Factor complexity of Arnoux-Rauzy and Poincaré S-adic sequences

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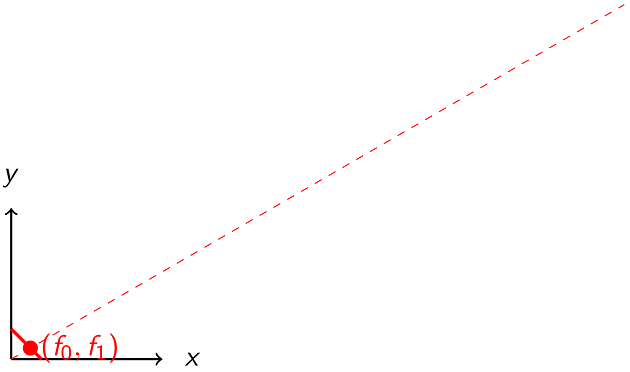
Joint work with Valérie Berthé and Pierre Arnoux

Outline

Plan

Question

Given $(f_0, f_1) \in \mathbb{R}^2$ *such that* $f_0 + f_1 = 1$, *can we construct an infinite sequence* $\mathbf{w} \in \mathcal{A}^{\mathbb{N}}$ *on the alphabet* $\mathcal{A} = \{0, 1\}$ *such that the frequency of digit* i *is* f_i *for all* $i \in \mathcal{A}$ *?*



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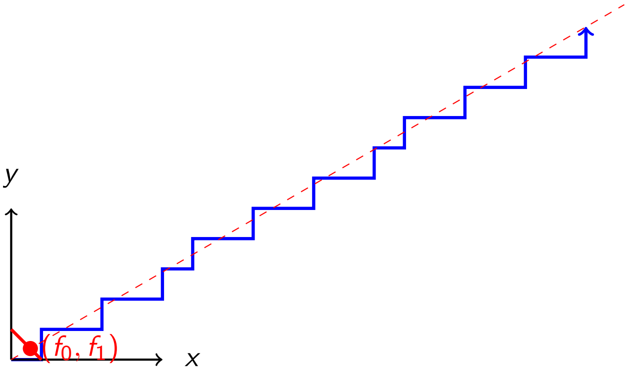

$0100100101001001001010010 01001 \cdots$

Factors, frequencies, vectors

Factor : finite string of consecutive digits. Let

$$w = 01\textcolor{red}{00100}10100100\textcolor{red}{1001}0100100\textcolor{red}{1001}.$$

Then 00100 and 1001 are factors of w. 1001 is a suffix of w.

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$$|w|_0 = 19, \quad |w|_1 = 11$$

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$\vec{u} = (|u|_0, |u|_1)$: the abelian vector of the factor u.

$$\overrightarrow{00100} = (4,1), \quad \overrightarrow{1001} = (2,2).$$

Normal numbers

Let **w** be the development in base b of a number $x \in \mathbb{R}$.
Let p be a finite prefix of **w**.

Definition

The number x is called normal in base b if

$$\lim_{|p|\to\infty} \frac{|p|_s}{|p|} = \frac{1}{b^k}$$

for every string s of length k.

The number

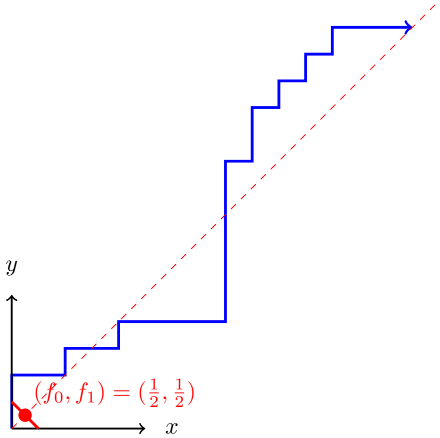
$$\pi = 11.00100100001111110110101010001\cdots$$

is conjectured normal in base 2.

Normal numbers

Normal number are such that :

- Really different factors appear in the sequence (ex : 0000 and 1111) ;
- All factors (b^k factors of length k) appear in the sequence.



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Definition

An infinite word $\mathbf{w} \in \mathcal{A}^{\mathbb{N}}$ is said to be finitely balanced or C-balanced or balanced if there exists a constant $C \in \mathbb{N}$ such that
for all pairs of factors u, v of $\mathbf{w}$ of the same length,
if $\vec{u} - \vec{v} = (a_0, a_1)$, then

$$|a_0| \leq C \quad \text{and} \quad |a_1| \leq C.$$

Base 2 development of π is not balanced for $C = 1$ because the factors

$$0000 \quad \text{and} \quad 1111$$

appear in the development. If π was proved normal, then 0^k and 1^k would also appear for all k, thus it would not be balanced.

Discrepancy

Definition

The ***discrepancy*** *of an infinite word* $\mathbf{w} \in \mathcal{A}^{\mathbb{N}}$ *is defined as*

$$\limsup_{i \in \mathcal{A}, p \text{ prefix of } \mathbf{w}} \left| f_i \cdot |p| - |p|_i \right|.$$

where f_i *is the* ***frequency of the letter*** $i \in \mathcal{A}$, *if it exists :*

$$f_i = \lim_{p \text{ prefix of } \mathbf{w}} \frac{|p|_i}{|p|}$$

Factor complexity

Let $w \in \mathcal{A}^{\mathbb{N}}$. The factor complexity is a function $p_w(n) : \mathbb{N} \to \mathbb{N}$ counting the number of factors of length n, noted $\mathrm{L}_w(n)$, in the sequence w.

$$w = 0001000100010010001000100010010001001$$

$$\mathrm{L}_w(4) = \{ \quad , \quad , \quad , \quad , \quad \}$$

n	$p_w(n)$
0	1
1	2
2	3
3	4
4	5

Upper bound : $p_w(n) \leq |\mathcal{A}|^n$.

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Theorem (Morse, Hedlund, 1940)

Let u be an infinite word and p its factor complexity function. Then either p is eventually periodic, or p is strictly increasing.

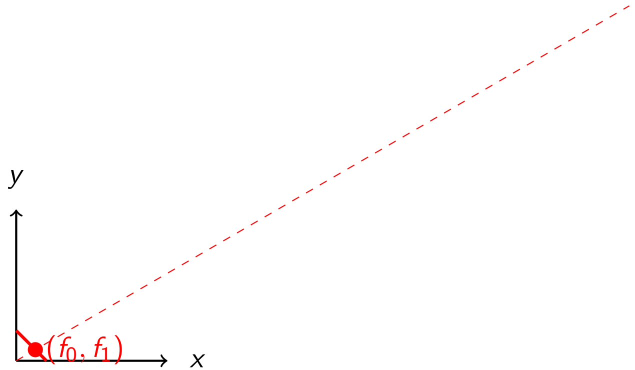
Corollary

Assume that for some integer n, $p(n) \leq n$. Then u is eventually periodic.

There exist infinite words with complexity $p(n) = n + 1$ for all n. Such words were called Sturmian words by Hedlund and Morse in honor of the mathematician Jacques Charles François Sturm (according to Wikipédia).

Question (updated)

Given $(f_0, f_1) \in \mathbb{R}^2$ such that $f_0 + f_1 = 1$, can we construct an infinite sequence $\mathbf{w} \in \mathcal{A}^{\mathbb{N}}$ on the alphabet $\mathcal{A} = \{0, 1\}$ such that the frequency of digit i is f_i for all $i \in \mathcal{A}$ and $\mathbf{w}$ is balanced and/or has a linear factor complexity ?



