Factor Complexity of Arnoux-Rauzy-Poincaré sequences

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Joint work with Valérie Berthé
Given \((f_0, f_1) \in \mathbb{R}^2\) such that \(f_0 + f_1 = 1\), can we construct an infinite sequence \(w \in \mathcal{A}^\mathbb{N}\) on the alphabet \(\mathcal{A} = \{0, 1\}\) such that the frequency of digit \(i\) is \(f_i\) for all \(i \in \mathcal{A}\)?
Given \((f_0, f_1) \in \mathbb{R}^2\) such that \(f_0 + f_1 = 1\), can we construct an infinite sequence \(w \in \mathcal{A}^\mathbb{N}\) on the alphabet \(\mathcal{A} = \{0, 1\}\) such that the frequency of digit \(i\) is \(f_i\) for all \(i \in \mathcal{A}\)?
One answer is the expansion of $\pi$ in base 2:

$$\pi = 11.0010010000111111011010101000 \cdots$$

It is not a good answer because:

- works only for (conjectured) $f_0 = f_1 = \frac{1}{2}$;
- Really different factors appear in the sequence (ex: 0000 and 1111);
- All factors ($2^k$ factors of length $k$) appear in the sequence.
Plan

1. Question
2. Partial answers
3. S-adic sequences and MCF algorithms
4. Results
5. Idea of proof on factor complexity for ARP sequences
6. ... Brun sequences
Question

Given a vector $\vec{f} \in \mathbb{R}_+^d$ with $\|\vec{f}\|_1 = 1$ can we construct an infinite word $w$ on the alphabet $A = \{1, 2, \cdots, d\}$ satisfying each of below conditions?

- **frequency of letters in $w$ exists and is equal to $\vec{f}$**, 
- **$w$ stays at bounded distance from $\mathbb{R}_+ \vec{f}$ ($w$ is balanced)**, 
- **$w$ has a linear factor complexity**.
Definition

A **sturmian word** is an infinite word having exactly \( p(n) = n + 1 \) factors of length \( n \).

Fact (Morse, Hedlund, 1940)

*Sturmian words are exactly the aperiodic 1-balanced sequences.*

For each \( \vec{f} \in \mathbb{R}_+^2 \) such that \( \| \vec{f} \|_1 = 1 \), there exists a sturmian word \( w \) such that the frequency of letters in \( w \) is \( \vec{f} \).
A bad answer

Consider the expansion of $\pi$ in base 2:

$$\pi = 11.00100100001111101101010000\cdots$$

It is not a good discrete line because:

- works only for (conjectured) $f_0 = f_1 = \frac{1}{2}$;
- Really different factors appear in the sequence (ex: 0000 and 1111);
- All factors ($2^k$ factors of length $k$) appear (conjecture: $\pi$ is normal).
Let $w \in \mathcal{A}^\mathbb{N}$. The factor complexity is a function $p_w(n): \mathbb{N} \rightarrow \mathbb{N}$ counting the number of factors of length $n$, noted $L_w(n)$, in the sequence $w$.

\[
w = 000100\underline{0100}010010000100010001001000100010010011\]

\[
L_w(4) = \{0001, 0010, 0100, 1000, 1001\}
\]

<table>
<thead>
<tr>
<th>$n$</th>
<th>$p_w(n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
</tr>
</tbody>
</table>

Upper bound: $p_w(n) \leq |\mathcal{A}|^n$. 
Factors, frequencies, vectors

**Factor**: finite string of consecutive digits. Let

\[ w = 010010010100100100101001001001. \]

Then 00100 and 1001 are factors of \( w \). 1001 is a **suffix** of \( w \).

\[ |w| : \text{the length of the factor } w. \quad |w| = 30. \]

\[ |w|_u : \text{the number of occurrences of the factor } u \text{ in } w. \]

\[ |w|_0 = 19, \quad |w|_1 = 11 \]

\[ |w|_{00} = 8, \quad |w|_{01} = 11, \quad |w|_{10} = 10, \quad |w|_{11} = 0. \]

\[ \vec{u} = (|u|_0, |u|_1) : \text{the abelian vector of the factor } u. \]

\[ \overrightarrow{00100} = (4, 1), \quad \overrightarrow{1001} = (2, 2). \]
Balanced sequences

**Definition**

An infinite word \( w \in \mathcal{A}^\mathbb{N} \) is said to be *finitely balanced* or \( C \)-balanced or *balanced* if there exists a constant \( C \in \mathbb{N} \) such that for all pairs of factors \( u, v \) of \( w \) of the same length,

\[
\|\vec{u} - \vec{v}\|_\infty \leq C.
\]

Base 2 development of \( \pi \) is not 1-balanced because

\[
\|\vec{0000} - \vec{1111}\|_\infty = \|(4, -4)\|_\infty = 4.
\]

If \( \pi \) was proven normal, then \( 0^k \) and \( 1^k \) would also appear for all \( k \), thus it would not be balanced.
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When $d = 3$ : Cutting and billiard sequences

Sturmian word are obtained from the cutting sequence of a line :

This can be generalized as billiard sequences :

