Factor Complexity of Arnoux-Rauzy-Poincaré sequences

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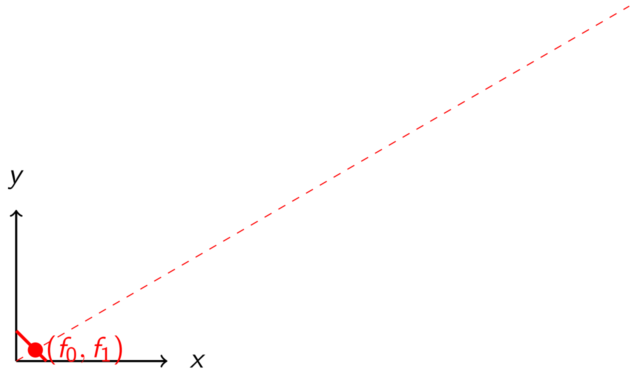
Joint work with Valérie Berthé

Outline

Question (from our talk at WORDS 2011)

Given $(f_0, f_1) \in \mathbb{R}^2$ *such that* $f_0 + f_1 = 1$*, can we construct an infinite sequence* $\mathbf{w} \in \mathcal{A}^{\mathbb{N}}$ *on the alphabet* $\mathcal{A} = \{0, 1\}$ *such that the frequency of digit* i *is* f_i *for all* $i \in \mathcal{A}$*?*



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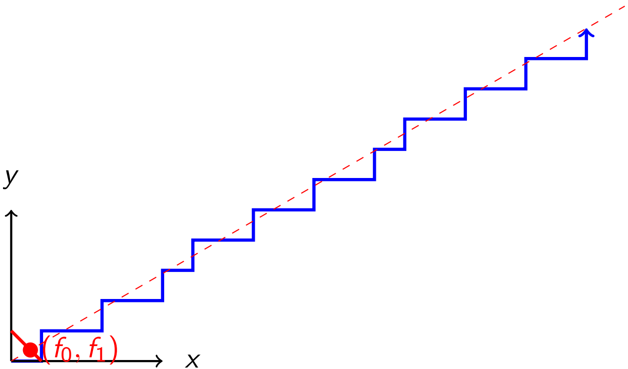
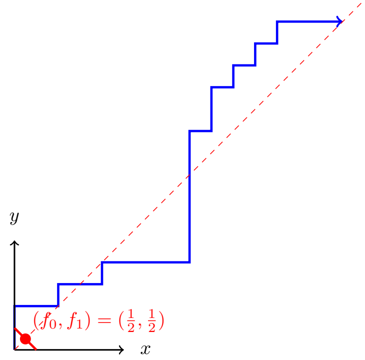


$0100100101001001001010010010 01\cdots$

One answer is the expansion of π in base 2 :

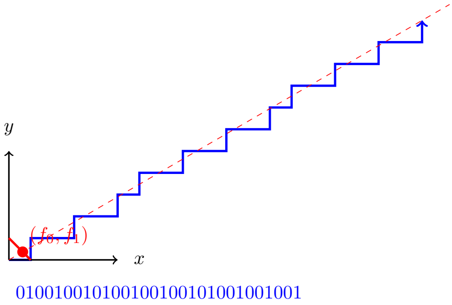
$$\pi = 11.00100100001111110110101010001000\cdots$$



It is not a good answer because :

- works only for (conjectured) $f_0 = f_1 = \frac{1}{2}$;
- Really different factors appear in the sequence (ex : 0000 and 1111) ;
- All factors (2^k factors of length k) appear in the sequence.

Plan



Question

*Given a vector $\vec{f} \in \mathbb{R}_+^d$ with $\|\vec{f}\|_1 = 1$ can we construct an infinite word **w** on the alphabet $\mathcal{A} = \{1, 2, \cdots, d\}$ satisfying each of below conditions?*

- *frequency of letters in **w** exists and is equal to $\vec{f}$,*
- ***w** stays at bounded distance from $\mathbb{R}_+\vec{f}$ (**w** is balanced),*
- ***w** has a linear factor complexity.*

Answer for $d = 2$: Sturmian sequences

Definition

A sturmian word is an infinite word having exactly $p(n) = n + 1$ factors of length n.

Fact (Morse, Hedlund, 1940)

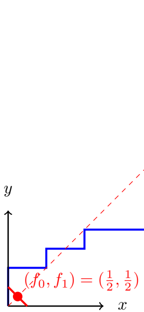
Sturmian words are exactly the aperiodic 1-balanced sequences.

For each $\vec{f} \in \mathbb{R}^2_+$ such that $\|\vec{f}\|_1 = 1$, there exists a sturmian word $\mathbf{w}$ such that the frequency of letters in $\mathbf{w}$ is $\vec{f}$.

A bad answer

Consider the expansion of π in base 2 :

$$\pi = 11.00100100001111110110101010001 \cdots$$



It is not a good discrete line because :

- works only for (conjectured) $f_0 = f_1 = \frac{1}{2}$;
- Really different factors appear in the sequence (ex : 0000 and 1111) ;
- All factors (2^k factors of length k) appear (conjecture : π is normal).

Factor complexity

Let $w \in \mathcal{A}^{\mathbb{N}}$. The factor complexity is a function $p_w(n) : \mathbb{N} \to \mathbb{N}$ counting the number of factors of length n, noted $\mathrm{L}_w(n)$, in the sequence w.

$$w = 000100\boxed{0100}0100100010001000100100010001001$$

$$\mathrm{L}_w(4) = \{0001, 0010, 0100, 1000, 1001\}$$

n	$p_w(n)$
0	1
1	2
2	3
3	4
4	5

Upper bound : $p_w(n) \leq |\mathcal{A}|^n$.

Factors, frequencies, vectors

Factor : finite string of consecutive digits. Let

$$w = 010010010100100100101001001001.$$

Then 00100 and 1001 are factors of w. 1001 is a suffix of w.