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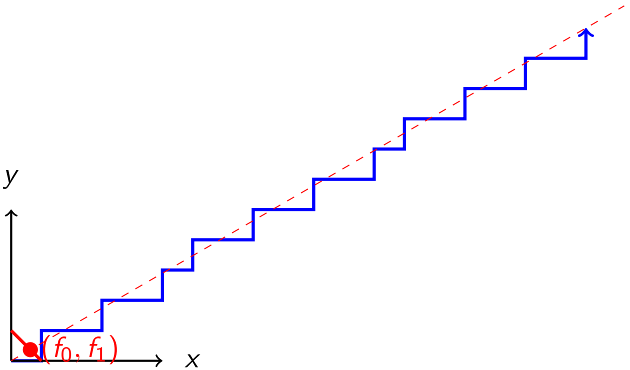


01001001010010010100100101001001$\cdots$

Answer for $d = 2$: Sturmian words

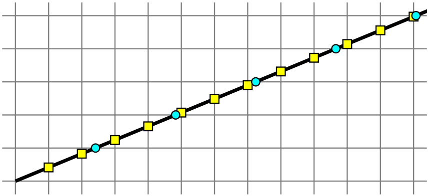
Proposition

Sturmian words are 1-balanced.

Sturmian word are also obtained from coding of rotations :

$$w_n = \begin{cases} 1 & \text{if } (\alpha + n\beta) \mod 1 \in [0, \beta[\\ 0 & \text{else} \end{cases} .$$

for some $\alpha, \beta \in [0, 1[$.
The Sturmian words of slope $\alpha \in \mathbb{R}_+$ is also obtained from the cutting sequence of a line.



Let $\Delta_d = \{(f_1, f_2, \ldots, f_d) \in \mathbb{R}_+^d : f_1 + f_2 + \cdots + f_d = 1\}$.

Question

Given a vector $(f_1, f_2, \cdots, f_d) \in \Delta_d$, can we construct an infinite word $\mathbf{w}$ on the alphabet $\mathcal{A} = \{1, 2, \cdots, d\}$ such that the frequency of each letter $i \in \mathcal{A}$ exists and is equal to f_i i.e.

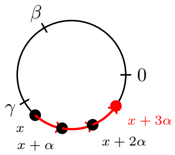
$$f_i = \lim_{p \text{ prefix of } \mathbf{w}} \frac{|p|_i}{|p|}.$$

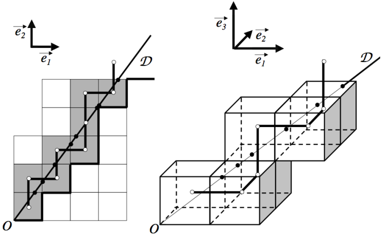
and $\mathbf{w}$ is balanced and/or has a linear factor complexity ?

Possible answers for $d > 2$

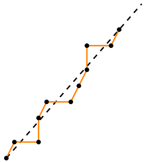
Typical answers to this question are :

	$\forall v \in \mathbb{R}^d_+$	$p(n)$ is linear	Balanced
Coding of Rotations	Yes	Yes	No
Coding of Interval Exchange Tr.	Yes	Yes	No
Billiard words	Yes	No	Yes
E. Andres discrete lines	Yes	No	Yes





Borel (2006)



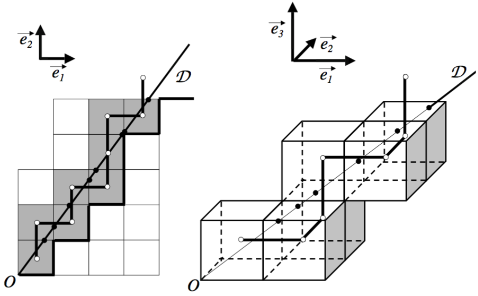
$$-7/2 \le 2x - 5z < 7/2$$
$$-8/2 \le 3x - 5y < 8/2$$
$$-5/2 \le 2y - 3z < 5/2$$

Andres (2003)

Factor Complexity of the Billiard word

Theorem (Baryshnikov, 1995 ; Bédaride, 2003)

If both the direction $(\alpha_1, \alpha_2, \alpha_3)$ *and* $(\alpha_1^{-1}, \alpha_2^{-1}, \alpha_3^{-1})$ *are* $\mathbb{Q}$ *independent, the number of factors appearing in the Billiard word in a cube is exactly* $p(n) = n^2 + n + 1$.



Source of image : J.-P. Borel, Complexity of Degenerated Three Dimensional Billiard Words, Developments in Language Theory 4036 (2006) 386-396.

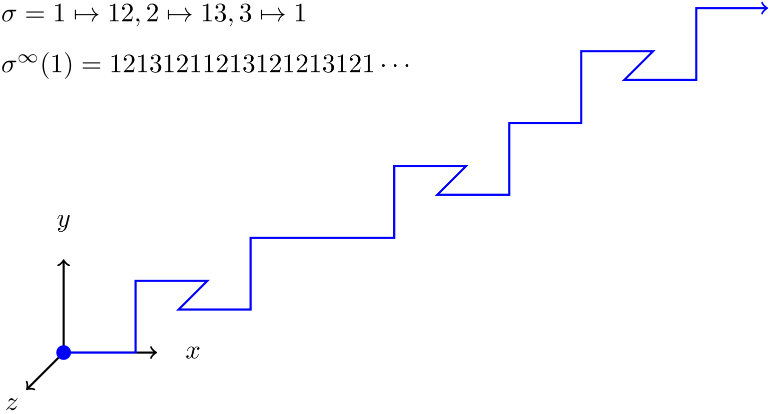
Hope : Tribonacci Example from Rauzy (1982)

The Tribonacci word is a infinite word over $\{1, 2, 3\}$ containing $2n + 1$ factors of length n and is balanced.

Can we generalize this to any 3D directions ?

$\sigma = 1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 1$

$\sigma^\infty(1) = 1213121121312121 3121 \cdots$

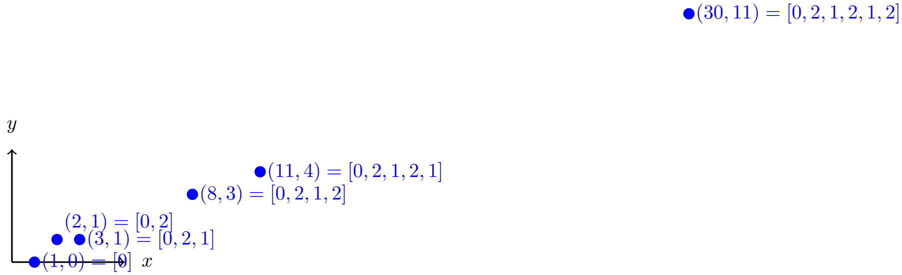


Let $\alpha = \frac{\sqrt{3}-1}{2} = 0.36602540\cdots$. We have

$$\alpha = 0 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cdots}}}} = [0; 2, 1, 2, 1, 2, 1, \cdots]$$

The convergents p_n/q_n are

$$0, \frac{1}{2}, \frac{1}{3}, \frac{3}{8}, \frac{4}{11}, \frac{11}{30}, \frac{15}{41}, \frac{41}{112}, \frac{56}{153}, \frac{153}{418}, \frac{209}{571}, \frac{571}{1560}, \frac{780}{2131}, \cdots$$



With

$$L = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

the convergents can be obtained as

$$\begin{aligned}
\begin{pmatrix} q_0 \\ p_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} &= R^0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
\begin{pmatrix} q_1 \\ p_1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} &= R^0 L^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
\begin{pmatrix} q_2 \\ p_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} &= R^0 L^2 R^1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
\begin{pmatrix} q_3 \\ p_3 \end{pmatrix} = \begin{pmatrix} 8 \\ 3 \end{pmatrix} &= R^0 L^2 R^1 L^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
\begin{pmatrix} q_4 \\ p_4 \end{pmatrix} = \begin{pmatrix} 11 \\ 4 \end{pmatrix} &= R^0 L^2 R^1 L^2 R^1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
\begin{pmatrix} q_5 \\ p_5 \end{pmatrix} = \begin{pmatrix} 30 \\ 11 \end{pmatrix} &= R^0 L^2 R^1 L^2 R^1 L^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\end{aligned}$$

