# A $d$-dimensional extension of Christoffel words


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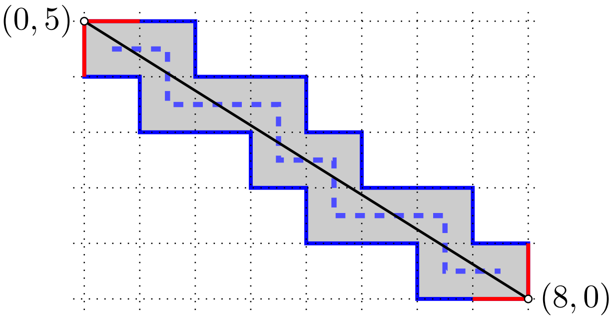
# Outline

1. Content of the talk

# Christoffel words

- Let $\mathbf{a} = (a_1, a_2) \in \mathbb{N}^2$ with $\gcd(\mathbf{a}) = 1$.
- A Christoffel path of normal vector $\mathbf{a}$ is a path from $(0, a_1)$ to $(a_2, 0)$ staying above the segment and closest to the segment.
- It is coded by a word $w_{\mathbf{a}} = amb \in \{a, b\}^*$ called Christoffel word.
- The word $m$ is well-known as the central word : it is a palindrome.




$$w_{(5,8)} = a \cdot abaababaaba \cdot b$$

# Motivation : Lyndon words and convexity

- Two words $w, w'$ are conjugate if $w = uv$ and $w' = vu$.
- A word $l$ is a Lyndon word if $l = uv$ implies that $l <_{lex} vu$.

**Theorem (Lyndon)**

*Any nonempty word $w$ admits a unique factorization as a sequence of lexicographically decreasing Lyndon words :*

$$w = l_1^{n_1} l_2^{n_2} \cdots l_k^{n_k}, \qquad l_1 > l_2 > \cdots > l_k,$$

*where $n_i \geq 1$ and $l_i$ is a Lyndon word, for all $i$ such that $1 \leq i \leq k$.*

```
sage: w = Word('journeesmontoisesareinnancythisyear')
sage: w.lyndon_factorization()
(journ, eesmontoises, areinn, ancythisyear)
```

# Motivation : Lyndon words and convexity

(note : in this slide, Christoffel words are drawn with positive slope)

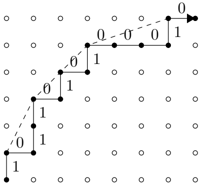
**Theorem (Brlek, Lachaud, Provençal, Reutenauer, 2009)**

*A word $w$ is NW-convex iff its unique Lyndon factorization $l_1^{n_1} l_2^{n_2} \cdots l_k^{n_k}$ is such that all $l_i$ are Christoffel words.*

Example :

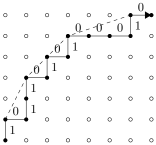
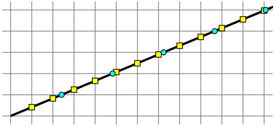
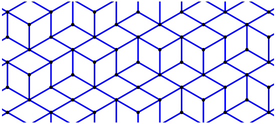
$$v = 1011010100010$$
$$= (1)^1 \cdot (011)^1 \cdot (01)^2 \cdot (0001)^1 \cdot (0)^1$$

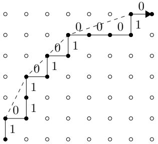
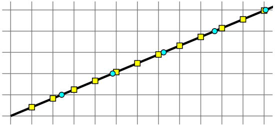
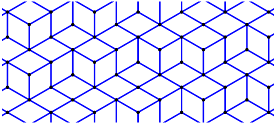
0, 011, 01, 0001 and 0 are all Christoffel words.