Borel (2006)
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This can be generalized as billiard sequences :

Borel (2006)

Theorem (Baryshnikov, 1995 ; Bédaride, 2003)

If both the direction $(\alpha_1, \alpha_2, \alpha_3)$ and $(\alpha_1^{-1}, \alpha_2^{-1}, \alpha_3^{-1})$ are $\mathbb{Q}$ independent, the number of factors appearing in the Billiard word in a cube is exactly $p(n) = n^2 + n + 1$. 
When $d = 3$ : Coding of rotations and IET

Sturmian word are obtained from coding of rotations:

This can be generalized to larger alphabet with coding of rotations on more intervals and more generally to interval exchange transformations (IET):
When \( d = 3 \): Coding of rotations and IET

Sturmian word are obtained from coding of rotations:

This can be generalized to larger alphabet with coding of rotations on more intervals and more generally to interval exchange transformations (IET):

Such sequences have linear factor complexity but are not balanced.

When $d \geq 2$ : Arnoux-Rauzy sequences

An infinite word $w \in \{1, 2, \ldots, d\}^\mathbb{N}$ is an Arnoux-Rauzy word if all its factors occur infinitely often, and if $p(n) = (d - 1)n + 1$ for all $n$, with exactly one left special and one right special factor of length $n$.

**Theorem (Delecroix, Hejda, Steiner, WORDS 2013)**

For $\mu$-almost every $f$ in the Rauzy gasket, the Arnoux-Rauzy word $w_{AR}(f)$ is finitely balanced.
When \( d \geq 2 \) : Arnoux-Rauzy sequences

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Definition

Let

\( A \) be an alphabet,

\((\sigma_i)_{i \geq 0}\) a sequence of morphisms such that \( \sigma_i : A^* \to A^* \),

\((a_i)_{i \geq 0}\) a sequence of letters with \( a_i \in A \).

Assume that the limit

\[
\mathbf{u} = \lim_{i \to \infty} (\sigma_0 \circ \sigma_1 \circ \cdots \circ \sigma_{i-1})(a_i)
\]

exists and is an infinite word \( \mathbf{u} \in A^\mathbb{N} \).

Then \((\sigma_i, a_i)_{i \geq 0}\) is called an \textit{s}-adic construction of \( \mathbf{u} \).
Let $\alpha = \frac{\sqrt{3} - 1}{2} = 0.36602540 \cdots$. We have

$$\alpha = 0 + \frac{1}{2 + \frac{1}{1 + \frac{1}{2 + \frac{1}{1 + \cdots}}}} = [0; 2, 1, 2, 1, 2, 1, \cdots]$$

The convergents $p_n/q_n$ are

$$0, \frac{1}{2}, \frac{1}{3}, \frac{3}{8}, \frac{11}{28}, \frac{11}{30}, \frac{15}{41}, \frac{41}{112}, \frac{56}{153}, \frac{153}{418}, \frac{209}{571}, \frac{571}{1560}, \frac{780}{2131}, \cdots$$

$$(30, 11) = [0, 2, 1, 2, 1, 2]$$
Continued fractions: matrices from convergents

With

\[ L = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \]

the convergents can be obtained as

\[
\begin{align*}
\begin{pmatrix} q_0 \\ p_0 \end{pmatrix} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} = R^0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
\begin{pmatrix} q_1 \\ p_1 \end{pmatrix} &= \begin{pmatrix} 2 \\ 1 \end{pmatrix} = R^0 L^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
\begin{pmatrix} q_2 \\ p_2 \end{pmatrix} &= \begin{pmatrix} 3 \\ 1 \end{pmatrix} = R^0 L^2 R^1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
\begin{pmatrix} q_3 \\ p_3 \end{pmatrix} &= \begin{pmatrix} 8 \\ 3 \end{pmatrix} = R^0 L^2 R^1 L^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
\begin{pmatrix} q_4 \\ p_4 \end{pmatrix} &= \begin{pmatrix} 11 \\ 4 \end{pmatrix} = R^0 L^2 R^1 L^2 R^1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
\begin{pmatrix} q_5 \\ p_5 \end{pmatrix} &= \begin{pmatrix} 30 \\ 11 \end{pmatrix} = R^0 L^2 R^1 L^2 R^1 L^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\end{align*}
\]
Continued fractions: substitutions from matrices

With

\[ L = \begin{pmatrix} 0 & \mapsto & 0 \\ 1 & \mapsto & 01 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 0 & \mapsto & 10 \\ 1 & \mapsto & 1 \end{pmatrix} \]

the convergents can be transformed into finite sequences over \( A \):

\[
\begin{align*}
 w_0 &= R^0(0) = 0 \\
 w_1 &= R^0L^2(1) = 001 \\
 w_2 &= R^0L^2R^1(0) = 0010 \\
 w_3 &= R^0L^2R^1L^2(1) = 00100010001 \\
 w_4 &= R^0L^2R^1L^2R^1(0) = 00100010001001000100010001001000100010001 \\
 w_5 &= R^0L^2R^1L^2R^1L^2(1) = 00100010001001000100010001001000100010001 \cdots
\end{align*}
\]
Sturmian sequences as $S$-adic sequences

From iteration of morphisms $L = \begin{pmatrix} 0 & 0 \\ 1 & 01 \end{pmatrix}$ and $R = \begin{pmatrix} 0 & 10 \\ 1 & 1 \end{pmatrix}$:

$$R^{d_0} L^{d_1} R^{d_2} L^{d_3} R^{d_4} L^{d_5} \ldots (1)$$

where $[d_0; d_1, d_2, d_3, \ldots]$ is the continued fraction expansion of the slope of the word drawn with horizontal and \textit{vertical} unitary steps.