Morphisms and incidence matrices

A morphism is a function $\sigma : \mathcal{A}^* \to \mathcal{A}^*$ such that $\sigma(uv) = \sigma(u)\sigma(v)$ for all $u, v \in \mathcal{A}^*$.
The incidence matrix M_σ of a morphism σ is the unique matrix over the non negative integers such that, for all $u \in \mathcal{A}^*$:

$$M_\sigma(\overrightarrow{u}) = \overrightarrow{\sigma(u)}.$$

$$\begin{array}{ccc} \mathcal{A}^* & \xrightarrow{\sigma} & \mathcal{A}^* \\ \downarrow \vec{\cdot} & & \downarrow \vec{\cdot} \\ \mathbb{N}^d & \xrightarrow{M_\sigma} & \mathbb{N}^d \end{array}$$

Example.

$$\sigma = \begin{array}{l} 0 \mapsto 0100111 \\ 1 \mapsto 101 \end{array} \qquad M_\sigma = \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}$$

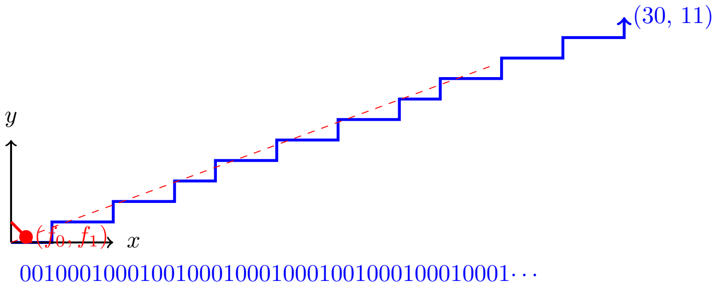
Continued fractions : substitutions from matrices

With

$$L = \begin{array}{l} 0 \mapsto 0 \\ 1 \mapsto 01 \end{array} \qquad \text{and} \qquad R = \begin{array}{l} 0 \mapsto 10 \\ 1 \mapsto 1 \end{array}$$

the convergents can be transformed into finite sequences over $\mathcal{A}$:

$$\begin{array}{lcll} w_0 & = & R^0(0) & = & 0 \\ w_1 & = & R^0L^2(1) & = & 001 \\ w_2 & = & R^0L^2R^1(0) & = & 0010 \\ w_3 & = & R^0L^2R^1L^2(1) & = & 00100010001 \\ w_4 & = & R^0L^2R^1L^2R^1(0) & = & 001000100010010 \\ w_5 & = & R^0L^2R^1L^2R^1L^2(1) & = & 00100010001001000100010001001000100010001 \end{array}$$



The previous command in Sage

Available in Sage since December 2009 :

```
sage: R = WordMorphism('0->01,1->1')
sage: L = WordMorphism('0->0,1->01')
sage: w5 = words.s_adic([L,L,R,L,L,R,L,L], '1')
sage: print w5
0001000100010010001000100010010001000100 1
```

Sturmian words

With

$$L = \begin{array}{l} 0 \mapsto 0 \\ 1 \mapsto 01 \end{array} \qquad \text{and} \qquad R = \begin{array}{l} 0 \mapsto 10 \\ 1 \mapsto 1 \end{array},$$

Sturmian words of slope

$$\alpha = [a_0; a_1, a_2, a_3, a_4, a_5, \cdots]$$

are also obtained from such sequences

$$\mathbf{w} = R^{a_0} L^{a_1} R^{a_2} L^{a_3} R^{a_4} L^{a_5} \cdots (1).$$

S-adic construction of a word

Definition

Let

$\mathcal{A}$ be an alphabet,

$(\sigma_i)_{i\geq 0}$ a sequence of morphisms such that $\sigma_i : \mathcal{A}^* \to \mathcal{A}^*$,

$(a_i)_{i\geq 0}$ a sequence of letters with $a_i \in \mathcal{A}$.

Assume that the limit

$$\mathbf{u} = \lim_{i\to\infty} (\sigma_0 \circ \sigma_1 \circ \cdots \circ \sigma_{i-1})(a_i)$$

exists and is an infinite word $\mathbf{u} \in \mathcal{A}^{\mathbb{N}}$.

Then $(\sigma_i, a_i)_{i\geq 0}$ is called an s-adic construction of $\mathbf{u}$.

The proposed approach

Generalize the Tribonacci word to any $v \in \mathbb{R}^3_+$ using multidimensional continued fractions algorithms and S-adic sequences.

There are other motivations for this approach :

- Study S-adic sequences, in the perspective of the S-adic conjecture concerning factor complexity.
- Study multidimensional continued fractions algorithms from substitutions and combinatorics on words point of view.
- Extend Pisot conjecture and Rauzy fractals (usually defined for fixed point of morphisms) to $\mathcal{S}$-adic sequences.

S-adic Conjecture

Conjecture

There exists a condition C such that a sequence has sub-linear complexity if and only if it is an S-adic sequence satisfying Condition C for some finite set S of morphisms.

Fabien Durand, Julien Leroy, and Gwénaël Richomme. Towards a statement of the s-adic conjecture through examples. *arXiv :1208.6376*, August 2012.

Sébastien Ferenczi. Rank and symbolic complexity. *Ergodic Theory Dynam. Systems*, 16(4) :663–682, 1996.

Plan

Continued fractions : from Euclid Algorithm

With

$$L = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad R = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

the execution of Euclid Algorithm appears as

$$\begin{aligned}
\begin{pmatrix} 30 \\ 11 \end{pmatrix} &= L^2 R^1 L^2 R^1 L^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
\begin{pmatrix} 8 \\ 11 \end{pmatrix} &= R^1 L^2 R^1 L^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
\begin{pmatrix} 8 \\ 3 \end{pmatrix} &= L^2 R^1 L^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
\begin{pmatrix} 2 \\ 3 \end{pmatrix} &= R^1 L^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
\begin{pmatrix} 2 \\ 1 \end{pmatrix} &= L^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\end{aligned}$$

2D : Euclid algorithm on $(11, 4)$

$$\boxed{\begin{array}{ccc} 11 & = & 2\cdot 4+3 \\ 4 & = & 1\cdot 3+1 \\ 3 & = & 3\cdot 1+0 \end{array}}$$

$$\frac{4}{11} = 0 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{3}}}$$

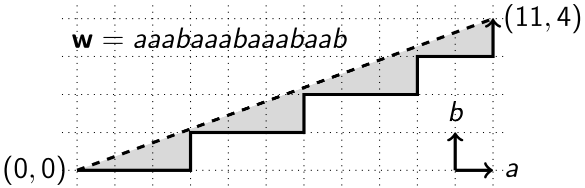
$$(11,4) \xleftarrow{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^2} (3,4) \xleftarrow{\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}} (3,1) \xleftarrow{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^3} (0,1)$$

$$\mathbf{w} = \mathbf{w}_0 \xleftarrow[\substack{a \mapsto a \\ b \mapsto aab}]{} \mathbf{w}_1 \xleftarrow[\substack{a \mapsto ab \\ b \mapsto b}]{} \mathbf{w}_2 \xleftarrow[\substack{a \mapsto a \\ b \mapsto aaab}]{} \mathbf{w}_3 = b$$

2D : Euclid algorithm on $(11, 4)$

11	$=$	$2 \cdot 4 + 3$
4	$=$	$1 \cdot 3 + 1$
3	$=$	$3 \cdot 1 + 0$

$$\frac{4}{11} = 0 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{3}}}$$



$$(11,4) \xleftarrow{\begin{pmatrix}1&1\\0&1\end{pmatrix}^2} (3,4) \xleftarrow{\begin{pmatrix}1&0\\1&1\end{pmatrix}} (3,1) \xleftarrow{\begin{pmatrix}1&1\\0&1\end{pmatrix}^3} (0,1)$$

$$\mathbf{w} = \mathbf{w}_0 \xleftarrow[\begin{array}{l}a \mapsto a\\ b \mapsto aab\end{array}]{} \mathbf{w}_1 \xleftarrow[\begin{array}{l}a \mapsto ab\\ b \mapsto b\end{array}]{} \mathbf{w}_2 \xleftarrow[\begin{array}{l}a \mapsto a\\ b \mapsto aaab\end{array}]{} \mathbf{w}_3 = b$$