S. Brlek, J.-O. Lachaud, X. Provençal, and C. Reutenauer. Lyndon + Christoffel = digitally convex. *Pattern Recognition*, 42(10) :2239–2246, October 2009.

| | $d = 2$ | $d \geq 3$ |
|---|---|---|
| $\mathbf{a} \in \mathbb{Z}^d$ | Christoffel words  | base element of some convex surface ? |
| $\mathbf{a} \in \mathbb{R}^d \setminus \mathbb{Q}^d$ | Stumian words  | Discrete hyperplanes  |

| | $d = 2$ | $d \geq 3$ |
|---|---|---|
| $\mathbf{a} \in \mathbb{Z}^d$ | Christoffel words  | base element of some convex surface ? |
| $\mathbf{a} \in \mathbb{R}^d \setminus \mathbb{Q}^d$ | Stumian words  | Discrete hyperplanes  |

Eric Domenjoud and Laurent Vuillon. Geometric palindromic closure. *Unif. Distrib. Theory*, 7(2) :109–140, 2012.

*"In fact, we propose with this construction a kind of geometric generalisation of Christoffel words in all dimensions."*

# Discrete lines and planes

Jean Berstel. Sturmian and episturmian words. In Symeon Bozapalidis and George Rahonis, editors, *Algebraic Informatics*, volume 4728 of *LNCS*, pages 23–47. Springer Berlin Heidelberg, 2007.

*"We list fourteen characterizations of central words."*

One of them is :

**Theorem (Pirillo)**

*A word $w = amb \in \{a, b\}^*$ is a Christoffel word if and only if amb and bma are conjugate.*

Giuseppe Pirillo. A curious characteristic property of standard Sturmian words. In *Algebraic combinatorics and computer science*, pages 541–546. Springer Italia, Milan, 2001.

# Pirillo's theorem

## Theorem (Pirillo)

*A word $w = amb \in \{a, b\}^*$ is a Christoffel word if and only if amb and bma are conjugate.*

Example of the necessity :

Suppose $amb$ is the $3^{rd}$ conjugate of $bma$ and suppose $|m| = 9$ :

$$\begin{array}{ccccccccccc} a & m_1 & m_2 & m_3 & m_4 & m_5 & m_6 & m_7 & m_8 & m_9 & b \\ m_3 & m_4 & m_5 & m_6 & m_7 & m_8 & m_9 & a & b & m_1 & m_2 \end{array}$$

Then

$$a = m_3 = m_6 = m_9 = m_1 = m_4 = m_7 = a \quad \text{and} \quad b = m_2 = m_5 = m_8 = b$$

and

$$amb = a \cdot abaabaaba \cdot b$$

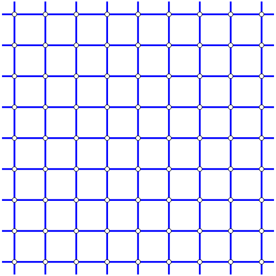
is a Christoffel word.

# Hypercubic lattice

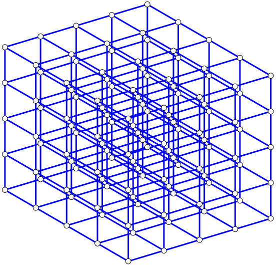
Oriented edges of the hypercubic lattice :

$$\mathbb{E}_d = \{(\mathbf{u}, \mathbf{u} + \mathbf{e}_i) : \mathbf{u} \in \mathbb{Z}^d \text{ and } 1 \leq i \leq d\}$$

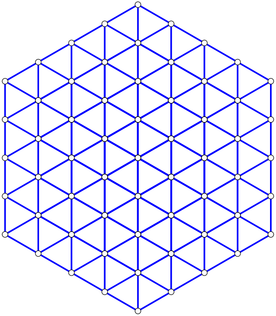
Note that the set $\mathbb{E}_d$ also corresponds to the Cayley graph of $\mathbb{Z}^d$ with generators $\mathbf{e}_i$ for all $i$ with $1 \leq i \leq d$.