This generalizes to $S$-adic sequences based on \textit{Multidimensional Continued Fraction (MCF) algorithms}...
Continued fractions: from Euclid Algorithm

With

\[ L = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \]

the execution of Euclid Algorithm appears as

\[
\begin{align*}
\left( \begin{array}{c} q_0 \\ p_0 \end{array} \right) &= \left( \begin{array}{c} 1 \\ 0 \end{array} \right) = R^0 \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \\
\left( \begin{array}{c} q_1 \\ p_1 \end{array} \right) &= \left( \begin{array}{c} 2 \\ 1 \end{array} \right) = R^0 L^2 \left( \begin{array}{c} 0 \\ 1 \end{array} \right) \\
\left( \begin{array}{c} q_2 \\ p_2 \end{array} \right) &= \left( \begin{array}{c} 3 \\ 1 \end{array} \right) = R^0 L^2 R^1 \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \\
\left( \begin{array}{c} q_3 \\ p_3 \end{array} \right) &= \left( \begin{array}{c} 8 \\ 3 \end{array} \right) = R^0 L^2 R^1 L^2 \left( \begin{array}{c} 0 \\ 1 \end{array} \right) \\
\left( \begin{array}{c} q_4 \\ p_4 \end{array} \right) &= \left( \begin{array}{c} 11 \\ 4 \end{array} \right) = R^0 L^2 R^1 L^2 R^1 \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \\
\left( \begin{array}{c} q_5 \\ p_5 \end{array} \right) &= \left( \begin{array}{c} 30 \\ 11 \end{array} \right) = R^0 L^2 R^1 L^2 R^1 L^2 \left( \begin{array}{c} 0 \\ 1 \end{array} \right)
\end{align*}
\]

\[
\begin{align*}
\left( \begin{array}{c} 30 \\ 11 \end{array} \right) &= L^2 R^1 L^2 R^1 L^2 \left( \begin{array}{c} 0 \\ 1 \end{array} \right) \\
\left( \begin{array}{c} 8 \\ 11 \end{array} \right) &= R^1 L^2 R^1 L^2 \left( \begin{array}{c} 0 \\ 1 \end{array} \right) \\
\left( \begin{array}{c} 8 \\ 3 \end{array} \right) &= L^2 R^1 L^2 \left( \begin{array}{c} 0 \\ 1 \end{array} \right) \\
\left( \begin{array}{c} 2 \\ 3 \end{array} \right) &= R^1 L^2 \left( \begin{array}{c} 0 \\ 1 \end{array} \right) \\
\left( \begin{array}{c} 2 \\ 1 \end{array} \right) &= L^2 \left( \begin{array}{c} 0 \\ 1 \end{array} \right)
\end{align*}
\]
3D Continued fraction algorithms

Brun’s Algorithm : Subtract the second largest to the largest.

\[ (7, 4, 6) \rightarrow (1, 4, 6) \rightarrow (1, 4, 2) \rightarrow (1, 2, 2) \rightarrow (1, 0, 2) \rightarrow (1, 0, 1) \rightarrow (0, 0, 1). \]

Selmer’s Algorithm : Subtract the smallest to the largest.

\[ (7, 4, 6) \rightarrow (3, 4, 6) \rightarrow (3, 4, 3) \rightarrow (3, 1, 3) \rightarrow (2, 1, 3) \rightarrow (2, 1, 2) \rightarrow (1, 1, 2) \rightarrow (1, 1, 1) \rightarrow (0, 1, 1) \rightarrow (0, 0, 1) \]

Poincaré’s Algorithm : Subtract the smallest to the mid and the mid to the largest.

\[ (7, 4, 6) \rightarrow (1, 4, 2) \rightarrow (1, 2, 1) \rightarrow (1, 1, 0) \rightarrow (1, 0, 0) \]

Arnoux-Rauzy’s Algorithm : Subtract the sum of the two smallest to the largest (not always possible).

\[ (7, 4, 6) \rightarrow \text{Impossible} \]

Fully subtractive’s Algorithm : Subtract the smallest to the other two.

\[ (7, 4, 6) \rightarrow (3, 4, 2) \rightarrow (1, 2, 2) \rightarrow (1, 1, 1) \rightarrow (1, 0, 0) \]
3D : Imitation of Euclid algorithm on $(7, 4, 6)$

\[
\left( \begin{array}{ccc}
1 & 0 & 1 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array} \right) \quad \left( \begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 1 & 1
\end{array} \right) \quad \left( \begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 2 \\
0 & 0 & 1
\end{array} \right) \quad \left( \begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
2 & 0 & 1
\end{array} \right)
\]