3D : Brun's algorithm on $(7, 4, 6)$

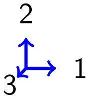
$$
\begin{array}{ccccccccc}
(7,4,6) & \xleftarrow{\begin{pmatrix}1&0&1\\0&1&0\\0&0&1\end{pmatrix}} & (1,4,6) & \xleftarrow{\begin{pmatrix}1&0&0\\0&1&0\\0&1&1\end{pmatrix}} & (1,4,2) & \xleftarrow{\begin{pmatrix}1&0&0\\0&1&2\\0&0&1\end{pmatrix}} & (1,0,2) & \xleftarrow{\begin{pmatrix}1&0&0\\0&1&0\\2&0&1\end{pmatrix}} & (1,0,0) \\
& \begin{array}{ccc}1&\mapsto&1\\2&\mapsto&2\\3&\mapsto&13\end{array} & & \begin{array}{ccc}1&\mapsto&1\\2&\mapsto&23\\3&\mapsto&3\end{array} & & \begin{array}{ccc}1&\mapsto&1\\2&\mapsto&2\\3&\mapsto&223\end{array} & & \begin{array}{ccc}1&\mapsto&133\\2&\mapsto&2\\3&\mapsto&3\end{array} & \\
\mathbf{w}_0 & \longleftarrow & \mathbf{w}_1 & \longleftarrow & \mathbf{w}_2 & \longleftarrow & \mathbf{w}_3 & \longleftarrow & \mathbf{w}_4
\end{array}
$$

Its (Hausdorff) distance to the euclidean line is 1.3680.

$$(7,4,6) \xleftarrow{\begin{pmatrix}1&0&1\\0&1&0\\0&0&1\end{pmatrix}} (1,4,6) \xleftarrow{\begin{pmatrix}1&0&0\\0&1&0\\0&1&1\end{pmatrix}} (1,4,2) \xleftarrow{\begin{pmatrix}1&0&0\\0&1&2\\0&0&1\end{pmatrix}} (1,0,2) \xleftarrow{\begin{pmatrix}1&0&0\\0&1&0\\2&0&1\end{pmatrix}} (1,0,0)$$

$$\mathbf{w}_0 \xleftarrow[\begin{array}{ccc}1&\mapsto&1\\2&\mapsto&2\\3&\mapsto&13\end{array}]{} \mathbf{w}_1 \xleftarrow[\begin{array}{ccc}1&\mapsto&1\\2&\mapsto&23\\3&\mapsto&3\end{array}]{} \mathbf{w}_2 \xleftarrow[\begin{array}{ccc}1&\mapsto&1\\2&\mapsto&2\\3&\mapsto&223\end{array}]{} \mathbf{w}_3 \xleftarrow[\begin{array}{ccc}1&\mapsto&133\\2&\mapsto&2\\3&\mapsto&3\end{array}]{} \mathbf{w}_4$$

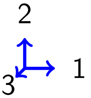
$$\mathbf{w} = \mathbf{w}_0 = 1213213132132131 3$$

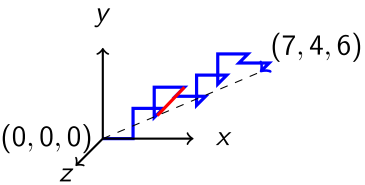


Its (Hausdorff) distance to the euclidean line is 1.3680.

$$
\begin{array}{ccccccccc}
(7,4,6) & \xleftarrow{\begin{pmatrix}1&0&1\\0&1&0\\0&0&1\end{pmatrix}} & (1,4,6) & \xleftarrow{\begin{pmatrix}1&0&0\\0&1&0\\0&1&1\end{pmatrix}} & (1,4,2) & \xleftarrow{\begin{pmatrix}1&0&0\\0&1&2\\0&0&1\end{pmatrix}} & (1,0,2) & \xleftarrow{\begin{pmatrix}1&0&0\\0&1&0\\2&0&1\end{pmatrix}} & (1,0,0) \\
\mathbf{w}_0 & \xleftarrow[\substack{1\mapsto 1\\2\mapsto 2\\3\mapsto 13}]{} & \mathbf{w}_1 & \xleftarrow[\substack{1\mapsto 1\\2\mapsto 23\\3\mapsto 3}]{} & \mathbf{w}_2 & \xleftarrow[\substack{1\mapsto 1\\2\mapsto 2\\3\mapsto 223}]{} & \mathbf{w}_3 & \xleftarrow[\substack{1\mapsto 133\\2\mapsto 2\\3\mapsto 3}]{} & \mathbf{w}_4
\end{array}
$$

$$\mathbf{w} = \mathbf{w}_0 = 12132131321321313$$





Its (Hausdorff) distance to the euclidean line is 1.3680.

3D Continued fraction algorithms

Brun's Algorithm : Subtract the second largest to the largest.

$(7,4,6) \to (1,4,6) \to (1,4,2) \to (1,2,2) \to (1,0,2) \to (1,0,1) \to (0,0,1).$

3D Continued fraction algorithms

Brun's Algorithm : Subtract the second largest to the largest.

$(7,4,6) \to (1,4,6) \to (1,4,2) \to (1,2,2) \to (1,0,2) \to (1,0,1) \to (0,0,1)$.

Selmer's Algorithm : Subtract the smallest to the largest.

$(7,4,6) \to (3,4,6) \to (3,4,3) \to (3,1,3) \to (2,1,3) \to (2,1,2) \to (1,1,2)$
$\to (1,1,1) \to (0,1,1) \to (0,0,1)$

3D Continued fraction algorithms

Brun's Algorithm : Subtract the second largest to the largest.

$(7,4,6) \to (1,4,6) \to (1,4,2) \to (1,2,2) \to (1,0,2) \to (1,0,1) \to (0,0,1).$

Selmer's Algorithm : Subtract the smallest to the largest.

$(7,4,6) \to (3,4,6) \to (3,4,3) \to (3,1,3) \to (2,1,3) \to (2,1,2) \to (1,1,2)$
$\to (1,1,1) \to (0,1,1) \to (0,0,1)$

Poincaré's Algorithm : Subtract the smallest to the mid and the mid to the largest.

$$(7,4,6) \to (1,4,2) \to (1,2,1) \to (1,1,0) \to (1,0,0)$$

3D Continued fraction algorithms

Brun's Algorithm : Subtract the second largest to the largest.

$(7,4,6) \to (1,4,6) \to (1,4,2) \to (1,2,2) \to (1,0,2) \to (1,0,1) \to (0,0,1)$.

Selmer's Algorithm : Subtract the smallest to the largest.

$(7,4,6) \to (3,4,6) \to (3,4,3) \to (3,1,3) \to (2,1,3) \to (2,1,2) \to (1,1,2)$
$\to (1,1,1) \to (0,1,1) \to (0,0,1)$

Poincaré's Algorithm : Subtract the smallest to the mid and the mid to the largest.

$$(7,4,6) \to (1,4,2) \to (1,2,1) \to (1,1,0) \to (1,0,0)$$

Arnoux-Rauzy's Algorithm : Subtract the sum of the two smallest to the largest (not always possible).

$$(7,4,6) \to \text{Impossible}$$

3D Continued fraction algorithms

Brun's Algorithm : Subtract the second largest to the largest.

$$(7,4,6) \to (1,4,6) \to (1,4,2) \to (1,2,2) \to (1,0,2) \to (1,0,1) \to (0,0,1).$$

Selmer's Algorithm : Subtract the smallest to the largest.

$$(7,4,6) \to (3,4,6) \to (3,4,3) \to (3,1,3) \to (2,1,3) \to (2,1,2) \to (1,1,2) \to (1,1,1) \to (0,1,1) \to (0,0,1)$$

Poincaré's Algorithm : Subtract the smallest to the mid and the mid to the largest.

$$(7,4,6) \to (1,4,2) \to (1,2,1) \to (1,1,0) \to (1,0,0)$$

Arnoux-Rauzy's Algorithm : Subtract the sum of the two smallest to the largest (not always possible).

