A d-dimensional extension of Christoffel words

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1103^{rd} AMS meeting
Halifax, October 18, 2014

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Computer Science > Discrete Mathematics

A d-dimensional extension of Christoffel words

Sébastien Labbé, Christophe Reutenauer

(Submitted on 15 Apr 2014)

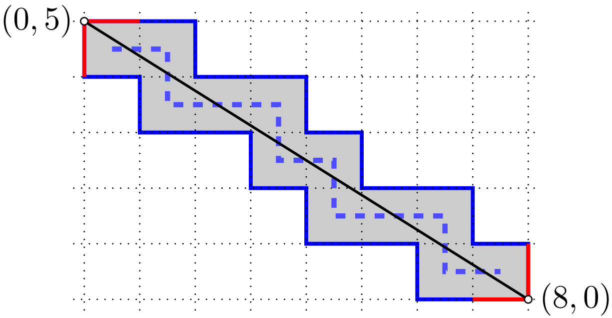
Available on :

`http://arxiv.org/abs/1404.4021`

Outline

Christoffel words

- Let $\mathbf{a} = (a_1, a_2) \in \mathbb{N}^2$ with $\gcd(\mathbf{a}) = 1$.
- A Christoffel path of normal vector $\mathbf{a}$ is a path from $(0, a_1)$ to $(a_2, 0)$ staying above the segment and closest to the segment.
- It is coded by a word $w_{\mathbf{a}} = amb \in \{a, b\}^*$ called Christoffel word.
- The word m is well-known as the central word : it is a palindrome.



$$w_{(5,8)} = a \cdot abaababaaba \cdot b$$

Motivation : Lyndon words and convexity

- Two words w, w' are conjugate if $w = uv$ and $w' = vu$.
- A word l is a Lyndon word if $l = uv$ implies that $l <_{lex} vu$.

Theorem (Lyndon)

Any nonempty word w admits a unique factorization as a sequence of lexicographically decreasing Lyndon words :

$$w = l_1^{n_1} l_2^{n_2} \cdots l_k^{n_k}, \qquad l_1 > l_2 > \cdots > l_k,$$

where $n_i \geq 1$ and l_i is a Lyndon word, for all i such that $1 \leq i \leq k$.

```
sage: w = Word('journeesmontoisesareinnancythisyear')
sage: w.lyndon_factorization()
(journ, eesmontoises, areinn, ancythisyear)
```

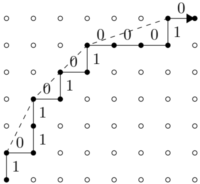
(note : in this slide, Christoffel words are drawn with positive slope)

Theorem (Brlek, Lachaud, Provençal, Reutenauer, 2009)

A word w is NW-convex iff its unique Lyndon factorization $l_1^{n_1} l_2^{n_2} \cdots l_k^{n_k}$ is such that all l_i are Christoffel words.

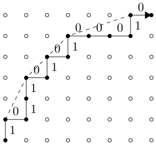
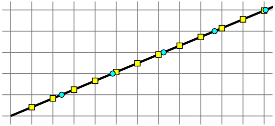
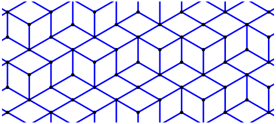
Example :

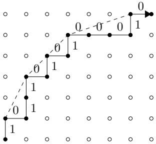
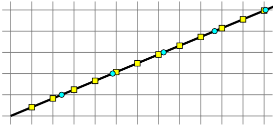
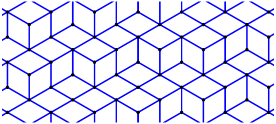
$$v = 1011010100010$$
$$= (1)^1 \cdot (011)^1 \cdot (01)^2 \cdot (0001)^1 \cdot (0)^1$$



0, 011, 01, 0001 and 0 are all Christoffel words.

S. Brlek, J.-O. Lachaud, X. Provençal, and C. Reutenauer. Lyndon + Christoffel = digitally convex. *Pattern Recognition*, 42(10) :2239–2246, October 2009.