$(7, 4, 6) \overset{1}{\leftrightarrow} (1, 4, 6) \overset{2}{\leftrightarrow} (1, 4, 2) \overset{3}{\leftrightarrow} (1, 0, 2) \overset{4}{\leftrightarrow} (1, 0, 0)$

$w_0 \overset{1}{\leftrightarrow} w_1 \overset{2}{\leftrightarrow} w_2 \overset{3}{\leftrightarrow} w_3 \overset{4}{\leftrightarrow} w_4$

Its (Hausdorff) distance to the euclidean line is $1.3680$. 
3D : Imitation of Euclid algorithm on \((7, 4, 6)\)

\[
\begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 2 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
2 & 0 & 1
\end{pmatrix}
\]

\((7, 4, 6) \leftrightarrow (1, 4, 6) \leftrightarrow (1, 4, 2) \leftrightarrow (1, 0, 2) \leftrightarrow (1, 0, 0)\)

\[
\begin{align*}
w_0 & \leftrightarrow 1 & w_1 & \leftrightarrow 1 & w_2 & \leftrightarrow 1 & w_3 & \leftrightarrow 1 & w_4 & \leftrightarrow 133 \\
2 & \leftrightarrow 2 & 2 & \leftrightarrow 23 & 2 & \leftrightarrow 2 & 2 & \leftrightarrow 2 \\
3 & \leftrightarrow 13 & 3 & \leftrightarrow 3 & 3 & \leftrightarrow 223 & 3 & \leftrightarrow 3
\end{align*}
\]

\[w = w_0 = 12132131321321313\]

Its (Hausdorff) distance to the euclidean line is 1.3680.
3D : Imitation of Euclid algorithm on (7, 4, 6)

\[
\begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 2 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
2 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

\[
(7, 4, 6) \leftrightarrow (1, 4, 6) \leftrightarrow (1, 4, 2) \leftrightarrow (1, 0, 2) \leftrightarrow (1, 0, 0)
\]

\[w_0 \leftrightarrow w_1 \leftrightarrow w_2 \leftrightarrow w_3 \leftrightarrow w_4\]

\[w = w_0 = 12132131321321313\]

Its (Hausdorff) distance to the euclidean line is 1.3680.
Experimentations on discrepancy

<table>
<thead>
<tr>
<th></th>
<th>Min</th>
<th>Mean</th>
<th>Max</th>
<th>Std</th>
</tr>
</thead>
<tbody>
<tr>
<td>Arnoux-Rauzy (AR)</td>
<td>0.6000</td>
<td>0.8922</td>
<td>1.200</td>
<td>0.09953</td>
</tr>
<tr>
<td>Fully subtractive</td>
<td>0.5000</td>
<td>6.047</td>
<td>14.21</td>
<td>4.385</td>
</tr>
<tr>
<td>Selmer</td>
<td>0.5000</td>
<td>2.151</td>
<td>12.75</td>
<td>2.076</td>
</tr>
<tr>
<td>Brun</td>
<td>0.5000</td>
<td>1.100</td>
<td>2.000</td>
<td>0.2625</td>
</tr>
<tr>
<td>Poincaré</td>
<td>0.5000</td>
<td>2.476</td>
<td>11.13</td>
<td>2.245</td>
</tr>
<tr>
<td>AR-Fully subtractive</td>
<td>0.5000</td>
<td>1.154</td>
<td>4.000</td>
<td>0.3759</td>
</tr>
<tr>
<td>AR-Selmer</td>
<td>0.5000</td>
<td>0.9991</td>
<td>1.600</td>
<td>0.1429</td>
</tr>
<tr>
<td>AR-Brun</td>
<td>0.5000</td>
<td>0.9169</td>
<td>1.520</td>
<td>0.1170</td>
</tr>
<tr>
<td>AR-Poincaré</td>
<td>0.5000</td>
<td>0.9066</td>
<td>1.320</td>
<td>0.1079</td>
</tr>
</tbody>
</table>

**Table:** Statistics for the discrepancy for strictly positive integer vectors \((a_1, a_2, a_3)\) such that \(a_1 + a_2 + a_3 = N\) and \(N = 100\).

(from talk at WORDS 2011)
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Theorem (Berthé, L., 2013)

Let $w = \lim_{n \to \infty} \sigma_0 \sigma_1 \cdots \sigma_n(1)$ be an $S$-adic word generated by the Arnoux-Rauzy-Poincaré algorithm from a totally irrational vector $x \in \Delta_3$. The factor complexity of $w$ is such that

- $p(n) \leq 3n + 1$ for all $n \geq 0$;
- $p(n+1) - p(n) \in \{2, 3\}$ for all $n \geq 0$;
- $\limsup_{n \to \infty} \frac{p(n)}{n} \leq \frac{5}{2} < 3$ (not sharp).
Results

Given a measure $\mu$ for which the cocycle is log-integrable, we say that $(A^N, T, A, \mu)$ has **Pisot spectrum** if the associated Lyapunov exponents satisfy $\Theta_1 > 0 > \Theta_2$. This property is related to the strong convergence of higher dimensional continued fraction algorithm [Lagarias, 93].