$$(7,4,6) \to \text{Impossible}$$

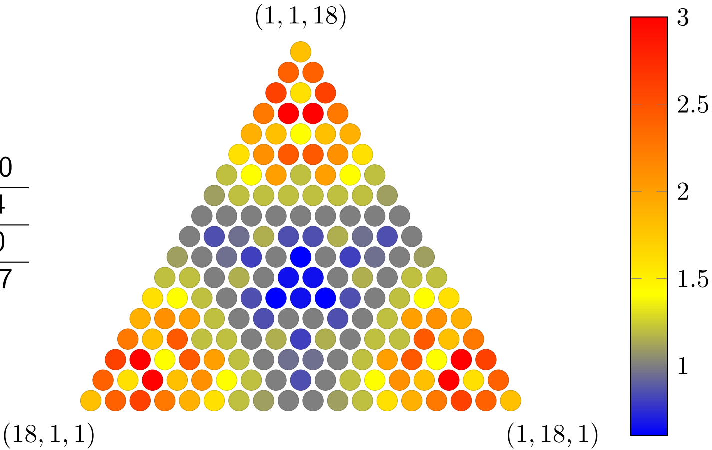
Fully subtractive's Algorithm : Subtract the smallest to the other two.

$$(7,4,6) \to (3,4,2) \to (1,2,2) \to (1,1,1) \to (1,0,0)$$

Plan

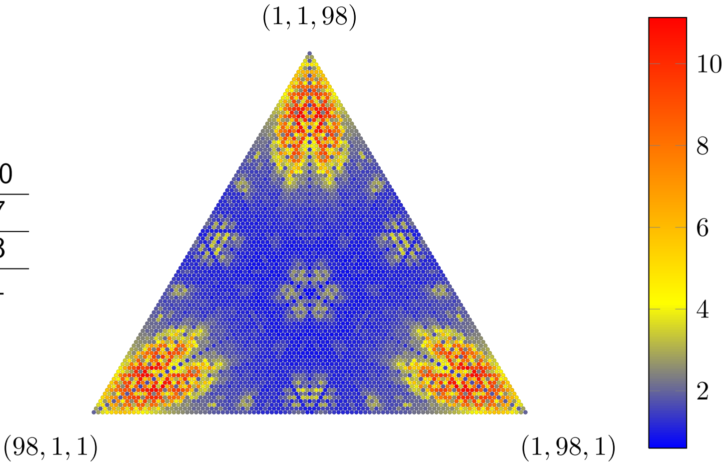
Discrépance pour les triplets d'entiers strictement positifs (a_1, a_2, a_3) tels que $a_1 + a_2 + a_3 = N$ et $N = 20$ pour l'algorithme **Poincaré**.

min	0.6000
moy.	1.484
max	3.000
E.T.	0.6137



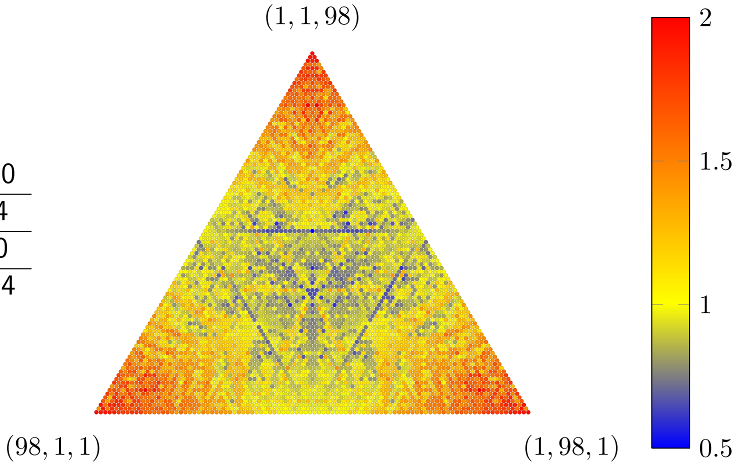
Discrépance pour les triplets d'entiers strictement positifs (a_1, a_2, a_3) tels que $a_1 + a_2 + a_3 = N$ et $N = 100$ pour l'algorithme **Poincaré**.

min	0.6000
moy.	2.527
max	11.13
E.T.	2.261



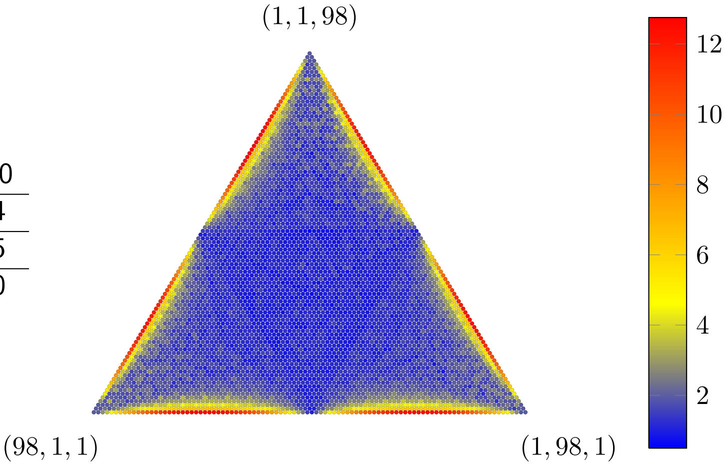
Discrépance pour les triplets d'entiers strictement positifs (a_1, a_2, a_3) tels que $a_1 + a_2 + a_3 = N$ et $N = 100$ pour l'algorithme **Brun**.

min	0.5000
moy.	1.114
max	2.000
E.T.	0.2664



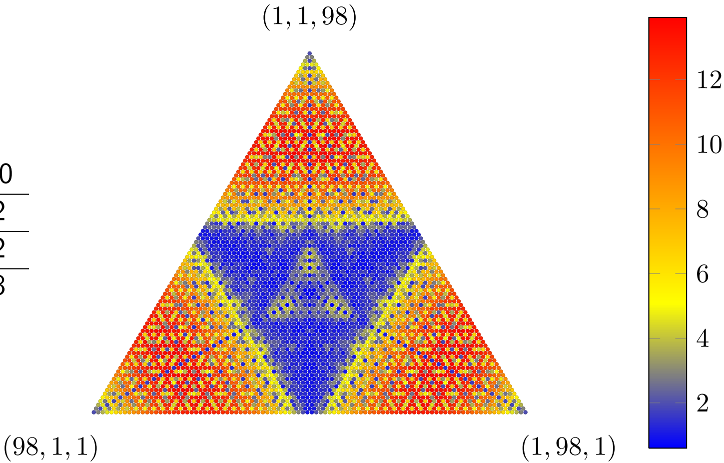
Discrépance pour les triplets d'entiers strictement positifs (a_1, a_2, a_3) tels que $a_1 + a_2 + a_3 = N$ et $N = 100$ pour l'algorithme **Selmer**.

min	0.5000
moy.	2.184
max	12.75
E.T.	2.070



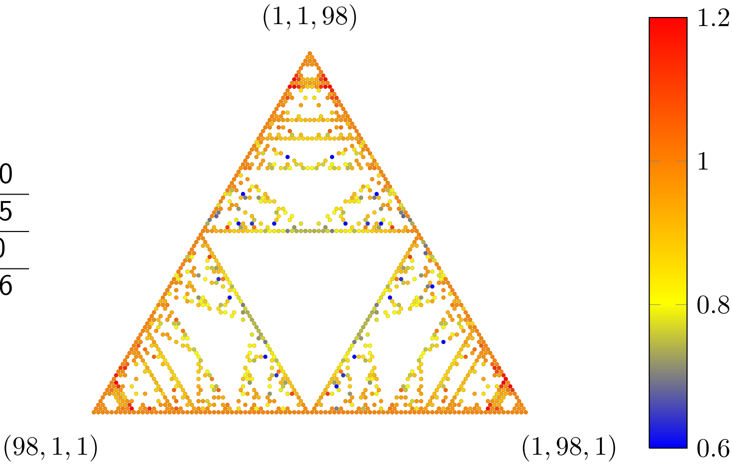
Discrépance pour les triplets d'entiers strictement positifs (a_1, a_2, a_3) tels que $a_1 + a_2 + a_3 = N$ et $N = 100$ pour l'algorithme **Fully subtractive**.

min	0.6000
moy.	5.982
max	13.92
E.T.	4.388



Discrépance pour les triplets d'entiers strictement positifs (a_1, a_2, a_3) tels que $a_1 + a_2 + a_3 = N$ et $N = 100$ pour l'algorithme **Arnoux-Rauzy**.

min	0.6000
moy.	0.9055
max	1.200
E.T.	0.1006



3D Continued fraction algorithms : fusions

Arnoux-Rauzy and Selmer Do Arnoux-Rauzy if possible, otherwise Selmer.