$|w|$: the length of the factor w. $|w| = 30$.

$|w|_u$: the number of occurences of the factor u in w.

$$|w|_0 = 19, \quad |w|_1 = 11$$

$$|w|_{00} = 8, \quad |w|_{01} = 11, \quad |w|_{10} = 10, \quad |w|_{11} = 0.$$

$\vec{u} = (|u|_0, |u|_1)$: the abelian vector of the factor u.

$$\overrightarrow{00100} = (4, 1), \quad \overrightarrow{1001} = (2, 2).$$

Definition

An infinite word $\mathbf{w} \in \mathcal{A}^{\mathbb{N}}$ is said to be finitely balanced or C-balanced or balanced if there exists a constant $C \in \mathbb{N}$ such that
for all pairs of factors u, v of $\mathbf{w}$ of the same length,

$$||\vec{u} - \vec{v}||_\infty \leq C.$$

Base 2 development of π is not 1-balanced because

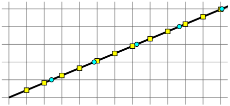
$$||\overrightarrow{0000} - \overrightarrow{1111}||_\infty = ||(4, -4)||_\infty = 4.$$

If π was proven normal, then 0^k and 1^k would also appear for all k, thus it would not be balanced.

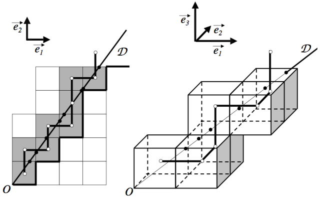
Plan

When $d = 3$: Cutting and billiard sequences

Sturmian word are obtained
from the cutting sequence of a line :



This can be generalized as billiard sequences :



Borel (2006)

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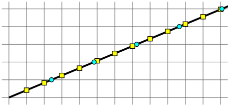

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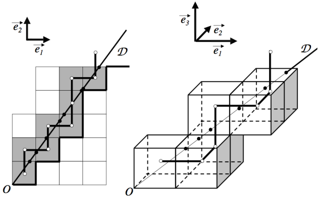


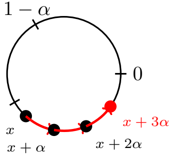
Borel (2006)

Theorem (Baryshnikov, 1995 ; Bédaride, 2003)

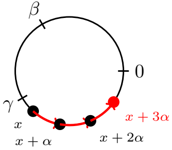
If both the direction $(\alpha_1, \alpha_2, \alpha_3)$ and $(\alpha_1^{-1}, \alpha_2^{-1}, \alpha_3^{-1})$ are $\mathbb{Q}$ independent, the number of factors appearing in the Billiard word in a cube is exactly $p(n) = n^2 + n + 1$.

When $d = 3$: Coding of rotations and IET

Sturmian word are obtained
from coding of rotations :



This can be generalized to larger alphabet with coding of rotations on more intervals and more generally to interval exchange transformations (IET) :



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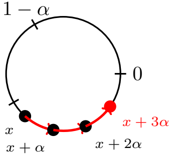

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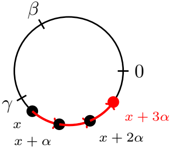

Such sequences have linear factor complexity but are not balanced.

Anton Zorich. Deviation for interval exchange transformations. *Ergodic Theory Dynam. Systems*, 17(6) :1477–1499, 1997.

When $d \geq 2$: Arnoux-Rauzy sequences

An infinite word $\mathbf{w} \in \{1, 2, \ldots, d\}^{\mathbb{N}}$ is an Arnoux-Rauzy word if all its factors occur infinitely often, and if $p(n) = (d-1)n + 1$ for all n, with exactly one left special and one right special factor of length n.

Theorem (Delecroix, Hejda, Steiner, WORDS 2013)

For μ-almost every $\mathbf{f}$ in the Rauzy gasket, the Arnoux-Rauzy word $w_{AR}(\mathbf{f})$ is finitely balanced.

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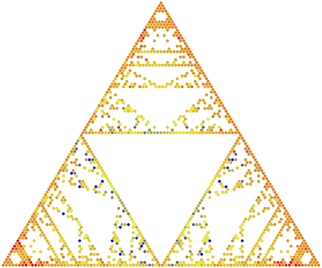
Pierre Arnoux and Štěpán Starosta. The Rauzy Gasket. In Julien Barral and Stéphane Seuret, editors, *Further Developments in Fractals and Related Fields*, Trends in Mathematics, pages 1–23. Birkhäuser Boston, 2013.



Plan

S-adic construction of a word

Definition

Let

$\mathcal{A}$ be an alphabet,

$(\sigma_i)_{i\geq 0}$ a sequence of morphisms such that $\sigma_i : \mathcal{A}^* \to \mathcal{A}^*$,

$(a_i)_{i\geq 0}$ a sequence of letters with $a_i \in \mathcal{A}$.

Assume that the limit

$$\mathbf{u} = \lim_{i\to\infty} (\sigma_0 \circ \sigma_1 \circ \cdots \circ \sigma_{i-1})(a_i)$$

exists and is an infinite word $\mathbf{u} \in \mathcal{A}^{\mathbb{N}}$.

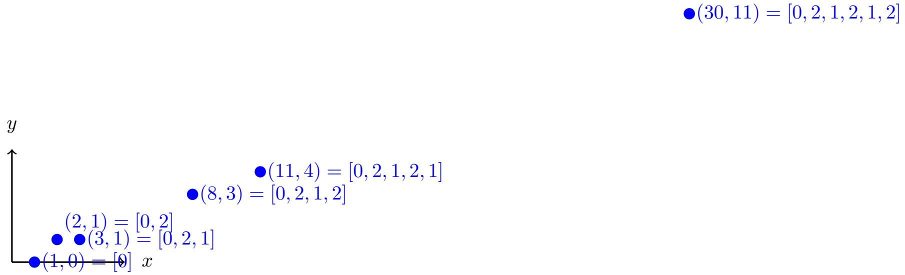
Then $(\sigma_i, a_i)_{i\geq 0}$ is called an s-adic construction of $\mathbf{u}$.

