorithm

$$F(x_1,x_2,x_3) = \begin{cases} (x_1 - x_3, x_3, x_2) & \text{if } x_1 > x_3 \\ (x_2, x_1, x_3 - x_1) & \text{if } x_1 < x_3. \end{cases}$$

$$M_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad M_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad \sigma_1 = \begin{cases} 1 \mapsto 1 \\ 2 \mapsto 13 \\ 3 \mapsto 2 \end{cases} \quad \sigma_2 = \begin{cases} 1 \mapsto 2 \\ 2 \mapsto 13 \\ 3 \mapsto 3 \end{cases}$$



Theorem (Cassaigne, 2015)

The map F is convergent and for every totally irrational vector $\mathbf{x} \in \Lambda = \mathbb{R}^3_+$ and $a \in \{1,2,3\}$, the factor complexity of $W(\mathbf{x}, a)$ is $p(n) = 2n + 1$.

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A square matrix with nonnegative entries is Pisot if its dominant eigenvalue is simple and all other eigenvalue have modulus less than one.

Theorem (Avila, Delecroix, 2015)

Let T be the Brun algorithm on $\Lambda = \mathbb{R}^3_+$ and M be its matrix function. If the matrix

$$M_n(\mathbf{x}) = M(\mathbf{x})M(T\mathbf{x})M(T^2\mathbf{x})\cdots M(T^{n-1}\mathbf{x})$$

is primitive, then it is Pisot.
The same holds for Fully subtractive algorithm on $\Lambda = \mathbb{R}^d_+$.

Avila, Delecroix, 2015

Recall that to a non-zero vector $v \in \mathbb{R}^d$, we associate its dual hyperplane

$$H_v = \{z \in \mathbb{R}^d \mid \langle v, z \rangle = 0\}.$$

Given a norm $\|.\|$ on $\mathbb{R}^d$ and a non-zero vector v in $\mathbb{R}^d$ we define the following semi-norm on $d \times d$ matrices

$$\|B\|_v = \sup_{z \in H_v \setminus \{0\}} \frac{\|Bz\|}{\|z\|}.$$

If $\Lambda \subset \mathbb{R}^d$ is a cone

$$\|B\|_\Lambda = \sup_{v \in \Lambda} \|B\|_v.$$

Avila, Delecroix, 2015 : Pisot property for Brun algo

Let $v \in \Lambda_{123}$. Then for some $k \in \mathbb{R}$ and $\mu_1, \mu_2, \mu_3 \geq 0$ such that $\mu_1 + \mu_2 + \mu_3 = 1$ we have

$$\begin{aligned} kv &= \mu_1(1,1,1) + \mu_2(0,1,1) + \mu_3(0,0,1) \\ &= (\mu_1, \mu_1 + \mu_2, 1) = (0,1,1) - (-\mu_1, \mu_3, 0). \end{aligned}$$

If $\langle v, z \rangle = 0$, then $(0,1,1) \cdot z = (-\mu_1, \mu_3, 0) \cdot z$ and

$$(M_{123})^T z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} z = \begin{pmatrix} 1 & 0 & 0 \\ -\mu_1 & \mu_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} z =: M' z$$

Therefore

$$\|(M_{123})^T\|_v = \sup_{z \in H_v \setminus \{0\}} \frac{\|(M_{123})^T z\|}{\|z\|} = \sup_{z \in H_v \setminus \{0\}} \frac{\|M' z\|}{\|z\|} \leq \|M'\|_\infty = 1.$$

Pisot property for ARP algorithm

Theorem (Arnoux, Berthé, L.)

Let T be the Arnoux-Rauzy-Poincaré algorithm on $\Lambda = \mathbb{R}^3_+$ and M be its matrix function. If the matrix

$$M_n(\mathbf{x}) = M(\mathbf{x})M(T\mathbf{x})M(T^2\mathbf{x}) \cdots M(T^{n-1}\mathbf{x})$$

is primitive, then it is Pisot.