$\mathbb{E}_2$



$\mathbb{E}_3$



$\mathbb{E}_3$

# Christoffel graph $\mathcal{H}_{\mathbf{a}}$

- Let $\mathbf{a} = (a_1, a_2, \ldots, a_d) \in \mathbb{N}^d$ such that $\gcd(\mathbf{a}) = 1$.
- Let $s = \|\mathbf{a}\|_1 = \sum a_i$.
- We define the mapping :

$$\begin{array}{rccc} \mathcal{F}_{\mathbf{a}} : & \mathbb{Z}^d & \to & \mathbb{Z}/s\mathbb{Z} \\ & \mathbf{x} & \mapsto & \mathbf{a} \cdot \mathbf{x} \bmod s. \end{array}$$

Note : we identify $\mathbb{Z}/s\mathbb{Z}$ and $\{0, 1, \ldots, s-1\}$ with its total order.

## Definition

The Christoffel graph $\mathcal{H}_{\mathbf{a}}$ of normal vector $\mathbf{a}$ is the subset of edges of $\mathbb{E}_d$ increasing for the function $\mathcal{F}_{\mathbf{a}}$ :

$$\mathcal{H}_{\mathbf{a}} = \{(\mathbf{u}, \mathbf{u} + \mathbf{e}_i) \in \mathbb{E}_d : \mathcal{F}_{\mathbf{a}}(\mathbf{u}) < \mathcal{F}_{\mathbf{a}}(\mathbf{u} + \mathbf{e}_i)\}.$$

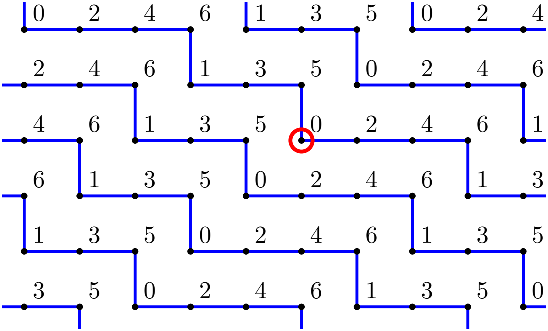



FIGURE: The graph $\mathcal{H}_{\mathbf{a}}$ with $\mathbf{a} = (2, 5)$ and $s = 7$.

**Lemma**

*The graph $\mathcal{H}_{\mathbf{a}}$ is invariant under the translation by the vector $\sum_{i=1}^{d} \mathbf{e}_i = (1, 1, \ldots, 1)$.*

# The graph $I_{\mathbf{a}}$

## Definition (Image)

Let $f : \mathbb{Z}^d \to S$ be an homomorphism of $\mathbb{Z}$-module. For some subset of edges $X \subseteq \mathbb{E}_d$, we define the image by $f$ of the edges $X$ by

$$f(X) = \{(f(\mathbf{u}), f(\mathbf{v})) \mid (\mathbf{u}, \mathbf{v}) \in X\}.$$

We define $I_{\mathbf{a}} = \pi(\mathcal{H}_{\mathbf{a}})$ where $\pi$ is the orthogonal projection from $\mathbb{R}^d$ onto the hyperplane $\mathcal{D}$ of equation $\sum x_i = 0$.

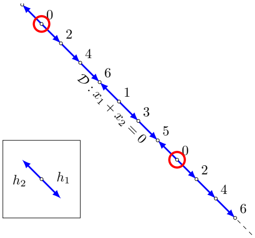
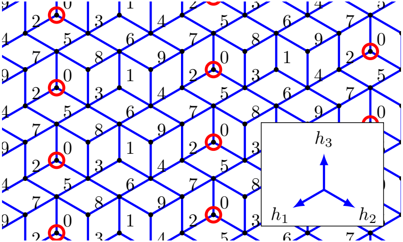



FIGURE: The graph $I_{\mathbf{a}}$ when $\mathbf{a} = (2, 5)$.