	$d = 2$	$d \geq 3$
$\mathbf{a} \in \mathbb{Z}^d$	Christoffel words 	atome of some convex surface ?
$\mathbf{a} \in \mathbb{R}^d \setminus \mathbb{Q}^d$	Stumian words 	Discrete hyperplanes 

	$d = 2$	$d \geq 3$
$\mathbf{a} \in \mathbb{Z}^d$	Christoffel words 	atome of some convex surface ?
$\mathbf{a} \in \mathbb{R}^d \setminus \mathbb{Q}^d$	Stumian words 	Discrete hyperplanes 

Eric Domenjoud and Laurent Vuillon. Geometric palindromic closure. *Unif. Distrib. Theory*, 7(2) :109–140, 2012.

“In fact, we propose with this construction a kind of geometric generalisation of Christoffel words in all dimensions.”

Jean Berstel. Sturmian and episturmian words. In Symeon Bozapalidis and George Rahonis, editors, *Algebraic Informatics*, volume 4728 of *LNCS*, pages 23–47. Springer Berlin Heidelberg, 2007.

"We list fourteen characterizations of central words."

One of them is :

Theorem (Pirillo)

A word $w = amb \in \{a, b\}^*$ is a Christoffel word if and only if amb and bma are conjugate.

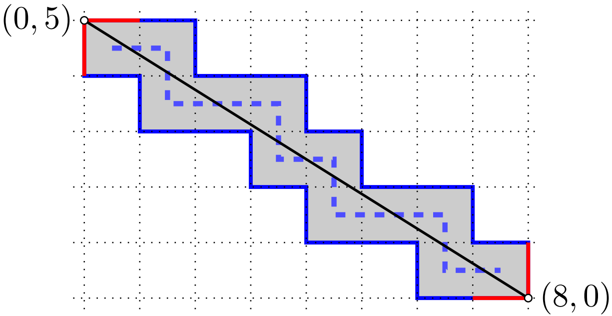
Giuseppe Pirillo. A curious characteristic property of standard Sturmian words. In *Algebraic combinatorics and computer science*, pages 541–546. Springer Italia, Milan, 2001.

Pirillo's theorem

Theorem (Pirillo)

A word $w = amb \in \{a,b\}^*$ is a Christoffel word if and only if amb and bma are conjugate.

Example of the sufficiency :



Pirillo's theorem

Theorem (Pirillo)

A word $w = amb \in \{a,b\}^*$ is a Christoffel word if and only if amb and bma are conjugate.

Example of the necessity :

Suppose amb is the 3^{rd} conjugate of bma and suppose $|m| = 9$:

$$\begin{array}{ccccccccccc} a & m_1 & m_2 & m_3 & m_4 & m_5 & m_6 & m_7 & m_8 & m_9 & b \\ m_3 & m_4 & m_5 & m_6 & m_7 & m_8 & m_9 & a & b & m_1 & m_2 \end{array}$$

Then

$$a = m_3 = m_6 = m_9 = m_1 = m_4 = m_7 = a \quad \text{and} \quad b = m_2 = m_5 = m_8 = b$$

and

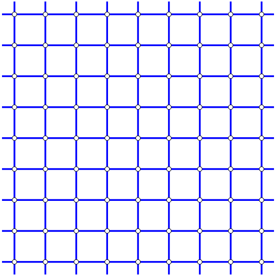
$$amb = a \cdot abaabaaba \cdot b$$

is a Christoffel word.

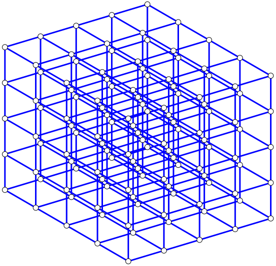
Oriented edges of the hypercubic lattice :

$$\mathbb{E}_d = \{(\mathbf{u}, \mathbf{u} + \mathbf{e}_i) : \mathbf{u} \in \mathbb{Z}^d \text{ and } 1 \leq i \leq d\}$$

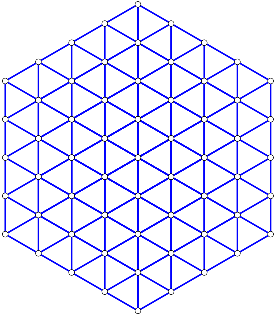
Note that the set $\mathbb{E}_d$ also corresponds to the Cayley graph of $\mathbb{Z}^d$ with generators $\mathbf{e}_i$ for all i with $1 \leq i \leq d$.