**Theorem (Berthé, Delecroix (Survey Thm. 6.4), 2013)**

Let $A$ be a MCF algorithm and $\Theta_1$ and $\Theta_2$ be the first two Lyapunov exponents of the algorithm $A$. If $\Theta_2 < 0$, then for $\mu$-almost all frequency vector $x \in \Delta$, the $S$-adic word $w(x)$ generated by the algorithm $A$ is **finitely balanced**.
Results

**Theorem (Avila, Delecroix, 2013)**

Let \((\Delta, T, A)\) be the cocycle associated to Brun map in dimension \(d = 2\) or the fully subtractive algorithm in any dimension. Then, for every \(T\)-invariant ergodic probability measure on \(\Delta\) such that there exists a cylinder \([w]\) of positive \(\mu\)-measure whose associated matrix \(A(w)\) is positive, the Lyapunov spectrum of \((\Delta, T, A, \mu)\) is Pisot.

Let \(\Delta_3 = \{(f_1, f_2, f_3) \in \mathbb{R}_+^3 : f_1 + f_2 + f_3 = 1\}\).

**Theorem (Delecroix, Hejda, Steiner, 2013)**

For Lebesgue almost all frequency vector \(x \in \Delta_3\), the \(S\)-adic word \(w_{AR}(x)\) generated by the Brun algorithm is finitely balanced.
### Summary

<table>
<thead>
<tr>
<th>Arnoux Rauzy words</th>
<th>$\forall v \in \mathbb{R}^3_+$</th>
<th>$p(n)$ linear</th>
<th>Balanced</th>
</tr>
</thead>
<tbody>
<tr>
<td>No</td>
<td>Yes</td>
<td>Almost always Yes</td>
<td></td>
</tr>
<tr>
<td>Billiard, Andres discrete line</td>
<td>Yes</td>
<td>No</td>
<td>Yes</td>
</tr>
<tr>
<td>Yes</td>
<td>Yes</td>
<td>No</td>
<td>No</td>
</tr>
<tr>
<td>Coding of rotations and of IET</td>
<td>Yes</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>Yes</td>
<td>$\approx 3n$?</td>
<td>Almost always Yes</td>
<td></td>
</tr>
<tr>
<td>Brun $S$-adic sequences</td>
<td>Yes</td>
<td>Yes : $\frac{5}{2}n$</td>
<td>?</td>
</tr>
<tr>
<td>Yes</td>
<td>$\frac{5}{2}n$</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td>ARP $S$-adic sequences</td>
<td>Yes</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td>Other $S$-adic sequences</td>
<td>?</td>
<td>?</td>
<td>?</td>
</tr>
</tbody>
</table>
Plan

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Arnoux-Rauzy and Poincaré substitutions

For all \( \{i, j, k\} = \{1, 2, 3\} \), we consider

\[
\pi_{jk} : i \mapsto ijk, j \mapsto jk, k \mapsto k \quad \text{(Poincaré substitutions)}
\]

\[
\alpha_k : i \mapsto ik, j \mapsto jk, k \mapsto k \quad \text{(Arnoux-Rauzy substitutions)}
\]

Namely,

\[
\pi_{23} = \begin{cases}
1 & \mapsto 123 \\
2 & \mapsto 23 \\
3 & \mapsto 3 \\
1 & \mapsto 12 \\
2 & \mapsto 2 \\
3 & \mapsto 312 \\
1 & \mapsto 1 \\
2 & \mapsto 231 \\
3 & \mapsto 31
\end{cases},
\]

\[
\pi_{13} = \begin{cases}
1 & \mapsto 13 \\
2 & \mapsto 213 \\
3 & \mapsto 3 \\
1 & \mapsto 132 \\
2 & \mapsto 3 \\
3 & \mapsto 32 \\
1 & \mapsto 1 \\
2 & \mapsto 21 \\
3 & \mapsto 321
\end{cases},
\]

\[
\pi_{32} = \begin{cases}
1 & \mapsto 13 \\
2 & \mapsto 23 \\
3 & \mapsto 32 \\
1 & \mapsto 1 \\
2 & \mapsto 2 \\
3 & \mapsto 32 \\
1 & \mapsto 1 \\
2 & \mapsto 21 \\
3 & \mapsto 31
\end{cases},
\]

\[
\alpha_3 = \begin{cases}
1 & \mapsto 13 \\
2 & \mapsto 23 \\
3 & \mapsto 3 \\
1 & \mapsto 132 \\
2 & \mapsto 32 \\
3 & \mapsto 32 \\
1 & \mapsto 1 \\
2 & \mapsto 21 \\
3 & \mapsto 31
\end{cases},
\]

\[
\alpha_2 = \begin{cases}
1 & \mapsto 12 \\
2 & \mapsto 2 \\
3 & \mapsto 312 \\
1 & \mapsto 1 \\
2 & \mapsto 231 \\
3 & \mapsto 31
\end{cases},
\]

\[
\alpha_1 = \begin{cases}
1 & \mapsto 12 \\
2 & \mapsto 21 \\
3 & \mapsto 312 \\
1 & \mapsto 1 \\
2 & \mapsto 21 \\
3 & \mapsto 31
\end{cases}.
\]
In general, it is possible that $p(n + 1) - p(n) > 3$ for some values of $n$. Let

\[ s = \pi_{23} \pi_{23} \pi_{13} \pi_{23} \pi_{23} \alpha_1 \alpha_3 \alpha_2(1). \]