The graph $I_{\mathbf{a}}$ when $\mathbf{a} = (2, 3, 5)$ :




## Lemma

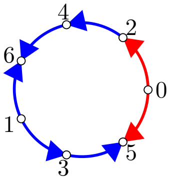
*The graph $I_{\mathbf{a}}$ produces a tiling of $\mathcal{D}$ by $d$ types of $(d-1)$-dimensional parallelotopes.*


Jean Françon. Sur la topologie d'un plan arithmétique. *Theoret. Comput. Sci.*, 156(1-2) :159–176, 1996.

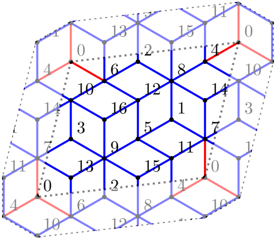
Pierre Arnoux, Valerie Berthé, and Shunji Ito. Discrete planes, $\mathbb{Z}^2$-actions, Jacobi-Perron algorithm and substitutions. *Ann. Inst. Fourier (Grenoble)*, 52(2) :305–349, 2002.

We define $\mathcal{G}_{\mathbf{a}} = \mathcal{F}_{\mathbf{a}}(\mathcal{H}_{\mathbf{a}})$. It is also equal to

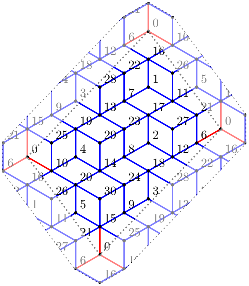
$$\mathcal{G}_{\mathbf{a}} = \{(k, k + a_i) \mid k \in \mathbb{Z}/s\mathbb{Z}, 1 \leq i \leq d \text{ and } k < k + a_i\}.$$




$\mathcal{G}_{(2,5)}$
*aaabaab*



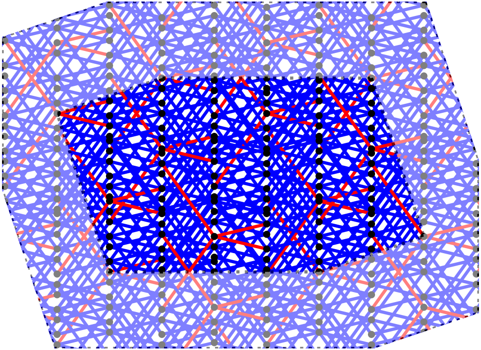
$\mathcal{G}_{(4,6,7)}$



$\mathcal{G}_{(6,10,15)}$

# Works for $d \geq 4$ : $\mathcal{G}_{(2,3,4,5)}$

$\mathcal{G}_{(2,3,4,5)}$



# Edges incident to zero

- Edges of $\mathbb{E}_d$ incident to zero :

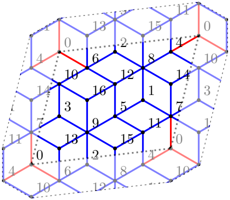
$$\mathcal{Q} = \{(\mathbf{u}, \mathbf{v}) \in \mathbb{E}_d : \mathcal{F}_{\mathbf{a}}(\mathbf{u}) = 0 \text{ or } \mathcal{F}_{\mathbf{a}}(\mathbf{v}) = 0\}.$$

- The legs of $X \subseteq \mathbb{E}_d$ are the edges of

$$X \cap \mathcal{Q}.$$

- The body of $X \subseteq \mathbb{E}_d$ is the set

$$X \setminus \mathcal{Q}.$$



# Flip, reversal and translation

Let $X \subseteq \mathbb{E}_d$. The reversal $-X$ is

$$-X = \{(-\mathbf{v}, -\mathbf{u}) \mid (\mathbf{u}, \mathbf{v}) \in X\}$$

Let $\mathbf{t} \in \mathbb{Z}^d$. The translate $X + \mathbf{t}$ is

$$X + \mathbf{t} = \{(\mathbf{u} + \mathbf{t}, \mathbf{v} + \mathbf{t}) \mid (\mathbf{u}, \mathbf{v}) \in X\}.$$