Pisot property for ARP algorithm

Let $v \in \Lambda_{321} \setminus \Lambda_1$. Then for some $k \in \mathbb{R}$ and $\mu_1, \mu_2, \mu_3 \geq 0$ such that $\mu_1 + \mu_2 + \mu_3 = 1$ we have

$$\begin{aligned} kv &= \mu_1(1,1,0) + \mu_2(1, \frac{1}{2}, \frac{1}{2}) + \mu_3(1,1,1) \\ &= (1,1,0) - \left(0, \frac{1}{2}\mu_2, -\frac{1}{2}\mu_2 - \mu_3\right) = (1,1,1) - \left(0, \frac{1}{2}\mu_2, \mu_1 + \frac{1}{2}\mu_2\right). \end{aligned}$$

If $\langle v, z \rangle = 0$ then

$$(1,1,0)\cdot z = (0, \frac{1}{2}\mu_2, -\frac{1}{2}\mu_2 - \mu_3)\cdot z \quad \text{and} \quad (1,1,1)\cdot z = (0, \frac{1}{2}\mu_2, \mu_1 + \frac{1}{2}\mu_2)\cdot z.$$

$$(M_{321})^T z = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2}\mu_2 & -\frac{1}{2}\mu_2 - \mu_3 \\ 0 & \frac{1}{2}\mu_2 & \mu_1 + \frac{1}{2}\mu_2 \end{pmatrix} z =: M' z.$$

We obtain

$$\|(M_{321})^T\|_v = \sup_{z \in H_v \setminus \{0\}} \frac{\|(M_{321})^T z\|}{\|z\|} = \sup_{z \in H_v \setminus \{0\}} \frac{\|M' z\|}{\|z\|} \leq \|M'\|_1 = 1.$$

$\Theta_2^\mu < 0$

Lemma (Avila, Delecroix, 2015)

Let $(A^{(i)})_{i\in\Sigma}$ be a finite or countable set of nonnegative matrices in $SL(d,\mathbb{Z})$. Let (Δ, T, A) be the associated full shift with its cocycle. Let $D \subset \mathbb{P}(\mathbb{R}^d)$ be adapted, i.e., it is nonempty, it is the closure of its interior and for all $i \in \Sigma$ we have $A_i D \subset D$. Assume that

$$\text{for all } i \in \Sigma, \qquad \left\| {A^{(i)}}^T \right\|_{A^{(i)} D} \leq 1$$

Let μ be a T-invariant and ergodic measure on D so that

- *the cocycle A_n is log-integrable,*
- *there exists a cylinder $[w]$ such that $\mu([w]) > 0$, $A_n(w)$ is positive and $\left\| A_n(w)^T \right\|_{A_n(w)D} < 1$.*

Then the two first Lyapunov exponents of the cocycle A_n for the measure μ satisfies $\Theta_1^\mu > 0 > \Theta_2^\mu$.

$\Theta_2^\mu < 0$ implies balanced

Theorem (Berthé, Delecroix (Survey Thm. 6.4), 2013)

Let $T : \Delta \to \Delta$ be a MCF algorithm and μ be a T-invariant and ergodic measure on Δ so that

- *the cocycle M_n is log-integrable,*
- *there exists a cylinder $[w]$ such that $\mu([w]) > 0$ and $M_n(w)$ is positive.*

Let Θ_1^μ and Θ_2^μ denote the two first Lyapunov exponents of M_n with respect to μ. If $\Theta_2^\mu < 0$, then for μ-almost all frequency vector $\mathbf{x} \in \Delta$ and letter $a \in \mathcal{A}$, the S-adic word $W(\mathbf{x}, a)$ generated by the algorithm T is finitely balanced.

Summary

Generalizations of Sturmian words :

	$\forall v \in \mathbb{R}^3_+$	$p(n)$ linear	Balanced
Arnoux Rauzy words	No	Yes	Almost always
Billiard, Andres discrete line	Yes	No	Yes
Coding of rotations and of IET	Yes	Yes	No
Brun S-adic sequences	Yes	?	Almost always
ARP S-adic sequences	Yes	Yes : $\frac{5}{2}n$	Almost always
Cassaigne S-adic sequences	Yes	Yes : $2n+1$	?