$\mathbb{E}_2$



$\mathbb{E}_3$



$\mathbb{E}_3$

Christoffel graph $\mathcal{H}_{\mathbf{a}}$

- Let $\mathbf{a} = (a_1, a_2, \ldots, a_d) \in \mathbb{N}^d$ such that $\gcd(\mathbf{a}) = 1$.
- Let $s = \|\mathbf{a}\|_1 = \sum a_i$.
- We define the mapping :

$$\begin{array}{rccl} \mathcal{F}_{\mathbf{a}} : & \mathbb{Z}^d & \to & \mathbb{Z}/s\mathbb{Z} \\ & \mathbf{x} & \mapsto & \mathbf{a} \cdot \mathbf{x} \bmod s. \end{array}$$

Note : we identify $\mathbb{Z}/s\mathbb{Z}$ and $\{0, 1, \ldots, s-1\}$ with its total order.

Definition

The Christoffel graph $\mathcal{H}_{\mathbf{a}}$ of normal vector $\mathbf{a}$ is the subset of edges of $\mathbb{E}_d$ increasing for the function $\mathcal{F}_{\mathbf{a}}$:

$$\mathcal{H}_{\mathbf{a}} = \{(\mathbf{u}, \mathbf{u} + \mathbf{e}_i) \in \mathbb{E}_d : \mathcal{F}_{\mathbf{a}}(\mathbf{u}) < \mathcal{F}_{\mathbf{a}}(\mathbf{u} + \mathbf{e}_i)\}.$$

Christoffel graph $\mathcal{H}_{\mathbf{a}}$

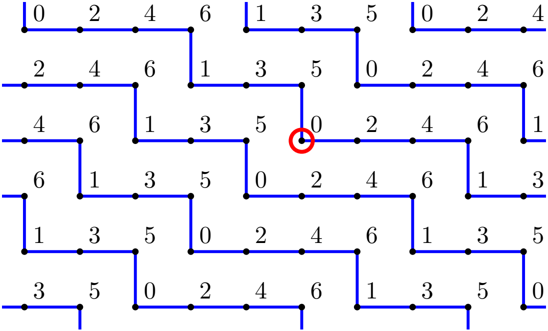


FIGURE: The graph $\mathcal{H}_{\mathbf{a}}$ with $\mathbf{a} = (2, 5)$ and $s = 7$.

Lemma

The graph $\mathcal{H}_{\mathbf{a}}$ is invariant under the translation by the vector $\sum_{i=1}^{d} \mathbf{e}_i = (1, 1, \ldots, 1)$.

The graph $I_{\mathbf{a}}$

Definition (Image)

Let $f : \mathbb{Z}^d \to S$ be an homomorphism of $\mathbb{Z}$-module. For some subset of edges $X \subseteq \mathbb{E}_d$, we define the image by f of the edges X by

$$f(X) = \{(f(\mathbf{u}), f(\mathbf{v})) \mid (\mathbf{u}, \mathbf{v}) \in X\}.$$

We define $I_{\mathbf{a}} = \pi(\mathcal{H}_{\mathbf{a}})$ where π is the orthogonal projection from $\mathbb{R}^d$ onto the hyperplane $\mathcal{D}$ of equation $\sum x_i = 0$.

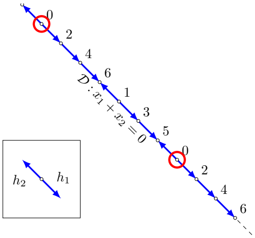
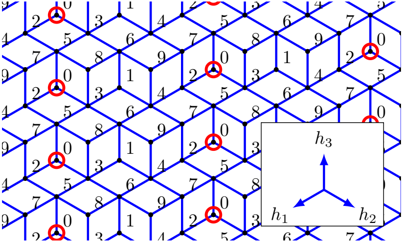


FIGURE: The graph $I_{\mathbf{a}}$ when $\mathbf{a} = (2, 5)$.