Let $\alpha = \frac{\sqrt{3}-1}{2} = 0.36602540\cdots$. We have

$$\alpha = 0 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cdots}}}} = [0; 2, 1, 2, 1, 2, 1, \cdots]$$

The convergents p_n/q_n are

$$0, \frac{1}{2}, \frac{1}{3}, \frac{3}{8}, \frac{4}{11}, \frac{11}{30}, \frac{15}{41}, \frac{41}{112}, \frac{56}{153}, \frac{153}{418}, \frac{209}{571}, \frac{571}{1560}, \frac{780}{2131}, \cdots$$



Continued fractions : matrices from convergents

With

$$L = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad R = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

the convergents can be obtained as

$$\begin{aligned}
\begin{pmatrix} q_0 \\ p_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} &= R^0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
\begin{pmatrix} q_1 \\ p_1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} &= R^0 L^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
\begin{pmatrix} q_2 \\ p_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} &= R^0 L^2 R^1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
\begin{pmatrix} q_3 \\ p_3 \end{pmatrix} = \begin{pmatrix} 8 \\ 3 \end{pmatrix} &= R^0 L^2 R^1 L^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
\begin{pmatrix} q_4 \\ p_4 \end{pmatrix} = \begin{pmatrix} 11 \\ 4 \end{pmatrix} &= R^0 L^2 R^1 L^2 R^1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
\begin{pmatrix} q_5 \\ p_5 \end{pmatrix} = \begin{pmatrix} 30 \\ 11 \end{pmatrix} &= R^0 L^2 R^1 L^2 R^1 L^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\end{aligned}$$

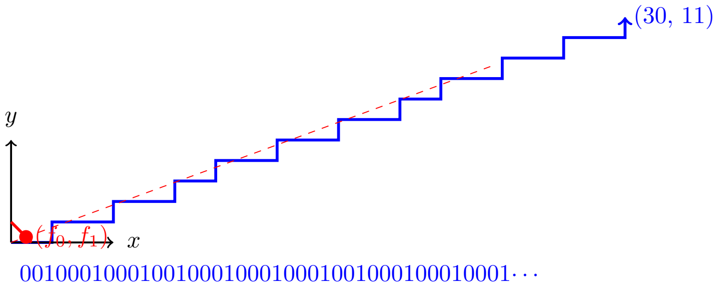
Continued fractions : substitutions from matrices

With

$$L = \begin{array}{l} 0 \mapsto 0 \\ 1 \mapsto 01 \end{array} \qquad \text{and} \qquad R = \begin{array}{l} 0 \mapsto 10 \\ 1 \mapsto 1 \end{array}$$

the convergents can be transformed into finite sequences over $\mathcal{A}$:

$$\begin{array}{lcllcl} w_0 & = & R^0(0) & = & 0 \\ w_1 & = & R^0L^2(1) & = & 001 \\ w_2 & = & R^0L^2R^1(0) & = & 0010 \\ w_3 & = & R^0L^2R^1L^2(1) & = & 00100010001 \\ w_4 & = & R^0L^2R^1L^2R^1(0) & = & 001000100010010 \\ w_5 & = & R^0L^2R^1L^2R^1L^2(1) & = & 00100010001001000100010001001000100010001 \end{array}$$



Sturmian sequences as S-adic sequences

From iteration of morphisms $L = \begin{array}{l} 0 \mapsto 0 \\ 1 \mapsto 01 \end{array}$ and $R = \begin{array}{l} 0 \mapsto 10 \\ 1 \mapsto 1 \end{array}$:

$$R^{d_0} L^{d_1} R^{d_2} L^{d_3} R^{d_4} L^{d_5} \cdots (1)$$

where $[d_0; d_1, d_2, d_3, \ldots]$ is the continued fraction expansion of the slope of the word drawn with horizontal and vertical unitary steps.

This generalizes to S-adic sequences based on Multidimensional Continued Fraction (MCF) algorithms...

Continued fractions : from Euclid Algorithm

With

$$L = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

the execution of Euclid Algorithm appears as

$$\begin{pmatrix} q_0 \\ p_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = R^0 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} q_1 \\ p_1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} = R^0 L^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} q_2 \\ p_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} = R^0 L^2 R^1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} q_3 \\ p_3 \end{pmatrix} = \begin{pmatrix} 8 \\ 3 \end{pmatrix} = R^0 L^2 R^1 L^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} q_4 \\ p_4 \end{pmatrix} = \begin{pmatrix} 11 \\ 4 \end{pmatrix} = R^0 L^2 R^1 L^2 R^1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} q_5 \\ p_5 \end{pmatrix} = \begin{pmatrix} 30 \\ 11 \end{pmatrix} = R^0 L^2 R^1 L^2 R^1 L^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 30 \\ 11 \end{pmatrix} = L^2 R^1 L^2 R^1 L^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 8 \\ 11 \end{pmatrix} = R^1 L^2 R^1 L^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 8 \\ 3 \end{pmatrix} = L^2 R^1 L^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix} = R^1 L^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} = L^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

3D Continued fraction algorithms

Brun's Algorithm : Subtract the second largest to the largest.

$$(7,4,6) \to (1,4,6) \to (1,4,2) \to (1,2,2) \to (1,0,2) \to (1,0,1) \to (0,0,1).$$

Selmer's Algorithm : Subtract the smallest to the largest.

$$(7,4,6) \to (3,4,6) \to (3,4,3) \to (3,1,3) \to (2,1,3) \to (2,1,2) \to (1,1,2) \\ \to (1,1,1) \to (0,1,1) \to (0,0,1)$$

Poincaré's Algorithm : Subtract the smallest to the mid and the mid to the largest.