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Bounded variation for ARP

The projective version of F on a codimension 1 compact domain

$$\Delta = \{\mathbf{x} \in \Lambda \mid \|\mathbf{x}\| = 1\}$$

for some norm $\|\cdot\|$ is explicitly given by :

$$\begin{array}{rccc} f: & \Delta & \to & \Delta \\ & \mathbf{x} & \mapsto & \frac{F(\mathbf{x})}{\|F(\mathbf{x})\|}. \end{array}$$

Theorem (Arnoux, Berthé, L.)

The function f for Arnoux-Rauzy-Poincaré algorithm admits a unique ergodic absolutely continuous invariant probability measure.

Proof : Bounded variation on n-cylinders of an accelerated version of ARP algorithm.

Natural extension of MCF algorithms

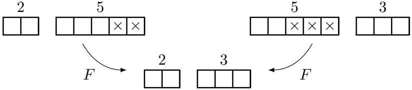
Explicit formulas for the invariant density of an abs. cont. invariant measure of f can be computed from a geometric model for the natural extension of f with good properties.
A potential geometric model for the *natural extension* $\widetilde{F}$ of F is

$$\begin{array}{rccc} \widetilde{F}: & \Lambda \times \mathbb{R}^d_+ & \to & \Lambda \times \mathbb{R}^d_+ \\ & \begin{pmatrix} \mathbf{x} \\ \mathbf{a} \end{pmatrix} & \mapsto & \begin{pmatrix} M(\mathbf{x})^{-1} & 0 \\ 0 & M(\mathbf{x})^\top \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{a} \end{pmatrix} \end{array}$$

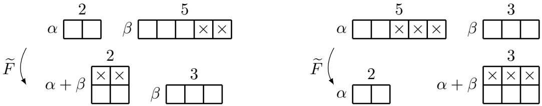
which has jacobian 1.
Main Problem :

find a conical domain $D \subset \Lambda \times \mathbb{R}^d_+$ on which $\widetilde{F}$ is a bijection.

The Farey map F is not a bijection :



One way to transform F into a bijection is to stack the removed part :



We thus constructed $\widetilde{F} : (x, y, \alpha, \beta) \mapsto \begin{cases} (x, y - x, \alpha + \beta, \beta) & \text{if } x < y, \\ (x - y, y, \alpha, \alpha + \beta) & \text{if } x > y. \end{cases}$

It is a bijection on the domain

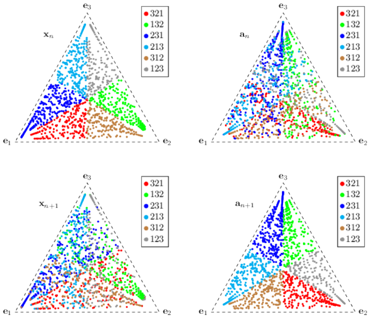
$$D = \{(x, y, \alpha, \beta) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid x > 0, y > 0, \alpha > 0, \beta > 0\}.$$

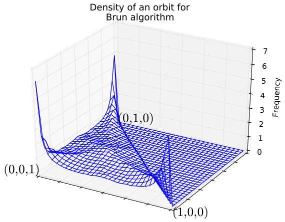
Skiping some details, we get the invariant density of $f = \frac{F(x)}{\|F(x)\|}$:

$$\delta(x) = \frac{1}{x(1-x)}.$$

Natural extension of Brun's algorithm

Sequences $\mathbf{x}_{n+1} = M(\mathbf{x}_n)^{-1}\mathbf{x}_n$ and $\mathbf{a}_{n+1} = M(\mathbf{x}_n)^{\top}\mathbf{a}_n$ helps to find D :





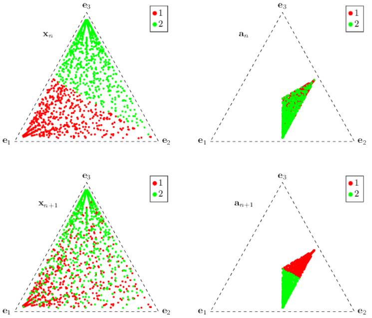
The density function of the invariant measure of $f : \Delta \to \Delta$ for the Brun algorithm is