The graph $I_{\mathbf{a}}$ when $\mathbf{a} = (2, 3, 5)$:



Lemma

The graph $I_{\mathbf{a}}$ produces a tiling of $\mathcal{D}$ by d types of $(d-1)$-dimensional parallelotopes.

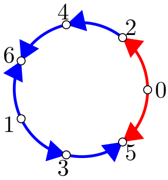
Jean Françon. Sur la topologie d'un plan arithmétique. *Theoret. Comput. Sci.*, 156(1-2) :159–176, 1996.

Pierre Arnoux, Valerie Berthé, and Shunji Ito. Discrete planes, $\mathbb{Z}^2$-actions, Jacobi-Perron algorithm and substitutions. *Ann. Inst. Fourier (Grenoble)*, 52(2) :305–349, 2002.

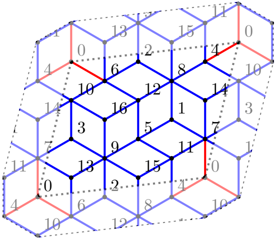
The graph $\mathcal{G}_{\mathbf{a}}$

We define $\mathcal{G}_{\mathbf{a}} = \mathcal{F}_{\mathbf{a}}(\mathcal{H}_{\mathbf{a}})$. It is also equal to

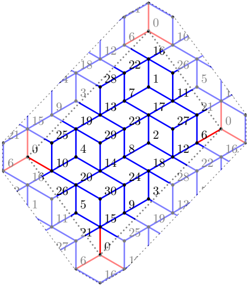
$$\mathcal{G}_{\mathbf{a}} = \{(k, k + a_i) \mid k \in \mathbb{Z}/s\mathbb{Z}, 1 \leq i \leq d \text{ and } k < k + a_i\}.$$



$\mathcal{G}_{(2,5)}$
aaabaab



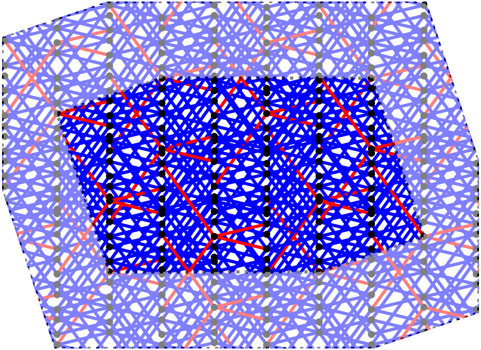
$\mathcal{G}_{(4,6,7)}$



$\mathcal{G}_{(6,10,15)}$

Works for $d \geq 4$: $\mathcal{G}_{(2,3,4,5)}$

$\mathcal{G}_{(2,3,4,5)}$



Edges incident to zero

- Edges of $\mathbb{E}_d$ incident to zero :

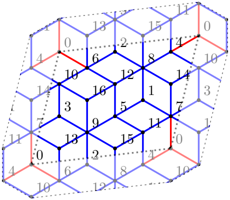
$$\mathcal{Q} = \{(\mathbf{u}, \mathbf{v}) \in \mathbb{E}_d : \mathcal{F}_\mathbf{a}(\mathbf{u}) = 0 \text{ or } \mathcal{F}_\mathbf{a}(\mathbf{v}) = 0\}.$$

- The legs of $X \subseteq \mathbb{E}_d$ are the edges of

$$X \cap \mathcal{Q}.$$

- The body of $X \subseteq \mathbb{E}_d$ is the set

$$X \setminus \mathcal{Q}.$$



Flip, reversal and translation

Let $X \subseteq \mathbb{E}_d$. The reversal $-X$ is

$$-X = \{(-\mathbf{v}, -\mathbf{u}) \mid (\mathbf{u}, \mathbf{v}) \in X\}$$

Let $\mathbf{t} \in \mathbb{Z}^d$. The translate $X + \mathbf{t}$ is

$$X + \mathbf{t} = \{(\mathbf{u} + \mathbf{t}, \mathbf{v} + \mathbf{t}) \mid (\mathbf{u}, \mathbf{v}) \in X\}.$$