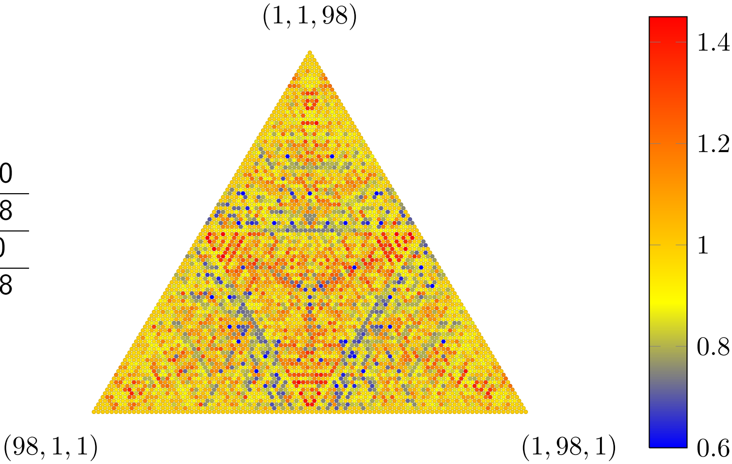
Arnoux-Rauzy and Fully Do Arnoux-Rauzy if possible, otherwise Fully subtractive.

Arnoux-Rauzy and Brun Do Arnoux-Rauzy if possible, otherwise Brun.

Arnoux-Rauzy and Poincaré Do Arnoux-Rauzy if possible, otherwise Poincaré.

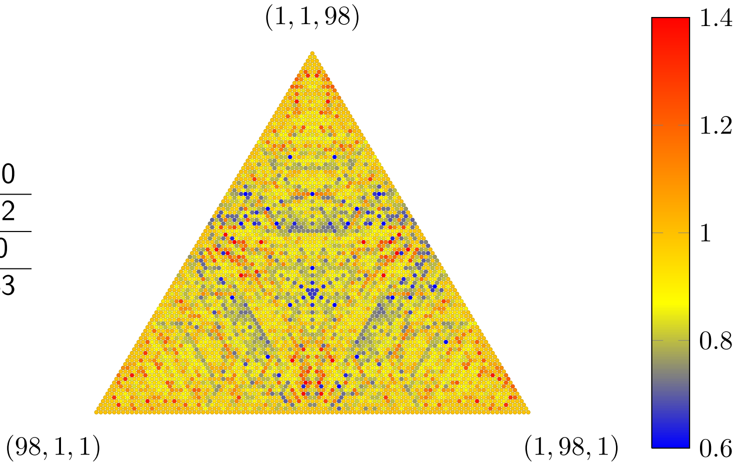
Discrépance pour les triplets d'entiers strictement positifs (a_1, a_2, a_3) tels que $a_1 + a_2 + a_3 = N$ et $N = 100$ pour l'algorithme **Arnoux-Rauzy-Selmer**.

min	0.6000
moy.	0.9678
max	1.450
E.T.	0.1438



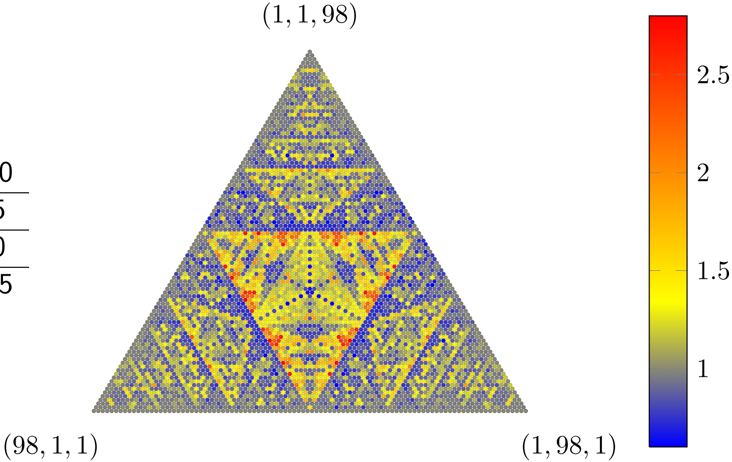
Discrépance pour les triplets d'entiers strictement positifs (a_1, a_2, a_3) tels que $a_1 + a_2 + a_3 = N$ et $N = 100$ pour l'algorithme **Arnoux-Rauzy-Brun**.

min	0.6000
moy.	0.9132
max	1.400
E.T.	0.1143



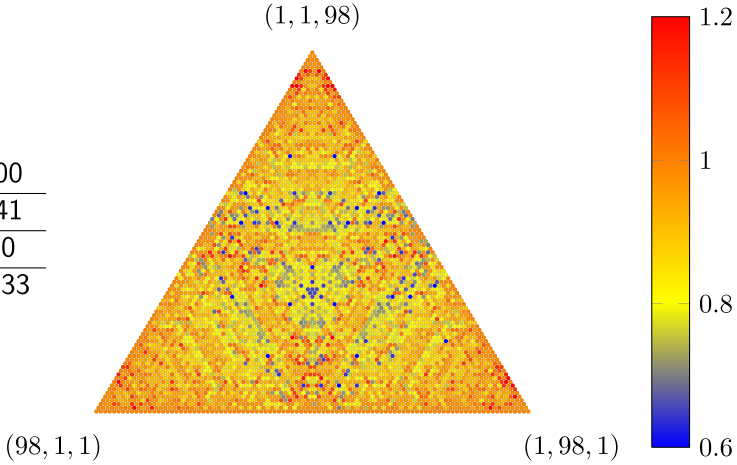
Discrépance pour les triplets d'entiers strictement positifs (a_1, a_2, a_3) tels que $a_1 + a_2 + a_3 = N$ et $N = 100$ pour l'algorithme **Arnoux-Rauzy-Fully subtractive**.

min	0.6000
moy.	1.095
max	2.800
E.T.	0.3105



Discrépance pour les triplets d'entiers strictement positifs (a_1, a_2, a_3) tels que $a_1 + a_2 + a_3 = N$ et $N = 100$ pour l'algorithme **Arnoux-Rauzy-Poincaré**.

min	0.6000
moy.	0.8941
max	1.200
E.T.	0.09733



Plan

Let $\mathcal{A} = \{1,2,3\}$, $\mathcal{S} = \{\sigma_i : i \in \mathcal{A}\}$ where

$$\sigma_i : i \mapsto i, j \mapsto ji \text{ for } j \notin \mathcal{A}.$$

An Arnoux-Rauzy word can be represented as an $\mathcal{S}$-adic sequence

$$\mathbf{w} = \lim_{n\to\infty} (\sigma_{i_0} \circ \sigma_{i_1} \circ \cdots \circ \sigma_{i_n})(1).$$

where every letter in $\mathcal{A}$ occurs infinitely often in $(i_m)_{m\geq 0}$. The sequence $(i_m)_{m\geq 0} \in \mathcal{A}^{\mathbb{N}}$ is called the $\mathcal{S}$-directive sequence.

Pierre Arnoux and Gérard Rauzy. Représentation géométrique de suites de complexité 2n+1. *Bull. Soc. Math. France*, 119(2) :199–215, 1991.