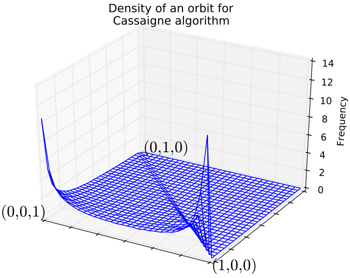
$$\frac{1}{2\,x_{\pi(2)}(1 - x_{\pi(2)})(1 - x_{\pi(1)} - x_{\pi(2)})}$$

on the part $\mathbf{x} = (x_1, x_2, x_3) \in \Lambda_\pi \cap \Delta$.

Natural extension of Cassaigne's algorithm

Sequences $\mathbf{x}_{n+1} = M(\mathbf{x}_n)^{-1}\mathbf{x}_n$ and $\mathbf{a}_{n+1} = M(\mathbf{x}_n)^\top \mathbf{a}_n$ helps to find D :





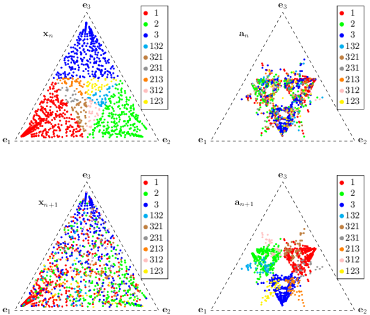
Theorem (Arnoux, L. 2015)

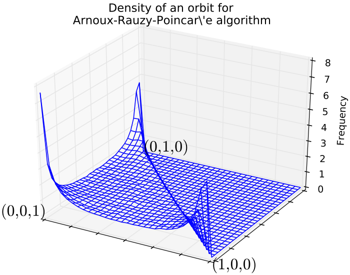
The density function of the invariant measure of $f : \Delta \to \Delta$ *for the Cassaigne algorithm is*

$$\frac{1}{(1-x)(1-z)}.$$

Natural extension of ARP algorithm

Sequences $\mathbf{x}_{n+1} = M(\mathbf{x}_n)^{-1}\mathbf{x}_n$ and $\mathbf{a}_{n+1} = M(\mathbf{x}_n)^{\top}\mathbf{a}_n$ helps to find D :

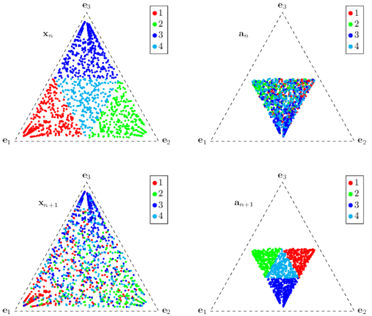


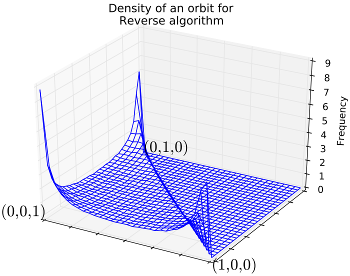


Question : Is this fractal domain D of positive mesure ?

Natural extension of Reverse algorithm

Sequences $\mathbf{x}_{n+1} = M(\mathbf{x}_n)^{-1}\mathbf{x}_n$ and $\mathbf{a}_{n+1} = M(\mathbf{x}_n)^{\top}\mathbf{a}_n$ helps to find D :





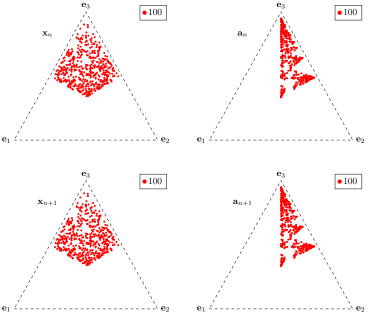
Theorem (Arnoux, L. 2015)

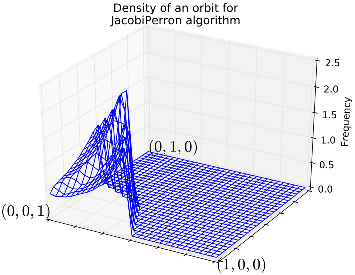
The density function of the invariant measure of $f : \Delta \to \Delta$ for the Reverse algorithm is

$$\frac{1}{(1-x)(1-y)(1-z)}.$$

Natural extension of Jacobi-Perron

Sequences $\mathbf{x}_{n+1} = M(\mathbf{x}_n)^{-1}\mathbf{x}_n$ and $\mathbf{a}_{n+1} = M(\mathbf{x}_n)^{\top}\mathbf{a}_n$ helps to find D :





Open Question : Find the density function of the invariant measure of Jacobi Perron.