The FLIP operation exchanges edges of $X \subseteq \mathbb{E}_d$ incident to zero :

$$\text{FLIP} : X \mapsto (X \setminus \mathcal{Q}) \cup (\mathcal{Q} \setminus X).$$

The FLIP generalizes the function $amb \mapsto bma$.
Note : the FLIP when $d = 3 \leftrightarrow$ a flip in a rhombus tiling


Pierre Arnoux, Valérie Berthé, Thomas Fernique, and Damien Jamet. Functional stepped surfaces, flips, and generalized substitutions. *Theoret. Comput. Sci.*, 380(3) :251–265, 2007.

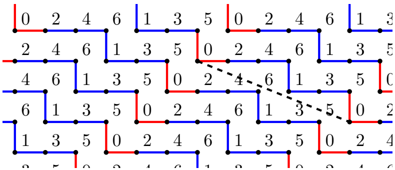
Olivier Bodini, Thomas Fernique, Michael Rao, and Éric Rémila. Distances on rhombus tilings. *Theoret. Comput. Sci.*, 412(36) :4787–4794, 2011.

# Flipping is translating

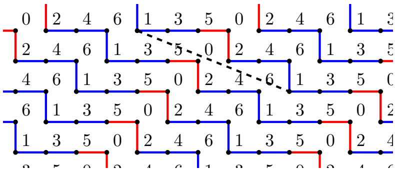
## Proposition

*Let* $\mathbf{t} \in \mathbb{Z}^d$ *be such that* $\mathcal{F}_{\mathbf{a}}(\mathbf{t}) = 1$ *(bezout vector). The* *translate* *by* $\mathbf{t}$ *of* $\mathcal{H}_{\mathbf{a}}$ *is* *equal to its flip*, *i.e.,*

$$\mathcal{H}_{\mathbf{a}} + \mathbf{t} = \textsc{flip}(\mathcal{H}_{\mathbf{a}}).$$



$\mathcal{H}_{\mathbf{a}}$ with $\mathbf{a} = (2, 5)$



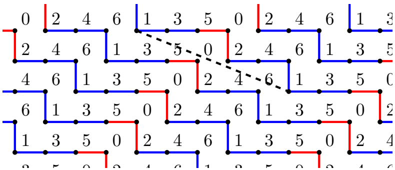
$\textsc{flip}(\mathcal{H}_{\mathbf{a}})$

This is a generalization of one implication of Pirillo Theorem (it generalizes the fact that a Christoffel word *amb* is conjugate to to *bma*).

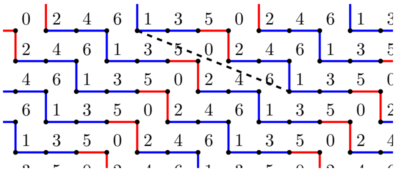
## Proposition

*Let* $\mathbf{t} \in \mathbb{Z}^d$ *be such that* $\mathcal{F}_{\mathbf{a}}(\mathbf{t}) = 1$ *(bezout vector). The translate by* $\mathbf{t}$ *of* $\mathcal{H}_{\mathbf{a}}$ *is equal to its flip, i.e.,*

$$\mathcal{H}_{\mathbf{a}} + \mathbf{t} = \text{FLIP}(\mathcal{H}_{\mathbf{a}}).$$



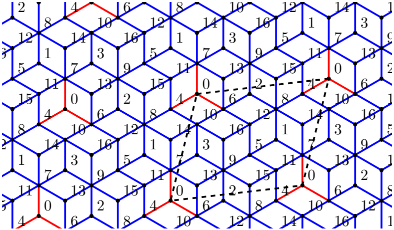

$\text{FLIP}(\mathcal{H}_{\mathbf{a}})$



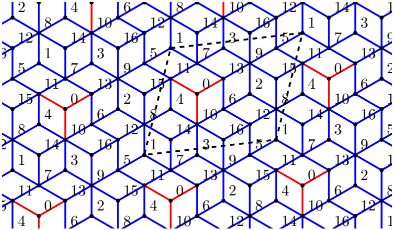

$\text{FLIP}(\mathcal{H}_{\mathbf{a}})$

This is a generalization of one implication of Pirillo Theorem (it generalizes the fact that a Christoffel word *amb* is conjugate to to *bma*).