J. Cassaigne, S. Ferenczi, and L. Q. Zamboni. Imbalances in Arnoux-Rauzy sequences. *Ann. Inst. Fourier (Grenoble)*, 50(4) :1265–1276, 2000.

Tribonacci Example from Rauzy (1982)

The Tribonacci word is the fixed point of

$$1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 1$$

and is also an Arnoux-Rauzy word with $\mathcal{S}$-directive sequence $123123\cdots$.

It has complexity $2n+1$ and is 2-balanced.

Gwénaël Richomme, Kalle Saari, and Luca Q. Zamboni. Balance and abelian complexity of the Tribonacci word. *Advances in Applied Mathematics*, 45(2) :212–231, August 2010.

Balance properties of Arnoux-Rauzy words

Let $\mathbf{w}$ be an Arnoux-Rauzy word with $\mathcal{S}$-directive sequence $(i_m)_{m\geq 0}$.

Theorem (Berthé, Cassaigne, Steiner, 2012)

If the weak partial quotients are bounded by h, i.e., if we do not have $i_m = i_{m+1} = \cdots = i_{m+h}$ for any $m \geq 0$, then $\mathbf{w}$ is $(2h+1)$-balanced.

Theorem (Berthé, Cassaigne, Steiner, 2012)

Let X be the set of words $\{1121, 1122, 12121, 12122\}$ together with all the words that are obtained from one of these four words by a permutation of the letters 1, 2, and 3. If $(i_m)_{m\geq 0}$ contains no factor in X, then $\mathbf{w}$ is 2-balanced.

Valérie Berthé, Julien Cassaigne, and Wolfgang Steiner. Balance properties of arnoux-rauzy words. *arXiv :1212.5106*, December 2012.

Plan

Arnoux-Rauzy and Poincaré substitutions

For all $\{i, j, k\} = \{1, 2, 3\}$, we consider

$$\begin{array}{ll} \pi_{jk} : i \mapsto ijk, j \mapsto jk, k \mapsto k & \text{(Poincaré substitutions)} \\ \alpha_k : i \mapsto ik, j \mapsto jk, k \mapsto k & \text{(Arnoux-Rauzy substitutions)} \end{array}$$

Namely,

$$\pi_{23} = \begin{cases} 1 \mapsto 123 \\ 2 \mapsto 23 \\ 3 \mapsto 3 \end{cases}, \quad \pi_{13} = \begin{cases} 1 \mapsto 13 \\ 2 \mapsto 213 \\ 3 \mapsto 3 \end{cases}, \quad \alpha_3 = \begin{cases} 1 \mapsto 13 \\ 2 \mapsto 23 \\ 3 \mapsto 3 \end{cases},$$

$$\pi_{12} = \begin{cases} 1 \mapsto 12 \\ 2 \mapsto 2 \\ 3 \mapsto 312 \end{cases}, \quad \pi_{32} = \begin{cases} 1 \mapsto 132 \\ 2 \mapsto 2 \\ 3 \mapsto 32 \end{cases}, \quad \alpha_2 = \begin{cases} 1 \mapsto 12 \\ 2 \mapsto 2 \\ 3 \mapsto 32 \end{cases},$$

$$\pi_{31} = \begin{cases} 1 \mapsto 1 \\ 2 \mapsto 231 \\ 3 \mapsto 31 \end{cases}, \quad \pi_{21} = \begin{cases} 1 \mapsto 1 \\ 2 \mapsto 21 \\ 3 \mapsto 321 \end{cases}, \quad \alpha_1 = \begin{cases} 1 \mapsto 1 \\ 2 \mapsto 21 \\ 3 \mapsto 31 \end{cases}.$$

Complexity examples

In general, it is possible that $p(n+1) - p(n) > 3$ for some values of n. Let

$$s = \pi_{23}\pi_{23}\pi_{13}\pi_{23}\pi_{23}\alpha_1\alpha_3\alpha_2(1).$$

Indeed,

$$p(7) = 23 > 22 = 3 \cdot 7 + 1 \quad \text{and} \quad p(6) - p(5) = 19 - 15 = 4.$$

Computations in the software Sage are shown below.

```
sage: p23 = WordMorphism({3:[3],2:[2,3],1:[1,2,3]})
sage: p13 = WordMorphism({3:[3],1:[1,3],2:[2,1,3]})
sage: a1 = WordMorphism({1:[1],2:[2,1],3:[3,1]})
sage: a2 = WordMorphism({1:[1,2],2:[2],3:[3,2]})
sage: a3 = WordMorphism({1:[1,3],2:[2,3],3:[3]})
sage: s = words.s_adic([p23,p23,p13,p23,p23,a1,a3,a2],[1]); s
word: 1232333233123233332331232333333123233323...
sage: map(s.number_of_factors, range(12))
[1, 3, 5, 8, 11, 15, 19, 23, 27, 31, 35, 38]
sage: [3*n+1 for n in range(12)]
[1, 4, 7, 10, 13, 16, 19, 22, 25, 28, 31, 34]
```

Quadratic complexity example

In fact the fixed point of

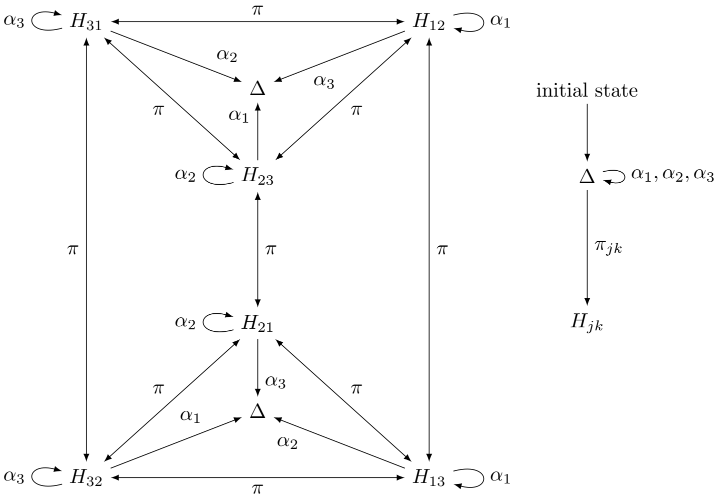
$$\pi_{13}\pi_{23} : \begin{cases} 1 \mapsto 132133 \\ 2 \mapsto 2133 \\ 3 \mapsto 3 \end{cases}$$

starting with letter 1 has a quadratic factor complexity.

```
sage: p23 = WordMorphism({3:[3],2:[2,3],1:[1,2,3]})
sage: p13 = WordMorphism({3:[3],1:[1,3],2:[2,1,3]})
sage: m = p13 * p23
sage: print m
WordMorphism: 1->132133, 2->2133, 3->3
sage: w = m.fixed_point(1); w
word: 1321332133132133332133132133313213332...
sage: p = w[:30000]
sage: map(p.number_of_factors, range(12))
[1, 3, 5, 8, 11, 15, 20, 25, 31, 38, 46, 54, 63]
```

Language of Arnoux-Rauzy Poincaré algorithm

Deterministic and minimized automaton recognizing the language $\mathcal{L} \subset \mathcal{S}^{\mathbb{N}}$ of ARP algorithm :



Result on complexity

Theorem (Berthé, L., 2013)

For (Lebesgue) almost all $(f_1, f_2, f_3) \in \Delta_3$, *there exists*

$$\mathbf{w} = \lim_{n\to\infty} \sigma_1 \circ \sigma_2 \circ \cdots \circ \sigma_n(a),$$

such that the frequency of letter i *in* $\mathbf{w}$ *is* f_i *and where* $a \in \{1, 2, 3\}$ *and* $(\sigma_i)_{i\geq 0} \in \mathcal{L} \subset \mathcal{S}^{\mathbb{N}}$.

Theorem (Berthé, L., 2013)

The factor complexity of $\mathbf{w}$ *is such that*

- $p(n+1) - p(n) \in \{2, 3\}$ *and*
- $\limsup_{n\to\infty} \frac{p(n)}{n} < 3$.