The FLIP operation exchanges edges of $X \subseteq \mathbb{E}_d$ incident to zero :

$$\textsc{flip} : X \mapsto (X \setminus \mathcal{Q}) \cup (\mathcal{Q} \setminus X).$$

The FLIP generalizes the function $amb \mapsto bma$.
Note : the FLIP when $d = 3 \leftrightarrow$ a flip in a rhombus tiling

Pierre Arnoux, Valérie Berthé, Thomas Fernique, and Damien Jamet. Functional stepped surfaces, flips, and generalized substitutions. *Theoret. Comput. Sci.*, 380(3) :251–265, 2007.

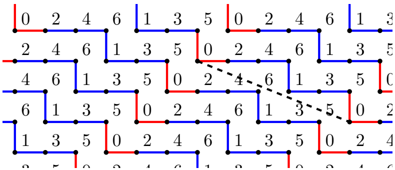
Olivier Bodini, Thomas Fernique, Michael Rao, and Éric Rémila. Distances on rhombus tilings. *Theoret. Comput. Sci.*, 412(36) :4787–4794, 2011.

Flipping is translating

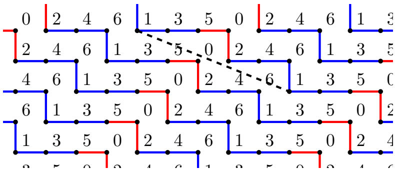
Proposition

Let $\mathbf{t} \in \mathbb{Z}^d$ *be such that* $\mathcal{F}_{\mathbf{a}}(\mathbf{t}) = 1$ *(bezout vector). The translate by* $\mathbf{t}$ *of* $\mathcal{H}_{\mathbf{a}}$ *is equal to its flip, i.e.,*

$$\mathcal{H}_{\mathbf{a}} + \mathbf{t} = \textsc{flip}(\mathcal{H}_{\mathbf{a}}).$$



$\mathcal{H}_{\mathbf{a}}$ with $\mathbf{a} = (2, 5)$



$\textsc{flip}(\mathcal{H}_{\mathbf{a}})$

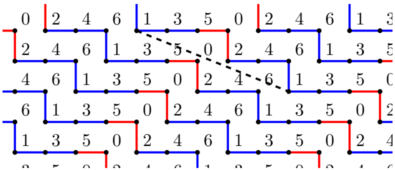
This is a generalization of one implication of Pirillo Theorem (it generalizes the fact that a Christoffel word *amb* is conjugate to to *bma*).

Flipping is translating

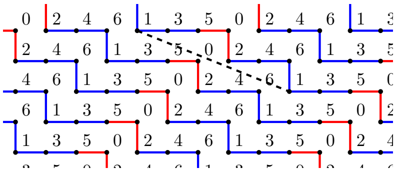
Proposition

Let $\mathbf{t} \in \mathbb{Z}^d$ *be such that* $\mathcal{F}_{\mathbf{a}}(\mathbf{t}) = 1$ *(bezout vector). The translate by* $\mathbf{t}$ *of* $\mathcal{H}_{\mathbf{a}}$ *is equal to its flip, i.e.,*

$$\mathcal{H}_{\mathbf{a}} + \mathbf{t} = \text{FLIP}(\mathcal{H}_{\mathbf{a}}).$$



$\text{FLIP}(\mathcal{H}_{\mathbf{a}})$



$\text{FLIP}(\mathcal{H}_{\mathbf{a}})$

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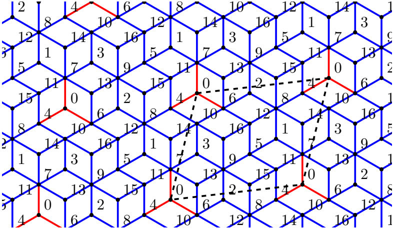
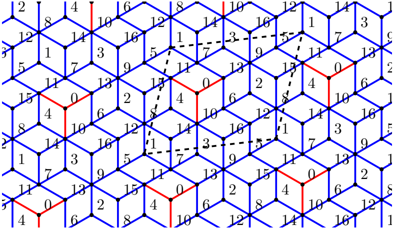
Flipping is translating