# Flipping is translating



$I_{\mathbf{a}}$ with $\mathbf{a} = (4, 6, 7)$



$\text{FLIP}(I_{\mathbf{a}})$

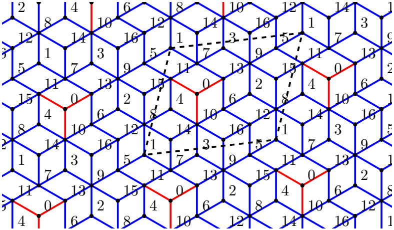
Thus, it verifies :

$$\mathcal{H}_{\mathbf{a}} + \mathbf{t} = \text{FLIP}(\mathcal{H}_{\mathbf{a}})$$

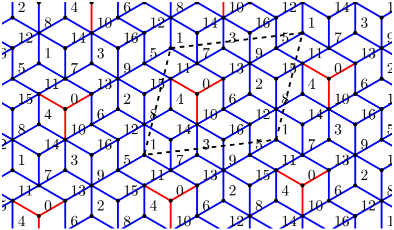
and also after projection by $\pi$ :

$$I_{\mathbf{a}} + \pi(\mathbf{t}) = \text{FLIP}(I_{\mathbf{a}}).$$

# Flipping is translating



$\textsc{flip}(l_{\mathbf{a}})$



$\textsc{flip}(l_{\mathbf{a}})$

Thus, it verifies :

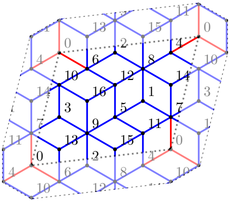
$$\mathcal{H}_{\mathbf{a}} + \mathbf{t} = \textsc{flip}(\mathcal{H}_{\mathbf{a}})$$

and also after projection by $\pi$ :

$$l_{\mathbf{a}} + \pi(\mathbf{t}) = \textsc{flip}(l_{\mathbf{a}}).$$

(Non-characteristic) properties like :

- Central words are palindromes.
- The reversal $\widetilde{amb}$ of a Christoffel word is equal to $bma$.
- A Christoffel word is conjugate to its reversal.



become :

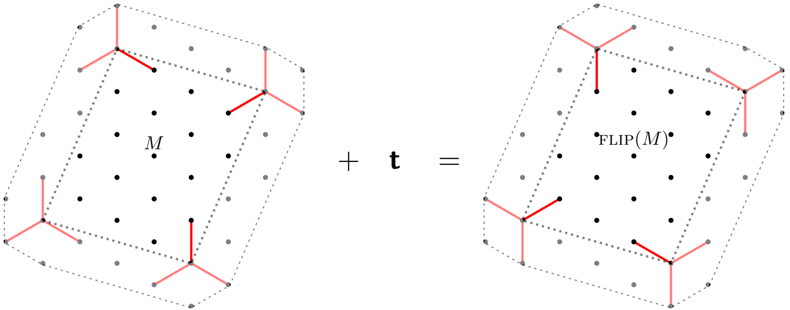
**Lemma**

- *The body of* $\mathcal{H}_{\mathbf{a}}$ *is symmetric, i.e.,* $-(\mathcal{H}_{\mathbf{a}} \setminus \mathcal{Q}) = \mathcal{H}_{\mathbf{a}} \setminus \mathcal{Q}$.
- *The reversal of* $\mathcal{H}_{\mathbf{a}}$ *is equal to its flip, i.e.,* $-\mathcal{H}_{\mathbf{a}} = \textsc{flip}(\mathcal{H}_{\mathbf{a}})$.
- *Let* $\mathbf{t} \in \mathbb{Z}^d$ *be such that* $\mathcal{F}_{\mathbf{a}}(\mathbf{t}) = 1$. *Then* $-\mathcal{H}_{\mathbf{a}} = \mathcal{H}_{\mathbf{a}} + \mathbf{t}$.

