Perron theorem and Coxeter groups

Comprehensible Seminar

SÉBASTIEN LABBÉ

Perron’s theorem says that the spectral radius of a positive matrix is a simple eigenvalue strictly greater than the modulus of the other eigenvalues. In the classical geometric representation of Coxeter groups, matrices are never positive but their spectral properties seem to be like positive matrices. In this talk, we show a criterion to check if the spectral radius of a real matrix (corresponding to an element of a Coxeter group) is a simple strictly dominant eigenvalue. If time allows, we will present open problems in the study of fractals and Coxeter groups that motivate this work.

This is a joint work with Jean-Philippe Labbé.
Reflection Group

V: n-dim. vector space
(\cdot, \cdot): symmetric bilinear form
If (\cdot, \cdot) is pos. definite, then \langle V, (\cdot, \cdot) \rangle is an Euclidean space.

Let \alpha \in V. A reflection is a linear map that sends \alpha \to -\alpha and fixes the hyperplane \{\lambda \in V \mid (\lambda, \alpha) = 0\} orthogonal to \alpha.

Formula: \quad S_{\alpha}(\lambda) = \lambda - \frac{2(\lambda, \alpha)}{(\alpha, \alpha)} \alpha

Orthogonal transformations are \text{O}(V) = \{\text{lin. map } f \mid (f(\alpha), f(\beta)) = (\alpha, \beta)\}

the set of lin map that preserves the bilinear form

A reflection group is a subgroup of \text{O}(V) generated by reflections.

Question: Classify all Reflection Groups

EX: \quad S_{\alpha} S_{\beta}, \quad S_{\beta} S_{\alpha}

Identity: \quad e

Groupe de Coxeter is a group with the presentation

\langle r_1, r_2, \ldots, r_n \mid (r_i r_j)^{m_{ij}} = 1 \rangle

where \( m_{ij} = 1 \) and \( m_{ij} \in \{2, 3, 4, \ldots, \infty\} \) if

Note: \quad \begin{align*}
& m_{ii} = 1 \implies r_i \text{ is an involution} \\
& m_{ij} = 2 \implies r_i \text{ and } r_j \text{ commute} \\
& m_{ij} = m_{ji} \neq 1
\end{align*}

EX: \quad \langle s, t \mid s^2 = t^2 = (st)^3 = 1 \rangle

Infinite dihedral group

Notation: \quad W = \text{group, } S = \{r_1, \ldots, r_n\} \text{ generators}
**Thm (Coxeter, 1934)** Every reflection group is a Coxeter group.

**Thm (Coxeter, 1935)** Every finite Coxeter group has a representation as a reflection group.

He also gave a classification of finite Coxeter groups, according to the matrix \((mst)\) of \(x_s x_t\) using Coxeter graphs.

"has a representation": \(S\) of elements of generators

\[ V = \text{vector space generated by vectors } \{ x_s | s \in S \} = \Delta \]

Symmetric bilinear form:

\[ B(x_s, x_t) = \begin{cases} -\cos \left( \frac{\pi}{m_{st}} \right) & \text{if } m_{st} \neq \infty \\ -1 & \text{if } m_{st} = \infty \end{cases} \]

\(\forall s \in S\), define the reflection

\[ o_s(\lambda) = \lambda - 2B(x_s, \lambda)x_s \]

The representation is:

\[ p: W \rightarrow GL(V) \]

Since \(p\) is injective, \(o_s\) preserves \(B\), then \(W\) isomorphic to a subgroup of \(O(V)\) generated by reflections. It is a reflection group.

**Root system**

\[ p: W \rightarrow GL(V) \]

\[ p(x_s) = o_s \]

\[ \Delta \leq V \]

\[ \Delta^+ \leq V^+ \]

\[ \Delta \cap \Delta^+ \leq \text{cone}(\Delta) \]

**Root system**

**Def** A root system \(\Delta\) is a finite set of nonzero vectors of \(V\) s.t.

1. \(\alpha + \alpha = \Delta\)
2. \(s_\alpha \Delta = \Delta\)
3. \(2(\alpha, \beta) / \beta \in \mathbb{Z}^+ \forall \alpha, \beta \in \Delta\) (crystallographic)

**Lemme (Humphreys, 1992)** Let \(G\) be a reflection group. A root system \(\Delta\) s.t. reflections of \(G\) are precisely \(\{ s_\alpha | \alpha \in \Delta \}\), i.e. reflections through hyperplanes perpendicular to the roots

**Lemme** \(\exists\) set \(\Delta\) of simple roots s.t. \(\Delta = W(\Delta)\)

and \(\exists\) set \(\Delta^+\) of positive roots s.t. \(\Delta = \Delta^+ \cup -\Delta^+\)

and \(\Delta \subset \Delta^+ \subset \text{cone}(\Delta)\).