$I_{\mathbf{a}}$ with $\mathbf{a} = (4, 6, 7)$



$\text{FLIP}(I_{\mathbf{a}})$

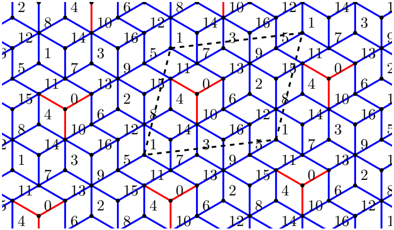
Thus, it verifies :

$$\mathcal{H}_{\mathbf{a}} + \mathbf{t} = \text{FLIP}(\mathcal{H}_{\mathbf{a}})$$

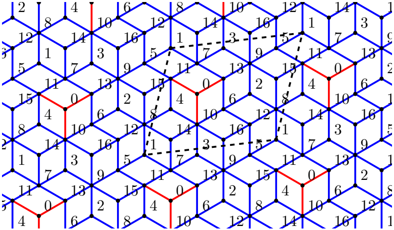
and also after projection by π :

$$I_{\mathbf{a}} + \pi(\mathbf{t}) = \text{FLIP}(I_{\mathbf{a}}).$$

Flipping is translating



$\text{FLIP}(I_{\mathbf{a}})$



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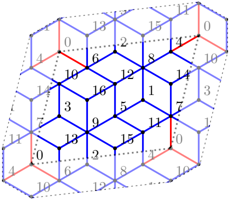
$$\mathcal{H}_{\mathbf{a}} + \mathbf{t} = \text{FLIP}(\mathcal{H}_{\mathbf{a}})$$

and also after projection by π :

$$I_{\mathbf{a}} + \pi(\mathbf{t}) = \text{FLIP}(I_{\mathbf{a}}).$$

(Non-characteristic) properties like :

- Central words are palindromes.
- The reversal $\widetilde{amb}$ of a Christoffel word is equal to bma.
- A Christoffel word is conjugate to its reversal.



become :

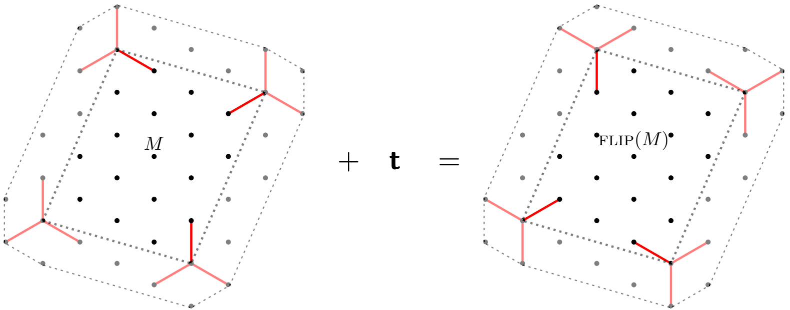
Lemma

- *The body of $\mathcal{H}_{\mathbf{a}}$ is symmetric, i.e.,* $-(\mathcal{H}_{\mathbf{a}} \setminus \mathcal{Q}) = \mathcal{H}_{\mathbf{a}} \setminus \mathcal{Q}$.
- *The reversal of $\mathcal{H}_{\mathbf{a}}$ is equal to its flip, i.e.,* $-\mathcal{H}_{\mathbf{a}} = \textsc{flip}(\mathcal{H}_{\mathbf{a}})$.
- *Let $\mathbf{t} \in \mathbb{Z}^d$ be such that $\mathcal{F}_{\mathbf{a}}(\mathbf{t}) = 1$. Then* $-\mathcal{H}_{\mathbf{a}} = \mathcal{H}_{\mathbf{a}} + \mathbf{t}$.

What about the converse ?

- Let K be a subgroup of finite index of $\mathbb{Z}^d$ such that $\sum_{i=1}^{d} \mathbf{e}_i \in K$.
- Let $\mathcal{Q} = \{(\mathbf{u}, \mathbf{v}) \in \mathbb{E}_d \mid \mathbf{u} \in K \text{ or } \mathbf{v} \in K\}$.