# What about the converse ?

- Let $K$ be a subgroup of finite index of $\mathbb{Z}^d$ such that $\sum_{i=1}^{d} \mathbf{e}_i \in K$.
- Let $\mathcal{Q} = \{(\mathbf{u}, \mathbf{v}) \in \mathbb{E}_d \mid \mathbf{u} \in K \text{ or } \mathbf{v} \in K\}$.

Question : For what subset of edges $M \subseteq \mathbb{E}_d$ invariant for the group of translations $K$ does there exists $\mathbf{t} \in \mathbb{Z}^d$ such that $M + \mathbf{t} = \textsc{flip}(M)$ ?




Question : Does $M$ has to be a Christoffel graph $\mathcal{H}_{\mathbf{a}}$ ?

# Christoffel graph $\mathcal{H}_{\mathbf{a},\omega}$

- Let $\mathbf{a} = (a_1, a_2, \ldots, a_d) \in \mathbb{N}^d$ such that $\gcd(\mathbf{a}) = 1$ with $s = \|\mathbf{a}\|_1$.
- Let $\omega \in \mathbb{N}$ be a divisor of $s$, called width, such that $0 < s/\omega < d$.
- We define the mapping :

$$\begin{array}{rccc} \mathcal{F}_{\mathbf{a},\omega} : & \mathbb{Z}^d & \to & \mathbb{Z}/\omega\mathbb{Z} \\ & \mathbf{x} & \mapsto & \mathbf{a}\cdot\mathbf{x} \bmod \omega. \end{array}$$

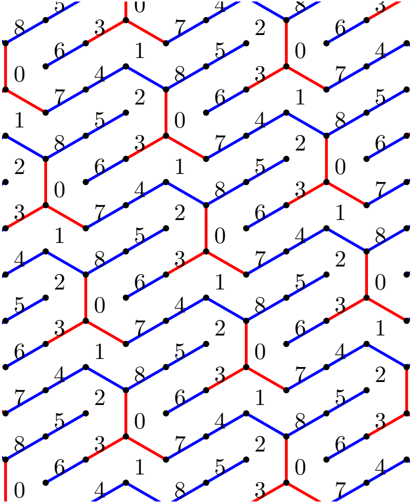
Note : we identify $\mathbb{Z}/\omega\mathbb{Z}$ and $\{0, 1, \ldots, \omega - 1\}$ with its total order.

**Definition**

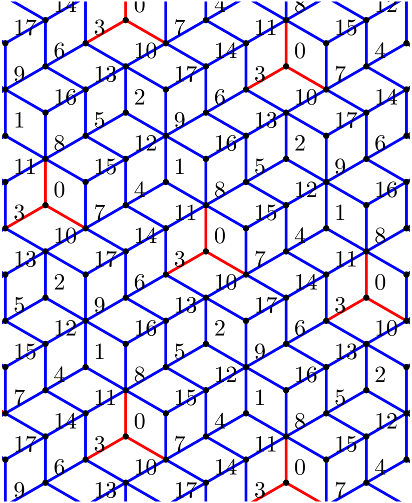
The Christoffel graph $\mathcal{H}_{\mathbf{a},\omega}$ of normal vector $\mathbf{a}$ and width $\omega$ is the subset of edges of $\mathbb{E}_d$ increasing for the function $\mathcal{F}_{\mathbf{a},\omega}$ :

$$\mathcal{H}_{\mathbf{a},\omega} = \{(\mathbf{u}, \mathbf{u} + \mathbf{e}_i) \in \mathbb{E}_d : \mathcal{F}_{\mathbf{a},\omega}(\mathbf{u}) < \mathcal{F}_{\mathbf{a},\omega}(\mathbf{u} + \mathbf{e}_i)\}.$$

Note : $\mathcal{H}_{\mathbf{a},\omega} \subseteq \mathcal{H}_{\mathbf{a}} \subseteq \mathbb{E}_d$.