Also \(\Delta\) is a base of \(V\).
Figure 1.11 Graïhe de Coxeter des groupes de Coxeter irréductibles finis. Les groupes de types $A, B$ et $D$ sont souvent appelés les familles infinies et les autres sont appelés groupes exceptionnels.
Figure 1.12 Graphe de Coxeter des groupes de Coxeter irréductibles affines.
$$M = (M_{st}) = \begin{pmatrix} M_{ss} & M_{st} \\ M_{ts} & M_{tt} \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$$

$$V = \text{v.s. generated by the base } \Delta = \{ \alpha_s, \alpha_t \} = \{ (0), (1) \}$$

$$B(\alpha_s, \alpha_s) = -\cos \left( \frac{\pi}{4} \right) = 1 = B(\alpha_t, \alpha_t)$$

$$B(\alpha_s, \alpha_t) = -\cos \left( \frac{\pi}{3} \right) = -\frac{1}{2} = B(\alpha_t, \alpha_s)$$

$$B(x, y) = \begin{pmatrix} x_1 & x_2 \\ -\frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$\sigma_3 (\lambda) = \lambda - 2 B(\alpha_s, \lambda) \alpha_s = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$$

$$\sigma_6 (\lambda) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$$

We get \( p(sututsutus) = \begin{pmatrix} -4.04 & 1 & 2.24 \ldots \\ -4.04 & 1.80 & 1.80 \ldots \\ -4.29 & 3.24 & 2.34 \ldots \end{pmatrix} \)

Remark: The set of columns \( \Xi \) is stable under \( W \): \( W(\Xi) = \Xi \). In fact \( \Xi = W(\Delta) \).

\( \Xi \) is a root system, fundamental in the study of Coxeter groups.

Lemma: Let \( G \) be a reflection group. A root system \( \Xi \) s.t. reflections of \( G \) are precisely the reflections through hyperplanes perpendicular to the roots \( \{ s_\alpha \mid \alpha \in \Xi \} \).

Positive roots: \( \Xi^+ = \text{cone}(\Delta) \cap \Xi \)

Negative roots: \( \Xi^- = -\Xi^+ \).

Partition: \( \Xi = \Xi^+ \cup \Xi^- \).

Therefore the entries of a column all have the same sign.
If $|W| = \infty$, then $|\mathbb{P}W| = \infty$.

**Definition:** Limit roots are the accumulation points of roots in the projective space $\mathbb{P}W$.

**Theorem (Hohloch, J. Plabür, Ripoll, 2019):** Limit roots are on the isotropic cone $\mathcal{C}(V) = \{x \in V | B(x, x) = 0\}$.

**Conjecture:** Limit roots $\equiv$ infinite reduced wads $/\nu$

**Subcase:** Periodic infinite reduced wads.

**Question:** $w \in W$. Is it true that columns of powers of $p(w)$ all have the same limit in $\mathbb{P}V$?

In other words, is $p(w)$ like a positive matrix for which Perron theorem applies?

**Theorem (Perron 1907):** Let $A \in \mathbb{R}^{n \times n}$ primitive, spectral radius $\lambda$. Then

1. $\lambda$ is a simple root of the char. poly.
2. $\lambda$ is strictly greater than the modulus of any other eigenvalue and
3. $\lambda$ has strictly positive eigenvectors $u$ and $v$.

Also, $\lim_{k \to \infty} \frac{1}{\lambda^k} A^k = WU$ is of rank one, where $UV = (1)$.