Question : For what subset of edges $M \subseteq \mathbb{E}_d$ invariant for the group of translations K does there exists $\mathbf{t} \in \mathbb{Z}^d$ such that $M + \mathbf{t} = \text{FLIP}(M)$?



Question : Does M has to be a Christoffel graph $\mathcal{H}_{\mathbf{a}}$?

Christoffel graph $\mathcal{H}_{\mathbf{a},\omega}$

- Let $\mathbf{a} = (a_1, a_2, \dots, a_d) \in \mathbb{N}^d$ such that $\gcd(\mathbf{a}) = 1$ with $s = \|\mathbf{a}\|_1$.
- Let $\omega \in \mathbb{N}$ be a divisor of s, called width, such that $0 < s/\omega < d$.
- We define the mapping :

$$\begin{array}{rccl} \mathcal{F}_{\mathbf{a},\omega} : & \mathbb{Z}^d & \to & \mathbb{Z}/\omega\mathbb{Z} \\ & \mathbf{x} & \mapsto & \mathbf{a} \cdot \mathbf{x} \bmod \omega. \end{array}$$

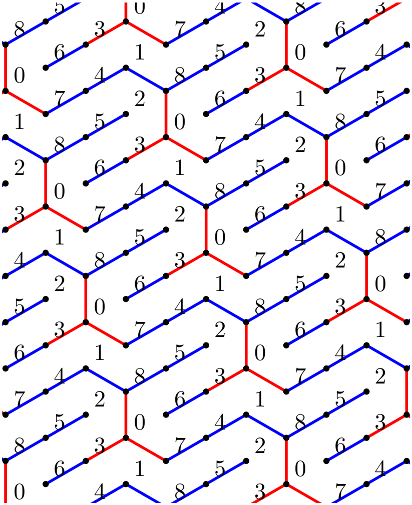
Note : we identify $\mathbb{Z}/\omega\mathbb{Z}$ and $\{0, 1, \dots, \omega - 1\}$ with its total order.

Definition

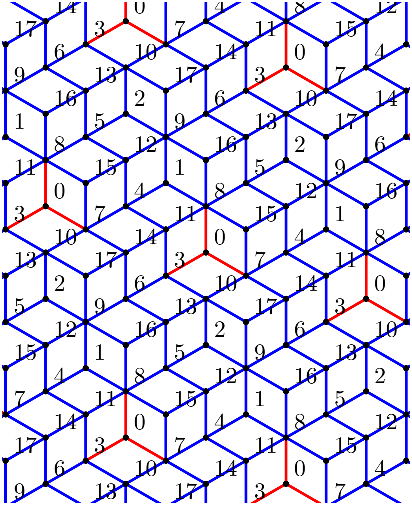
The Christoffel graph $\mathcal{H}_{\mathbf{a},\omega}$ of normal vector $\mathbf{a}$ and width ω is the subset of edges of $\mathbb{E}_d$ increasing for the function $\mathcal{F}_{\mathbf{a},\omega}$:

$$\mathcal{H}_{\mathbf{a},\omega} = \{(\mathbf{u}, \mathbf{u} + \mathbf{e}_i) \in \mathbb{E}_d : \mathcal{F}_{\mathbf{a},\omega}(\mathbf{u}) < \mathcal{F}_{\mathbf{a},\omega}(\mathbf{u} + \mathbf{e}_i)\}.$$

Note : $\mathcal{H}_{\mathbf{a},\omega} \subseteq \mathcal{H}_{\mathbf{a}} \subseteq \mathbb{E}_d$.