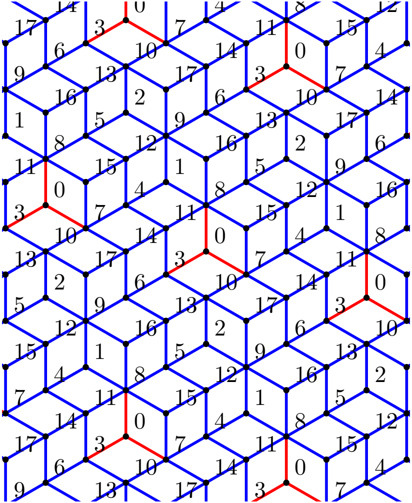
$I_{(3,7,8),9}$



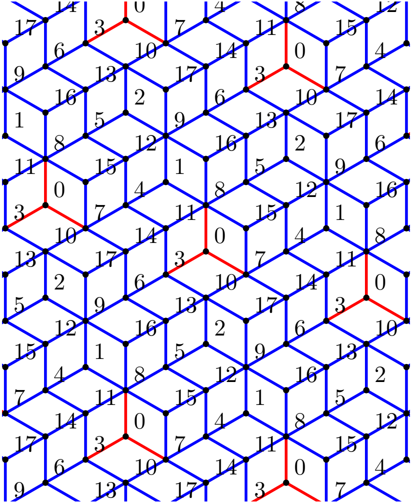
$I_{(3,7,8)}$

TODO : explain double discrete plane

Note : $\mathcal{H}_{\mathbf{a},\omega} \subseteq \mathcal{H}_{\mathbf{a}} \subseteq \mathbb{E}_d$.



$I_{(3,7,8)}$



$I_{(3,7,8)}$

TODO : explain double discrete plane

# $d$-dimensional Pirillo's theorem

## Theorem

- *Let $K$ be a subgroup of finite index of $\mathbb{Z}^d$ such that $\sum_{i=1}^{d} \mathbf{e}_i \in K$.*
- *Let $\mathcal{Q} = \{(\mathbf{u}, \mathbf{v}) \in \mathbb{E}_d \mid \mathbf{u} \in K \text{ or } \mathbf{v} \in K\}$.*
- *Suppose the legs of $M$ are positive, i.e.,*

$$M \cap \mathcal{Q} = \{(\mathbf{0}, \mathbf{e}_i) \mid 1 \leq i \leq d\} + K.$$

*There exists $\mathbf{t} \in \mathbb{Z}^d$ such that $M + \mathbf{t} = \text{FLIP}(M)$ if and only if $M = \mathcal{H}_{\mathbf{a},\omega}$ is the Christoffel graph of normal vector $\mathbf{a}$ and width $\omega$ where $\omega$ is a divisor of $s$ such that $0 < s/\omega < d$.*

# Credit :)

The images of this presentation were done using Sage and the new optional Sage package slabbe-0.1.spkg. Install :

```
sage -i http://www.liafa.univ-paris-diderot.fr/~labbe/Sage/slabbe-0.1.spkg
```

Import :

```
sage: from slabbe import *
```

And use :

```
sage: G = ChristoffelGraph((6,8,9)); G
Christoffel set of edges for normal vector v=(6, 8, 9)
sage: latex.add_to_preamble("\\usepackage{tikz}")
sage: view(G.tikz_kernel(), tightpage=True)
```