$$(7,4,6) \to (1,4,2) \to (1,2,1) \to (1,1,0) \to (1,0,0)$$

Arnoux-Rauzy's Algorithm : Subtract the sum of the two smallest to the largest (not always possible).

$$(7,4,6) \to \text{Impossible}$$

Fully subtractive's Algorithm : Subtract the smallest to the other two.

$$(7,4,6) \to (3,4,2) \to (1,2,2) \to (1,1,1) \to (1,0,0)$$

3D : Imitation of Euclid algorithm on $(7, 4, 6)$

$$
\begin{array}{ccccccccc}
(7,4,6) & \xleftarrow{\begin{pmatrix}1&0&1\\0&1&0\\0&0&1\end{pmatrix}} & (1,4,6) & \xleftarrow{\begin{pmatrix}1&0&0\\0&1&0\\0&1&1\end{pmatrix}} & (1,4,2) & \xleftarrow{\begin{pmatrix}1&0&0\\0&1&2\\0&0&1\end{pmatrix}} & (1,0,2) & \xleftarrow{\begin{pmatrix}1&0&0\\0&1&0\\2&0&1\end{pmatrix}} & (1,0,0) \\
 & \begin{array}{ccc}1&\mapsto&1\\2&\mapsto&2\\3&\mapsto&13\end{array} & & \begin{array}{ccc}1&\mapsto&1\\2&\mapsto&23\\3&\mapsto&3\end{array} & & \begin{array}{ccc}1&\mapsto&1\\2&\mapsto&2\\3&\mapsto&223\end{array} & & \begin{array}{ccc}1&\mapsto&133\\2&\mapsto&2\\3&\mapsto&3\end{array} & \\
\mathbf{w}_0 & \longleftarrow & \mathbf{w}_1 & \longleftarrow & \mathbf{w}_2 & \longleftarrow & \mathbf{w}_3 & \longleftarrow & \mathbf{w}_4
\end{array}
$$

Its (Hausdorff) distance to the euclidean line is 1.3680.

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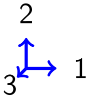
\end{array}
$$

$$\mathbf{w} = \mathbf{w}_0 = 1213213132132131$$



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$$

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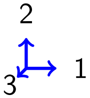

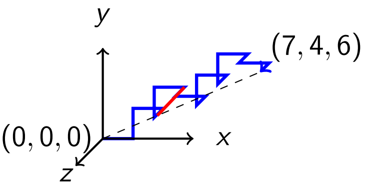

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Experimentations on discrepancy

	Min	Mean	Max	Std
Arnoux-Rauzy (AR)	0.6000	0.8922	1.200	0.09953
Fully subtractive	0.5000	6.047	14.21	4.385
Selmer	0.5000	2.151	12.75	2.076
Brun	0.5000	1.100	2.000	0.2625
Poincaré	0.5000	2.476	11.13	2.245
AR-Fully subtractive	0.5000	1.154	4.000	0.3759
AR-Selmer	0.5000	0.9991	1.600	0.1429
AR-Brun	0.5000	0.9169	1.520	0.1170
AR-Poincaré	0.5000	0.9066	1.320	0.1079

TABLE: Statistics for the discrepancy for strictly positive integer vectors (a_1, a_2, a_3) such that $a_1 + a_2 + a_3 = N$ and $N = 100$.

(from talk at WORDS 2011)

Plan

Results

Theorem (Berthé, L., 2013)

Let $\mathbf{w} = \lim_{n\to\infty} \sigma_0\sigma_1 \cdots \sigma_n(1)$ *be an S-adic word generated by the Arnoux-Rauzy-Poincaré algorithm from a totally irrational vector* $\mathbf{x} \in \Delta_3$*. The factor complexity of* $\mathbf{w}$ *is such that*

- $p(n) \leq 3n + 1$ *for all* $n \geq 0$ *;*
- $p(n+1) - p(n) \in \{2, 3\}$ *for all* $n \geq 0$ *;*
- $\limsup_{n\to\infty} \frac{p(n)}{n} \leq \frac{5}{2} < 3$ *(not sharp).*

Results

Given a measure μ for which the cocycle is log-integrable, we say that ($(A^{\mathbb{N}}, T, A, \mu)$ has Pisot spectrum if the associated Lyapunov exponents satisfy $\Theta_1 > 0 > \Theta_2$. This property is related to the strong convergence of higher dimensional continued fraction algorithm [Lagarias, 93].

Theorem (Berthé, Delecroix (Survey Thm. 6.4), 2013)

Let $\mathcal{A}$ be a MCF algorithm and Θ_1 and Θ_2 be the first two Lyapunov exponents of the algorithm $\mathcal{A}$. If $\Theta_2 < 0$, then for μ-almost all frequency vector $\mathbf{x} \in \Delta$, the S-adic word $w(\mathbf{x})$ generated by the algorithm $\mathcal{A}$ is finitely balanced.

Results

Theorem (Avila, Delecroix, 2013)

Let (Δ, T, A) be the cocycle associated to Brun map in dimension $d = 2$ or the fully subtractive algorithm in any dimension. Then, for every T-invariant ergodic probability measure on Δ such that there exists a cylinder $[w]$ of positive μ-measure whose associated matrix $A(w)$ is positive, the Lyapunov spectrum of (Δ, T, A, μ) is Pisot.

Let $\Delta_3 = \{(f_1, f_2, f_3) \in \mathbb{R}^3_+ : f_1 + f_2 + f_3 = 1\}$.

Theorem (Delecroix, Hejda, Steiner, 2013)

For Lebesgue almost all frequency vector $\mathbf{x} \in \Delta_3$, the S-adic word $w_{AR}(\mathbf{x})$ generated by the Brun algorithm is finitely balanced.