Indeed,

\[ p_s(n) = (1, 3, 5, 8, 11, 15, 19, 23, 27, 31, 35, 38, \ldots) \]
In general, it is possible that $p(n+1) - p(n) > 3$ for some values of $n$. Let

$$s = \pi_{23}\pi_{23}\pi_{13}\pi_{23}\alpha_1\alpha_3\alpha_2(1).$$

Indeed,

$$p_s(n) = (1, 3, 5, 8, 11, 15, 19, 23, 27, 31, 35, 38, \cdots)$$

Even worse, the fixed point of

$$\pi_{13}\pi_{23} : \begin{cases} 1 \mapsto 132133 \\ 2 \mapsto 2133 \\ 3 \mapsto 3 \end{cases}$$

starting with letter 1 has a quadratic factor complexity.
Language of Arnoux-Rauzy Poincaré algorithm

Deterministic and minimized automaton recognizing the language $\mathcal{L} \subset S^\mathbb{N}$ of ARP algorithm:

\[
\begin{align*}
\alpha_3 & \xrightarrow{} H_{31} \\
\alpha_1 & \xrightarrow{} H_{12} \\
\alpha_2 & \xrightarrow{} H_{23} \\
\alpha_1 & \xrightarrow{} H_{21} \\
\alpha_2 & \xrightarrow{} H_{32} \\
\alpha_3 & \xrightarrow{} H_{13}
\end{align*}
\]
Let \( w = \lim_{n \to \infty} \sigma_0 \sigma_1 \cdots \sigma_n(1) \) be an \( S \)-adic word generated by the Arnoux-Rauzy-Poincaré algorithm from a totally irrational vector \( x \in \Delta_3 \).

**Theorem (Factor Complexity)**

The factor complexity of \( w \) is such that

- \( p(n) \leq 3n + 1 \) for all \( n \geq 0 \);
- \( p(n + 1) - p(n) \in \{2, 3\} \) for all \( n \geq 0 \);
- \( \limsup_{n \to \infty} \frac{p(n)}{n} \leq \frac{5}{2} < 3 \) (not sharp).
Let \( w = \lim_{n \to \infty} \sigma_0 \sigma_1 \cdots \sigma_n(1) \) be an \( S \)-adic word generated by the Arnoux-Rauzy-Poincaré algorithm from a totally irrational vector \( x \in \Delta_3 \).

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**Theorem (Frequencies and Convergence)**

The symbolic dynamical system generated by \( w \) is uniquely ergodic, and the frequencies of letters are proved to exist in \( w \) and to be equal to the coordinates of \( x \).

Furthermore, the Arnoux-Rauzy-Poincaré algorithm is a weakly convergent algorithm, that is, for Lebesgue almost every \( x \in \Delta \), if \( (M_n)_n \) stands for the sequence of matrices produced by the Arnoux-Rauzy-Poincaré algorithm, then one has \( \cap_n M_0 \cdots M_n(\mathbb{R}_+^3) = \mathbb{R}_+x \).
Idea of the proof on complexity

Let $p(n)$ be the factor complexity function of $w$. Let $s(n)$ and $b(n)$ be its sequences of finite differences of order 1 and 2:

$$
p(n) = 1, 3, 5, 7, 9, 11, 14, 17, 20, 22, 24, 26, 28,
$$

$$
s(n) = p(n + 1) - p(n) = 2, 2, 2, 2, 2, 3, 3, 3, 2, 2, 2,
$$

$$
b(n) = s(n + 1) - s(n) = 0, 0, 0, 0, +1, 0, 0, -1, 0, 0, 0,
$$
Idea of the proof on complexity

Let $p(n)$ be the factor complexity function of $w$. Let $s(n)$ and $b(n)$ be its sequences of finite differences of order 1 and 2:

$$
p(n) = 1, 3, 5, 7, 9, 11, 14, 17, 20, 22, 24, 26, 28,$$
$$s(n) = p(n + 1) - p(n) = 2, 2, 2, 2, 2, 3, 3, 3, 2, 2, 2,$$
$$b(n) = s(n + 1) - s(n) = 0, 0, 0, 0, +1, 0, 0, -1, 0, 0, 0,$$

Functions $s$ and $b$ are related to special and bispecial factors of $w$.