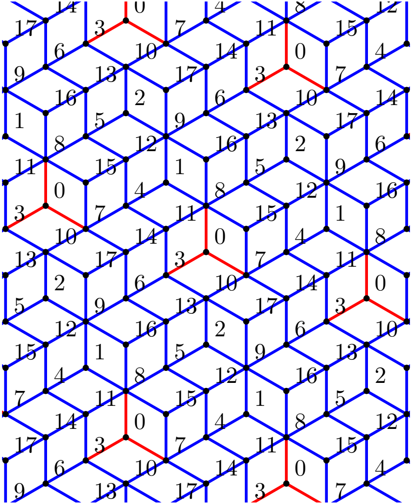
$\mathcal{H}_{(3,7,8),9}$



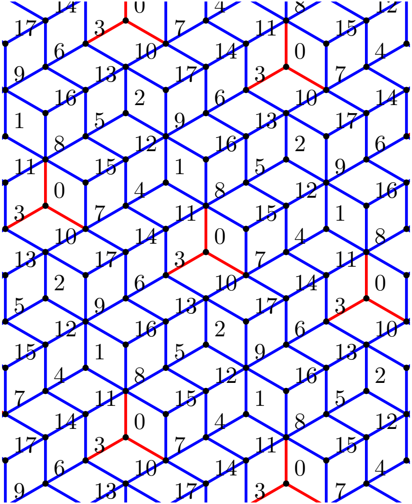
$\mathcal{H}_{(3,7,8)}$

Note : $\mathcal{H}_{\mathbf{a},\omega}$ is the union of two thin discrete plane.

Note : $\mathcal{H}_{\mathbf{a},\omega} \subseteq \mathcal{H}_{\mathbf{a}} \subseteq \mathbb{E}_d$.



$\mathcal{H}_{(3,7,8)}$



$\mathcal{H}_{(3,7,8)}$

Note : $\mathcal{H}_{\mathbf{a},\omega}$ is the union of two thin discrete plane.

d-dimensional Pirillo's theorem

Theorem

- *Let K be a subgroup of finite index of $\mathbb{Z}^d$ such that $\sum_{i=1}^{d} \mathbf{e}_i \in K$.*
- *Let $\mathcal{Q} = \{(\mathbf{u}, \mathbf{v}) \in \mathbb{E}_d \mid \mathbf{u} \in K \text{ or } \mathbf{v} \in K\}$.*
- *Suppose the legs of M are positive, i.e.,*

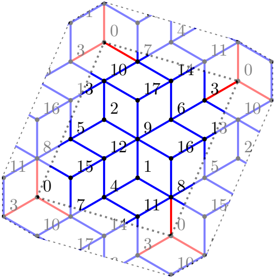
$$M \cap \mathcal{Q} = \{(\mathbf{0}, \mathbf{e}_i) \mid 1 \leq i \leq d\} + K.$$

There exists $\mathbf{t} \in \mathbb{Z}^d$ such that $M + \mathbf{t} = \text{FLIP}(M)$ if and only if $M = \mathcal{H}_{\mathbf{a},\omega}$ is the Christoffel graph of normal vector $\mathbf{a}$ and width ω where ω is a divisor of s such that $0 < s/\omega < d$.

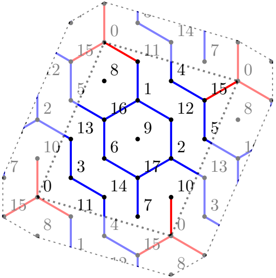
Let K be the subgroup of $\mathbb{Z}^3$ given by

$$K = \langle (0,4,1), (-2,0,3), (1,1,1) \rangle.$$

The only two possibilities for a pattern $M \subseteq \mathbb{E}_d$ invariant under translations of K satisfying $M = \text{FLIP}(M + \mathbf{t})$ are :



$\mathcal{H}_{(3,7,8)}$
$\mathbf{t} = \mathbf{e}_3 - \mathbf{e}_2$



$\mathcal{H}_{(15,11,10),18}$
$\mathbf{t} = \mathbf{e}_2 - \mathbf{e}_3$

Credit :)

The images of this presentation were done using Sage and the new optional Sage package slabbe-0.1.spkg. Install :

```
sage -i http://www.liafa.univ-paris-diderot.fr/~labbe/Sage/slabbe-0.1.spkg
```

Import :

```
sage: from slabbe import *
```

And use :

```
sage: G = ChristoffelGraph((6,8,9)); G
Christoffel set of edges for normal vector v=(6, 8, 9)
sage: latex.add_to_preamble("\\usepackage{tikz}")
sage: view(G.tikz_kernel(), tightpage=True)
```