Summary

	$\forall v \in \mathbb{R}^3_+$	$p(n)$ linear	Balanced
Arnoux Rauzy words	No	Yes	Almost always
Billiard, Andres discrete line	Yes	No	Yes
Coding of rotations and of IET	Yes	Yes	No
Brun S-adic sequences	Yes	$\approx 3n$?	Almost always
ARP S-adic sequences	Yes	Yes : $\frac{5}{2}n$	?
Other S-adic sequences	?	?	?

Plan

Arnoux-Rauzy and Poincaré substitutions

For all $\{i, j, k\} = \{1, 2, 3\}$, we consider

$$\begin{array}{ll} \pi_{jk} : i \mapsto ijk, j \mapsto jk, k \mapsto k & \text{(Poincaré substitutions)} \\ \alpha_k : i \mapsto ik, j \mapsto jk, k \mapsto k & \text{(Arnoux-Rauzy substitutions)} \end{array}$$

Namely,

$$\pi_{23} = \begin{cases} 1 \mapsto 123 \\ 2 \mapsto 23 \\ 3 \mapsto 3 \end{cases}, \quad \pi_{13} = \begin{cases} 1 \mapsto 13 \\ 2 \mapsto 213 \\ 3 \mapsto 3 \end{cases}, \quad \alpha_3 = \begin{cases} 1 \mapsto 13 \\ 2 \mapsto 23 \\ 3 \mapsto 3 \end{cases},$$

$$\pi_{12} = \begin{cases} 1 \mapsto 12 \\ 2 \mapsto 2 \\ 3 \mapsto 312 \end{cases}, \quad \pi_{32} = \begin{cases} 1 \mapsto 132 \\ 2 \mapsto 2 \\ 3 \mapsto 32 \end{cases}, \quad \alpha_2 = \begin{cases} 1 \mapsto 12 \\ 2 \mapsto 2 \\ 3 \mapsto 32 \end{cases},$$

$$\pi_{31} = \begin{cases} 1 \mapsto 1 \\ 2 \mapsto 231 \\ 3 \mapsto 31 \end{cases}, \quad \pi_{21} = \begin{cases} 1 \mapsto 1 \\ 2 \mapsto 21 \\ 3 \mapsto 321 \end{cases}, \quad \alpha_1 = \begin{cases} 1 \mapsto 1 \\ 2 \mapsto 21 \\ 3 \mapsto 31 \end{cases}.$$

Quadratic complexity for ARP sequences

In general, it is possible that $p(n+1) - p(n) > 3$ for some values of n. Let

$$s = \pi_{23}\pi_{23}\pi_{13}\pi_{23}\pi_{23}\alpha_1\alpha_3\alpha_2(1).$$

Indeed,

$$p_s(n) = (1, 3, 5, 8, 11, 15, 19, 23, 27, 31, 35, 38, \cdots)$$

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Indeed,

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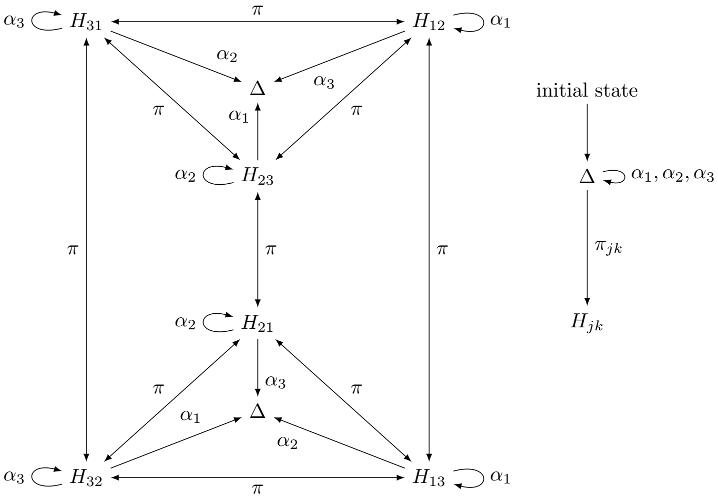
Even worse, the fixed point of

$$\pi_{13}\pi_{23} : \begin{cases} 1 \mapsto 132133 \\ 2 \mapsto 2133 \\ 3 \mapsto 3 \end{cases}$$

starting with letter 1 has a quadratic factor complexity.

Language of Arnoux-Rauzy Poincaré algorithm

Deterministic and minimized automaton recognizing the language $\mathcal{L} \subset \mathcal{S}^{\mathbb{N}}$ of ARP algorithm :



Let $\mathbf{w} = \lim_{n\to\infty} \sigma_0\sigma_1 \cdots \sigma_n(1)$ be an S-adic word generated by the Arnoux-Rauzy-Poincaré algorithm from a totally irrational vector $\mathbf{x} \in \Delta_3$.

Theorem (Factor Complexity)

The factor complexity of $\mathbf{w}$ *is such that*

- $p(n) \leq 3n + 1$ *for all* $n \geq 0$ *;*
- $p(n+1) - p(n) \in \{2, 3\}$ *for all* $n \geq 0$ *;*
- $\limsup_{n\to\infty} \frac{p(n)}{n} \leq \frac{5}{2} < 3$ *(not sharp).*

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Theorem (Frequencies and Convergence)

The symbolic dynamical system generated by $\mathbf{w}$ *is uniquely ergodic, and the frequencies of letters are proved to exist in* $\mathbf{w}$ *and to be equal to the coordinates of* $\mathbf{x}$*.*
Furthermore, the Arnoux-Rauzy-Poincaré algorithm is a weakly convergent algorithm, that is, for Lebesgue almost every $\mathbf{x} \in \Delta$*, if* $(M_n)_n$ *stands for the sequence of matrices produced by the Arnoux-Rauzy-Poincaré algorithm, then one has* $\cap_n M_0 \cdots M_n(\mathbb{R}^3_+) = \mathbb{R}_+\mathbf{x}$*.*

Idea of the proof on complexity

Let $p(n)$ be the factor complexity function of $\mathbf{w}$. Let $s(n)$ and $b(n)$ be its sequences of finite differences of order 1 and 2 :

$$\begin{aligned} p(n) &= 1, 3, 5, 7, 9, 11, 14, 17, 20, 22, 24, 26, 28, \\ s(n) = p(n+1) - p(n) &= 2, 2, 2, 2, 2, 3, 3, 3, 2, 2, 2, 2, \\ b(n) = s(n+1) - s(n) &= 0, 0, 0, 0, +1, 0, 0, -1, 0, 0, 0, \end{aligned}$$

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$$\begin{aligned} p(n) &= 1,3,5,7,9,11,14,17,20,22,24,26,28, \\ s(n) = p(n+1) - p(n) &= 2,2,2,2,2,3,3,3,2,2,2,2, \\ b(n) = s(n+1) - s(n) &= 0,0,0,0,+1,0,0,-1,0,0,0, \end{aligned}$$

Functions s and b are related to special and bispecial factors of $\mathbf{w}$.