**Theorem (Cassaigne, 1997; Cassaigne, Nicolas, 2010)**

Let $u \in A^\mathbb{N}$ be a infinite [recurrent] word. Then, for all $n \in \mathbb{N}$:

$$s(n) = \sum_{w \in RS_n(u)} (d^+(w) - 1) \quad \text{and} \quad b(n) = \sum_{w \in BS_n(u)} m(w)$$
Special and bispecial words

Let $u$ be an infinite word $L(u)$ be its language.

**Right extensions and right valence:**

$$E^+(w) = \{x \in A | wx \in L(u)\} \quad d^+(w) = \text{Card}E^+(w).$$

**Left extensions and left valence:**

$$E^-(w) = \{x \in A | xw \in L(u)\} \quad d^-(w) = \text{Card}E^-(w).$$

A factor $w$ is
- **right special** if $d^+(w) \geq 2$,
- **left special** if $d^-(w) \geq 2$,
- **bispecial** if it is left and right special.
Special and bispecial words

The extension type of a factor \( w \) of \( u \) is

\[
E(w) = \{(a, b) \in A \times A | awb \in L(u)\}.
\]

The bilateral multiplicity of a factor \( w \) is

\[
m(w) = \text{Card } E(w) - d^-(w) - d^+(w) + 1.
\]

A bispecial factor is said

- weak if \( m(w) < 0 \),
- neutral if \( m(w) = 0 \),
- strong if \( m(w) > 0 \).
Examples of extension types $E(w)$ of bispecial factors $w$

$$m(w) = \text{Card } E(w) - d^-(w) - d^+(w) + 1.$$ 

\begin{tabular}{c|cc}
 & 1 & 2 \\
\hline
1 & \times & \\
2 & \times & \\
m(w) = -1 & \\
\text{weak} & \\
\end{tabular}

\begin{tabular}{c|cc}
 & 1 & 2 \\
\hline
1 & \times & \times \\
2 & \times & \\
m(w) = 0 & \\
\text{neutral and ordinary} & \\
\end{tabular}

\begin{tabular}{c|cc}
 & 1 & 2 \\
\hline
1 & \times & \times \\
2 & \times & \times \\
m(w) = 1 & \\
\text{strong} & \\
\end{tabular}
Examples of extension types $E(w)$ of bispecial factors $w$

$$m(w) = \text{Card } E(w) - d^-(w) - d^+(w) + 1.$$ 

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$m(w) = -1$ weak

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$m(w) = 0$ neutral and ordinary

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$m(w) = 1$ strong

A bispecial factor $w$ is **ordinary** if $\exists$ row $\exists$ column s.t.

$$\text{row } \cap \text{column } \subseteq E(w) \subseteq \text{row } \cup \text{column}.$$ 

**Lemma**

*If a bispecial factor is ordinary, then it is neutral.*

On a binary alphabet, the reciprocal also holds.
Examples of extension types $E(w)$ of bispecial factors $w$

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$m(w) = 0$
neutral and ordinary

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$m(w) = 0$
neutral and ordinary

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$m(w) = 0$
neutral but not ordinary

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$m(w) = 1$
strong
Some synchronization lemmas allows to define uniquely the antecedent $w_{k+1}$ of a (bispecial) factor $w_k = s \cdot \sigma_k(w_{k+1}) \cdot p$.

- $w_k$ is an extended image of $w_{k+1}$

\[
\begin{align*}
& w = w_0 \xrightarrow{\sigma_0} w_1 \xrightarrow{\sigma_1} w_2 \ldots \xrightarrow{\sigma_{n-1}} w_{n-1} \xrightarrow{\sigma_n} w_n = \varepsilon
\end{align*}
\]
Some synchronization lemmas allows to define uniquely the antecedent $w_{k+1}$ of a (bispecial) factor $w_k = s \cdot \sigma_k(w_{k+1}) \cdot p$.

- $w_k$ is an extended image of $w_{k+1}$

We always have $|w_k| > |w_{k+1}|$.

- The history of $w$ is $\sigma_0 \sigma_1 \cdots \sigma_n$.
- The life of $w$ is $(w_k)_{0 \leq k \leq n}$.
- $w_{k+1}$ has only one bispecial extended image $w_k$ under an Arnoux-Rauzy substitution.
- $w_{k+1}$ has one or two extended images under Poincaré substitution.
Ordinary bispecial factor from an ordinary

Let

\[ S_\alpha = \{ \alpha_1, \alpha_2, \alpha_3 \}, \]
\[ S_\pi = \{ \pi_{12}, \pi_{13}, \pi_{23}, \pi_{21}, \pi_{31}, \pi_{32} \} \]
\[ S = S_\alpha \cup S_\pi. \]