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These are good MCF algorithms, i.e.,

- weakly convergent,
- have a finite invariant measure, thus conservative,
- $F^k(\mathbf{x}) \to 0$ for every rationaly independent $\mathbf{x} \in \Lambda$.

Farey	Brun	Selmer	ARP
$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix}$

They are of the good form

$$A = \begin{pmatrix} I_{d-1} & 0 \\ \mathbf{a} & 1 \end{pmatrix}$$

where I_{d-1} is a $(d-1) \times (d-1)$ identity matrix and $\mathbf{a} = (a_1, a_2, \ldots, a_{d-1}) \in \mathbb{R}^{d-1}$ is some vector taking only negative values with at least one of them being non zero.

Not good MCF algorithms and not good form :

Poincaré ($d = 3$) Fully subtractive

$$\left(\begin{array}{cc|c} 1 & 0 & 0 \\ -1 & 1 & 0 \\ \hline 0 & -1 & 1 \end{array}\right) \qquad \left(\begin{array}{cc|c} 1 & 0 & 0 \\ -1 & 1 & 0 \\ \hline -1 & 0 & 1 \end{array}\right)$$

Good by blocks :

Double Farey Poincaré ($d = 4$) Brun-Selmer

$$\left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{array}\right) \qquad \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ \hline 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{array}\right) \qquad \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{array}\right)$$

Let $\Lambda = \{(x_1, x_2, \ldots, x_d) \in \mathbb{R}^d \mid 0 < x_1 < x_2 < \ldots < x_d\}$.

Conjecture

A sorted MCF algorithm $F : \Lambda \to \Lambda : \mathbf{x} \mapsto M(\mathbf{x})^{-1}\mathbf{x}$ is good if and only if its matrix $M(\mathbf{e}_d)$ is decomposed by blocks into a lower triangular matrix such that the diagonal block B_i are of the good form :

$$M(\mathbf{e}_d) = \begin{pmatrix} B_1 & 0 & 0 & 0 \\ * & B_2 & 0 & 0 \\ * & * & \cdots & 0 \\ * & * & * & B_k \end{pmatrix} \quad \text{and} \quad B_i = \begin{pmatrix} I & 0 \\ \mathbf{a} & 1 \end{pmatrix}$$

Conjecture (Nogueira 1995)

Let F be the Poincaré algorithm on $\Lambda \subset \mathbb{R}^d_+$ and $f(\mathbf{x}) = \frac{F(\mathbf{x})}{\|F(\mathbf{x})\|}$.

- *if d is even : F is ergodic and f is ergodic and conservative.*
- *if d is odd : F is nonergodic and f is ergodic, although nonconservative, and f has an attractor, $f^k(\mathbf{x}) \to (0, \ldots 0, 1)$ as $k \to \infty$, for a.e. $\mathbf{x}$.*

Proved for $d = 3$ (Nogueira 1995) and $d = 4, 5$ (Nogueira 2001).

Results on the factor complexity and invariant measures are available here :

V. Berthé and S. Labbé. Factor complexity of S-adic words generated by the Arnoux-Rauzy-Poincaré algorithm. *Adv. in Appl. Math.*, 63 :90–130, 2015.

Pierre Arnoux and Sébastien Labbé. On some symmetric multidimensional continued fraction algorithms. *arXiv:1508.07814*, August 2015.

Images (and code to reproduce them) are available in those Cheat Sheets :

Sébastien Labbé. 3-dimensional Continued Fraction Algorithms Cheat Sheets. *arXiv:1511.08399*, November 2015.