Theorem (Cassaigne, 1997 ; Cassaigne, Nicolas, 2010)

Let $\mathbf{u} \in A^{\mathbb{N}}$ be a infinite [recurrent] word. Then, for all $n \in \mathbb{N}$:

$$s(n) = \sum_{w \in RS_n(\mathbf{u})} (d^+(w) - 1) \qquad \text{and} \qquad b(n) = \sum_{w \in BS_n(\mathbf{u})} m(w)$$

Special and bispecial words

Let $\mathbf{u}$ be an infinite word $L(\mathbf{u})$ be its language.

Right extensions and right valence :

$$E^+(w) = \{x \in \mathcal{A} | wx \in L(\mathbf{u})\} \qquad d^+(w) = \mathrm{Card} E^+(w).$$

Left extensions and left valence :

$$E^-(w) = \{x \in \mathcal{A} | xw \in L(\mathbf{u})\} \qquad d^-(w) = \mathrm{Card} E^-(w).$$

A factor w is

- right special if $d^+(w) \geq 2$,
- left special if $d^-(w) \geq 2$,
- bispecial if it is left and right special.

Special and bispecial words

The extension type of a factor w of $\mathbf{u}$ is

$$E(w) = \{(a,b) \in \mathcal{A} \times \mathcal{A} | awb \in L(\mathbf{u})\}.$$

The bilateral multiplicity of a factor w is

$$m(w) = \mathrm{Card}\, E(w) - d^-(w) - d^+(w) + 1.$$

A bispecial factor is said

weak if $m(w) < 0$, neutral if $m(w) = 0$, strong if $m(w) > 0$.

Examples of extension types $E(w)$ of bispecial factors w

$$m(w) = \mathrm{Card}\, E(w) - d^-(w) - d^+(w) + 1.$$

$$\begin{array}{c|cc} & 1 & 2 \\ \hline 1 & & \times \\ 2 & \times & \end{array}$$

$m(w) = -1$

weak

$$\begin{array}{c|cc} & 1 & 2 \\ \hline 1 & \times & \times \\ 2 & \times & \end{array}$$

$m(w) = 0$

neutral and ordinary

$$\begin{array}{c|cc} & 1 & 2 \\ \hline 1 & \times & \times \\ 2 & \times & \times \end{array}$$

$m(w) = 1$

strong

Examples of extension types $E(w)$ of bispecial factors w

$$m(w) = \mathrm{Card}\, E(w) - d^-(w) - d^+(w) + 1.$$

	1	2
1		$\times$
2	$\times$	

$m(w) = -1$
weak

	1	2
1	$\times$	$\times$
2	$\times$	

$m(w) = 0$
neutral and ordinary

	1	2
1	$\times$	$\times$
2	$\times$	$\times$

$m(w) = 1$
strong

A bispecial factor w is ordinary if $\exists$ row $\exists$ column s.t.

$$\text{row} \cap \text{column} \quad \subseteq \quad E(w) \quad \subseteq \quad \text{row} \cup \text{column}.$$

Lemma

If a bispecial factor is ordinary, then it is neutral.

On a binary alphabet, the reciprocal also holds.

Examples of extension types $E(w)$ of bispecial factors w

	1	2	3
1		$\times$	
2		$\times$	
3	$\times$	$\times$	$\times$

$m(w) = 0$
neutral and ordinary

	1	2	3
1		$\times$	
2			
3	$\times$	$\times$	$\times$

$m(w) = 0$
neutral and ordinary

	1	2	3
1		$\times$	
2			$\times$
3	$\times$	$\times$	$\times$

$m(w) = 0$
neutral but not ordinary

	1	2	3
1		$\times$	
2		$\times$	
3	$\times$		$\times$

$m(w) = -1$
weak

	1	2	3
1		$\times$	
2			
3			$\times$

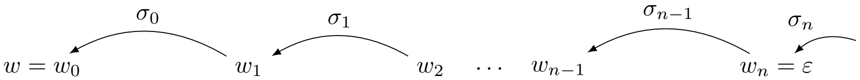
$m(w) = -1$
weak

	1	2	3
1			
2		$\times$	$\times$
3	$\times$	$\times$	$\times$

$m(w) = 1$
strong

Life of a bispecial factor under ARP substitutions

- Some synchronization lemmas allows to define uniquely the antecedent w_{k+1} of a (bispecial) factor $w_k = s \cdot \sigma_k(w_{k+1}) \cdot p$.
- w_k is an extended image of w_{k+1}



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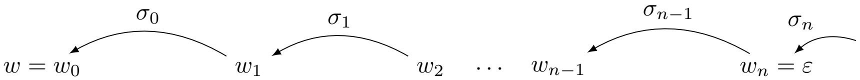


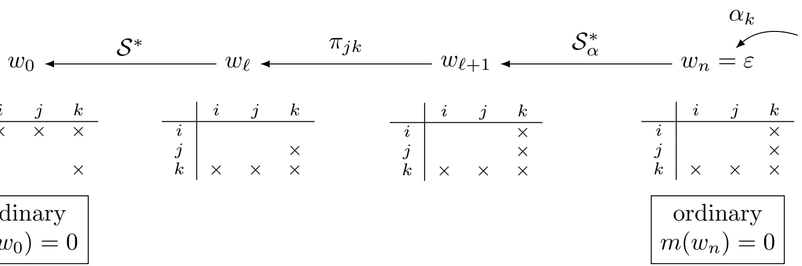
- We always have $|w_k| > |w_{k+1}|$.
- The history of w is $\sigma_0\sigma_1 \cdots \sigma_n$.
- The life of w is $(w_k)_{0 \leq k \leq n}$.
- w_{k+1} has only one bispecial extended image w_k under an Arnoux-Rauzy substitution.
- w_{k+1} has one or two extended images under Poincaré substitution.