**CASE 1**: If \( \text{history}(w_0) \in S^* \pi_{jk} S_\alpha^{*}\{\alpha_k\} \), then \( m(w_0) = 0 \):

\[ w_0 \quad S^* \quad w_\ell \quad \pi_{jk} \quad w_{\ell+1} \quad S_\alpha^* \quad w_n = \varepsilon \]

\begin{align*}
\begin{array}{c|ccc}
i & j & k \\
\hline 
i & \times & \times & \times \\
j & \times \\
k & \times & \times & \times \\
\end{array} & 
\begin{array}{c|ccc}
i & j & k \\
\hline 
i & \times \\
j & \times & \times \\
k & \times & \times & \times \\
\end{array} & 
\begin{array}{c|ccc}
i & j & k \\
\hline 
i & \times \\
j & \times \\
k & \times & \times & \times \\
\end{array} & 
\begin{array}{c|ccc}
i & j & k \\
\hline 
i & \times \\
j & \times \\
k & \times & \times & \times \\
\end{array}
\end{align*}

ordinary \[ m(w_0) = 0 \]

ordinary \[ m(w_n) = 0 \]
Two ordinary bispecial factors from an ordinary

**CASE 2** : If \( \text{history}(w_0) \in S^* \pi_{jk} S^*_\alpha \{\alpha_i, \alpha_j\} \), then \( m(w_0) = 0 \):

\[
\begin{array}{c}
\text{ordinary} \\
m(w'_0) = 0
\end{array}
\]

\[
\begin{array}{c|c}
\hline
i & j & k \\
\hline
i & \times \\
\hline
j & \times & \times \\
\hline
k & \times & \times & \times \\
\hline
\end{array}
\]

\[
\begin{array}{c|c}
\hline
i & j & k \\
\hline
i & \times \\
\hline
j & \times & \times & \times \\
\hline
k & \times \\
\hline
\end{array}
\]

\[
\begin{array}{c|c}
\hline
i & j & k \\
\hline
i & \times \\
\hline
j & \times & \times & \times \\
\hline
k & \times & \times & \times \\
\hline
\end{array}
\]

\[
\begin{array}{c|c}
\hline
i & j & k \\
\hline
i & \times \\
\hline
j & \times & \times & \times \\
\hline
k & \times \\
\hline
\end{array}
\]

\[
\begin{array}{c|c}
\hline
i & j & k \\
\hline
i & \times \\
\hline
j & \times & \times & \times \\
\hline
k & \times \\
\hline
\end{array}
\]
CASE 3: If \( \text{history}(w_0) \in S^* \pi_{jk} S^*_\alpha \{\pi_{ji}, \pi_{ki}, \pi_{ij}, \pi_{kj}\} \), then \( m(w_0) = 0 \):

\[
\begin{array}{c|ccc}
\text{ordinary} \\
m(w'_0) = 0 \\
\hline
i & j & k & \times & \times & \times & \times \\
\hline
i & j & k & \times & \times & \times & \times \\
\end{array}
\]

\[
\begin{array}{c|ccc}
\text{neutral} \\
\text{not ordinary} \\
m(wn) = 0 \\
\hline
i & j & k & \times & \times & \times & \times \\
\hline
i & j & k & \times & \times & \times & \times \\
\end{array}
\]

\[
\begin{array}{c|ccc}
\text{ordinary} \\
m(w_0) = 0 \\
\hline
i & j & k & \times & \times & \times & \times \\
\hline
i & j & k & \times & \times & \times & \times \\
\end{array}
\]

\[
\begin{array}{c|ccc}
\text{neutral} \\
\text{not ordinary} \\
m(wn) = 0 \\
\hline
i & j & k & \times & \times & \times & \times \\
\hline
i & j & k & \times & \times & \times & \times \\
\end{array}
\]
**CASE 4** : If \( \text{history}(w_0) \in S^* \pi_{jk} S^*_\alpha\{\pi_{jk}, \pi_{ik}\} \), then \( m(w_0) = \pm 1 \):

**Strong**: \( m(w_0^+) = +1 \)

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**Weak**: \( m(w_0^-) = -1 \)

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**Neutral** not ordinary: \( m(w_n) = 0 \)

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When it does not work

Age 0 (0) ε
Age 1 (+1) 3 π23 ε π23
(−1) 23 π23 ε π23
Age 2 (+1) 33 π23 3 π23 ε π13
(−1) 3233 π23 23 π13 ε π23
Age 3 (+1) 333 π23 33 π23 3 π13 ε π23
(−1) 331232333 π23 31233 π23 13 π13 ε π23
Age 4 (+1) 3333 π23 333 π23 33 π13 3 π23 ε π23
(−1) 33323312323333 π23 * π23 32133 π13 23 π23 ε π23
(0) 33333 π23 3333 π23 333 π13 33 π23 3 π13 ε α1 π23
(0) 3333233123233333 π23 * π23 * π13 3233 π23 23 π13 ε α1
Idea of the proof on complexity

Show that the lifes of two pairs of strong and weak bispecial factors do not intersect, i.e. the following equality is preserved:

$$|z^+| < |z^-| < |w^+| < |w^-|.$$
Plan

1. Question
2. Partial answers
3. S-adic sequences and MCF algorithms
4. Results
5. Idea of proof on factor complexity for ARP sequences
6. ... Brun sequences
Extension type of the empty word do not depend only on the previous substitution

We must consider Extended extension type (2 letters on the left)

There are 5 types of extension type for the empty word.

Ordered Version (full shift, 3 substitutions) vs Not ordered (SFT, 6 substitutions)

...still we think it is s.t. $p(n) < 3n$