Can not be generalized to matrices with pos. and negative columns while keeping conclusions (1), (2), and (3):

**Example:**

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 2 & 3 & 1 \\ -3 & 1 & 3 & 1 \\ -3 & 1 & 3 & 1 \end{pmatrix}, \quad \chi_A(\lambda) = (\lambda - 2)^2 (\lambda - 1)$$

But we have this criteria:

**Theorem (Labbe 2)** Let $A \in \mathbb{R}^{n \times n}$ s.t. $AV = \lambda V, uA = \lambda u, u, V > 0$.

1. The fol. cond. are equivalent: $1^TV = (1)$ and $uv = (1)$.

2. $\lim_{k \to \infty} \frac{1}{\lambda^k} A^k = UV$

3. $\lambda > 1$ and $1^TV = (1)$, and $uv = (1)$.

**Example:**

$$B = \begin{pmatrix} -1 & 14 \\ -24 & 29 \end{pmatrix}, \quad V = \frac{1}{20} \begin{pmatrix} 7 \\ 13 \end{pmatrix}, \quad u = \frac{1}{\lambda} \begin{pmatrix} 20 & 20 \end{pmatrix}, \quad \lambda = 15.$$

We have $BV = \lambda V, uB = \lambda u, 1^TV = (1)$, and $uv = (1).

We compute:

$$\lambda V 1^T + B - V 1^T B = 15 \cdot \begin{pmatrix} 7 & 7 \\ 13 & 13 \end{pmatrix} + \begin{pmatrix} 7 & 7 \\ 13 & 13 \end{pmatrix} = \begin{pmatrix} 36 & 36 \\ 54 & 54 \end{pmatrix} > 0$$

$\Rightarrow$ 15 is simple w.r.t. $v$, of $B$ strictly dominating.
```python
In [40]:
def roots(G, depth):
    L = []
    for i in range(depth):
        for w in G.elements_of_length(i):
            for column in w.canonical_matrix().columns():
                column = tuple(entry.real() for entry in column.n())
                L.append(column)
    return L

def draw_roots(G, depth):
    R = roots(G, depth)
    P = points(R)
    if len(R[0]) == 2:
        P.show()
    elif len(R[0]) == 3:
        P.show(viewer='tachyon')
    else:
        raise NotImplementedError

In [22]: G = CoxeterGroup([[1,2,3],[2,1,7],[3,7,1]])
G
Out[22]: Coxeter group over Universal Cyclotomic Field with Coxeter matrix:
[[1 2 3]
 [2 1 7]
 [3 7 1]]

In [23]: draw_roots(G, 15)
```

Limit roots of Coxeter groups

http://localhost:8888/nbconvert/html/Limit roots o...
def normalized_roots(G, depth):
    R = roots(G, depth)
    L = [vector(root)/sum(root) for root in R]
    return L

sqrt2 = sqrt(2)
sqrt3 = sqrt(3)
M3to2 = matrix(2,[-sqrt3,sqrt3,0,-1,-1,2],ring=RR)/2
M4to3 = matrix([[1, 0, -1/sqrt2],
                 [-1, 0, -1/sqrt2],
                 [0, 1, 1/sqrt2],
                 [0, -1, 1/sqrt2]], ring=RR).transpose()

def draw_normalized_roots(G, depth):
    R = normalized_roots(G, depth)
    if len(R[0]) == 3:
        P = points([M3to2*root for root in R])
        P.show(frame=False)
    elif len(R[0]) == 4:
        P = points([M4to3*root for root in R])
        P.show(viewer='tachyon')
    else:
        raise Not ImplementedError

In [39]: draw_normalized_roots(G, 15)
Appendix A

Some root systems of rank 3 & 4

Here are some images of normalized root systems of rank 3 and 4 with small labels.

Figure A.1: In the first column: type $A_3$, $H_3$ and the triangle group $\{2,3,7\}$. In the second column: $B_3$, $I_2(6)$ and type $\tilde{A}_2$. 
Figure A.2: In the first column: type $\tilde{B}_2$ and the triangle group $\{3, 3, 4\}$. In the second column: the triangle group $\{2, 4, 5\}$ and the triangle group $\{3, 4, 4\}$.

Figure A.3: In the top right image: type $\tilde{C}_3$ and three different groups two of which give rise to fractal limits.