Ordinary bispecial factor from an ordinary

Let

$$\begin{aligned}\mathcal{S}_\alpha &= \{\alpha_1, \alpha_2, \alpha_3\},\\ \mathcal{S}_\pi &= \{\pi_{12}, \pi_{13}, \pi_{23}, \pi_{21}, \pi_{31}, \pi_{32}\}\\ \mathcal{S} &= \mathcal{S}_\alpha \cup \mathcal{S}_\pi.\end{aligned}$$

CASE 1 : If history$(w_0) \in \mathcal{S}^* \, \pi_{jk} \, \mathcal{S}_\alpha^* \{\alpha_k\}$, then $m(w_0) = 0$:



w_0	i	j	k
i	×	×	×
j			
k			×

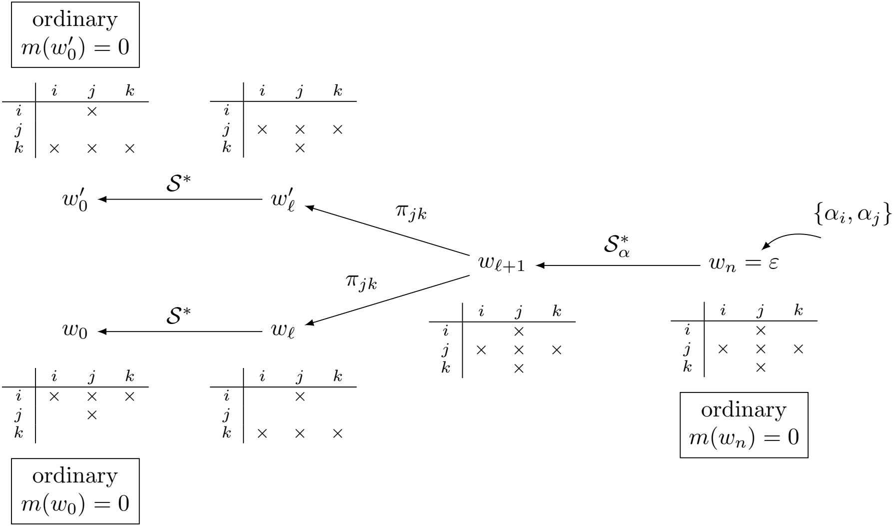
w_ℓ	i	j	k
i			
j			×
k	×	×	×

$w_{\ell+1}$	i	j	k
i			×
j			×
k	×	×	×

w_n	i	j	k
i			×
j			×
k	×	×	×

Two ordinary bispecial factors from an ordinary

CASE 2 : If $\text{history}(w_0) \in \mathcal{S}^* \pi_{jk} \mathcal{S}_\alpha^* \{\alpha_i, \alpha_j\}$, then $m(w_0) = 0$:



w'_0:

	i	j	k
i		×	
j			
k	×	×	×

w'_ℓ:

	i	j	k
i			
j	×	×	×
k		×	

w_0:

	i	j	k
i	×	×	×
j		×	
k			

w_ℓ:

	i	j	k
i		×	
j			
k	×	×	×

$w_{\ell+1}$:

	i	j	k
i		×	
j	×	×	×
k		×	

w_n:

	i	j	k
i		×	
j	×	×	×
k		×	

Two ordinary bispecial factors from an non ordinary

CASE 3 : If $\text{history}(w_0) \in \mathcal{S}^* \pi_{jk} \mathcal{S}_\alpha^* \{\pi_{ji}, \pi_{ki}, \pi_{ij}, \pi_{kj}\}$, then $m(w_0) = 0$:

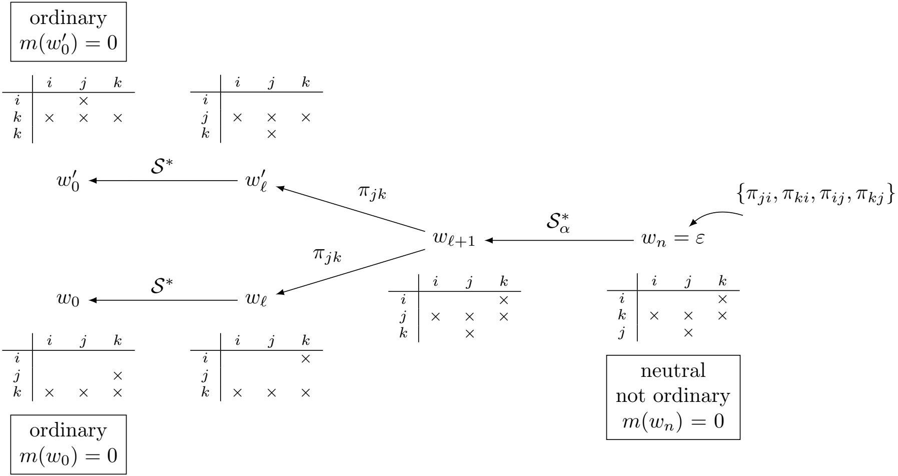


Table for w_0':

	i	j	k
i		$\times$	
k	$\times$	$\times$	$\times$
k			

Table for w_ℓ':

	i	j	k
i			
j	$\times$	$\times$	$\times$
k		$\times$	

Table for $w_{\ell+1}$:

	i	j	k
i			$\times$
j	$\times$	$\times$	$\times$
k		$\times$	

Table for w_n:

	i	j	k
i			$\times$
k	$\times$	$\times$	$\times$
j		$\times$	

Table for w_0:

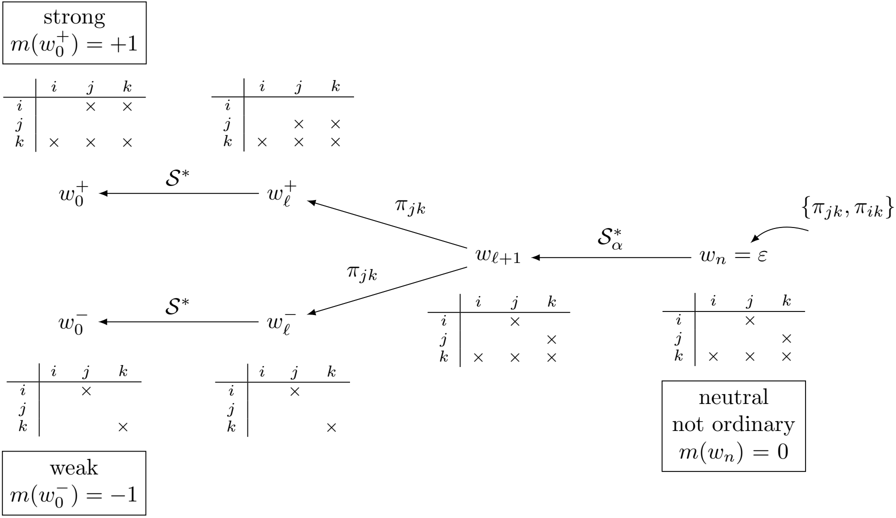
	i	j	k
i			
j			$\times$
k	$\times$	$\times$	$\times$

Table for w_ℓ:

	i	j	k
i			$\times$
j			
k	$\times$	$\times$	$\times$

Strong and weak bispecial factors from an non ordinary

CASE 4 : If history$(w_0) \in \mathcal{S}^* \pi_{jk} \mathcal{S}_\alpha^* \{\pi_{jk}, \pi_{ik}\}$, then $m(w_0) = \pm 1$:



	i	j	k
i		×	×
j			
k	×	×	×

	i	j	k
i			
j		×	×
k	×	×	×

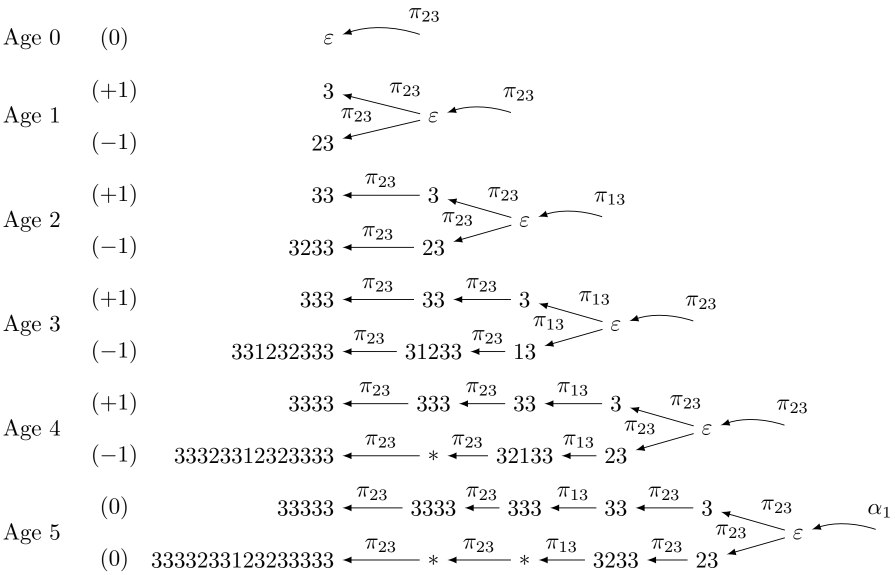
	i	j	k
i		×	
j			×
k	×	×	×

	i	j	k
i		×	
j			×
k	×	×	×

	i	j	k
i		×	
j			
k			×

	i	j	k
i		×	
j			
k			×

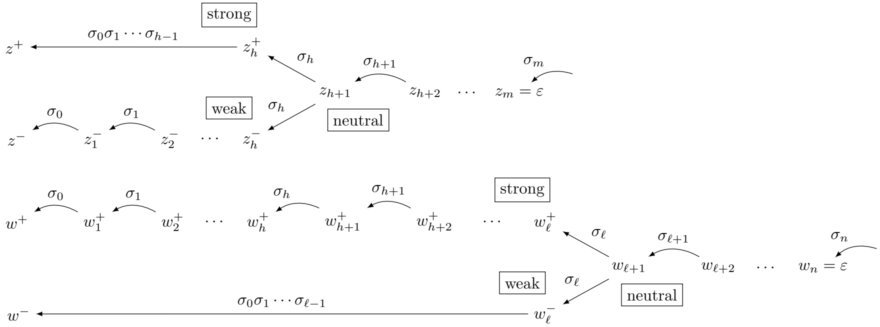
When it does not work



Idea of the proof on complexity

Show that the lifes of two pairs of strong and weak bispecial factors do not intersect, i.e. the following equality is preserved :

$$|z^+| < |z^-| < |w^+| < |w^-|.$$



Plan

...Brun sequences

- Extension type of the empty word do not depend only on the previous substitution
- We must consider Extended extension type (2 letters on the left)
- There are 5 types of extension type for the empty word.
- Ordered Version (full shift, 3 substitutions) vs Not ordered (SFT, 6 substitutions)
- ...still we think it is s.t. $p(n) < 3n$