Arnoux-Rauzy Poincaré algorithm

Theorem (Boshernitzan, 1984)

A minimal symbolic system (X, S) *such that* $\limsup_{n\to\infty} \frac{p(n)}{n} < 3$ *is uniquely ergodic.*

Theorem (see CANT, Prop. 7.2.10)

A symbolic system (X_x, S) *is uniquely ergodic if, and only if,* x *has uniform frequencies.*

Sébastien Ferenczi and Thierry Monteil. Infinite words with uniform frequencies, and invariant measures. In *Combinatorics, automata and number theory*, volume 135 of *Encyclopedia Math. Appl.*, pages 373–409. Cambridge Univ. Press, Cambridge, 2010.

Corollary

The frequencies of factors and letters in $\mathbf{w}$ *exist.*

Special and bispecial words

A language is a subset of the free monoid $\mathcal{A}^*$. A language L is factorial if

$$w \in L \quad \text{and} \quad u \text{ factor of } w \implies u \in L$$

The language of factors an infinite word $\mathbf{u} \in \mathcal{A}^{\mathbb{N}}$ is

$$L(\mathbf{u}) = \{w : w \text{ is factor of } \mathbf{u}\}.$$

Right extensions and right valence :

$$E^+(w) = \{x \in \mathcal{A} | wx \in L(\mathbf{u})\} \qquad d^+(w) = \mathrm{Card} E^+(w).$$

Left extensions and left valence :

$$E^-(w) = \{x \in \mathcal{A} | xw \in L(\mathbf{u})\} \qquad d^-(w) = \mathrm{Card} E^-(w).$$

A factor w is

- right special if $d^+(w) \geq 2$,
- left special if $d^-(w) \geq 2$,
- bispecial if it is left and right special.

Special and bispecial words

The set of right special, left special and bispecial factors of length n are identified respectively by $RS_n(\mathbf{u})$, $LS_n(\mathbf{u})$ and $BS_n(\mathbf{u})$. The extension type of a factor w of $\mathbf{u}$ is the set of pairs (a, b) of $\mathcal{A} \times \mathcal{A}$ such that w can be extended in both directions as awb :

$$E(w) = \{(a, b) \in \mathcal{A} \times \mathcal{A} | awb \in L(\mathbf{u})\}.$$

The bilateral multiplicity of a factor w is

$$m(w) = \mathrm{Card}\, E(w) - d^{-}(w) - d^{+}(w) + 1.$$

(Assuming u is recurrent, $m(w) \neq 0$ implies that w is bispecial.)
A bispecial factor is said

- strong if $m(w) > 0$,
- weak if $m(w) < 0$,
- neutral if $m(w) = 0$.

Idea of the proof on complexity

Let $p(n)$ be the factor complexity function of $\mathbf{w}$. Let $s(n)$ and $b(n)$ be its sequences of finite differences of order 1 and 2 :

$$s(n) = p(n+1) - p(n),$$
$$b(n) = s(n+1) - s(n).$$

Functions s and b are related to special and bispecial factors of $\mathbf{w}$.

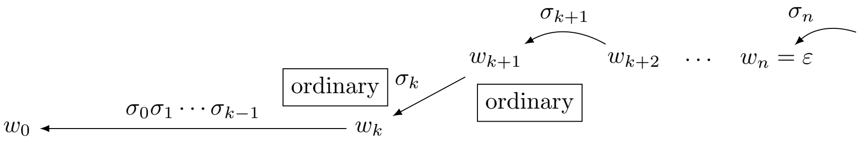
Theorem (Cassaigne, Nicolas, 2010)

Let $\mathbf{u} \in A^{\mathbb{N}}$ be a infinite [recurrent] word. Then, for all $n \in \mathbb{N}$:

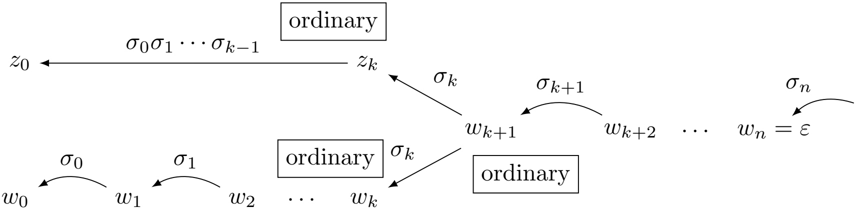
1. $s(n) = \sum_{w \in RS_n(\mathbf{u})} (d^+(w) - 1)$
2. $s(n) = \sum_{w \in LS_n(\mathbf{u})} (d^-(w) - 1)$
3. $b(n) = \sum_{w \in BS_n(\mathbf{u})} m(w)$

Typical life of ordinary bispecial factors

If $\sigma_0\sigma_1\cdots\sigma_n \in \mathcal{S}_\alpha^*\{\alpha_k\} \cup \mathcal{S}^*\{\pi_{ij},\pi_{kj},\pi_{ji},\pi_{ki}\}\mathcal{S}_\alpha^*\{\alpha_k\}$:

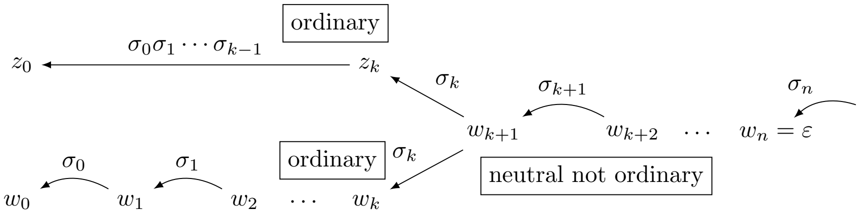


If $\sigma_0\sigma_1\cdots\sigma_n \in \mathcal{S}^*\{\pi_{ik},\pi_{jk}\}\mathcal{S}_\alpha^*\{\alpha_k\}$:

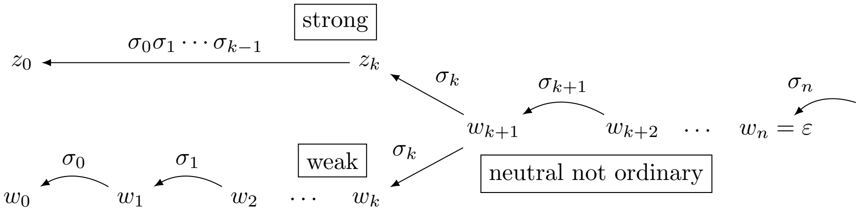


Typical life of neutral not ordinary bispecial factors

If $\sigma_0\sigma_1\cdots\sigma_n \in \mathcal{S}^*\{\pi_{ij},\pi_{kj},\pi_{ji},\pi_{ki}\}\mathcal{S}_\alpha^*\{\pi_{jk}\}$:

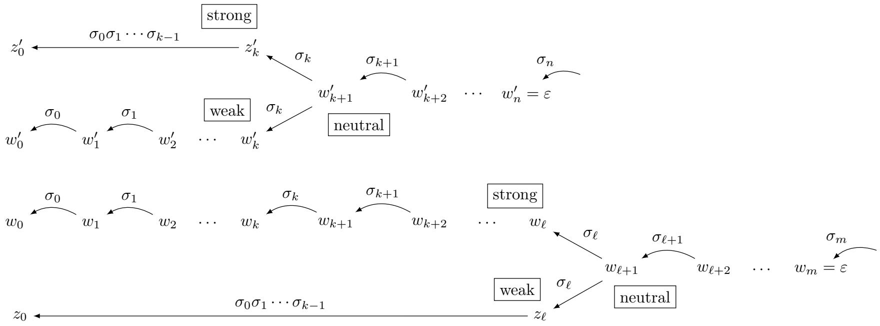


If $\sigma_0\sigma_1\cdots\sigma_n \in \mathcal{S}^*\{\pi_{ik},\pi_{jk}\}\mathcal{S}_\alpha^*\{\pi_{jk}\}$:



Idea of the proof on complexity

The proof consists in a complete characterization of the life of bispecial factors (strong, neutral and weak), not all of them being ordinary.



Plan

Evaluate the invariant measure

From a starting point

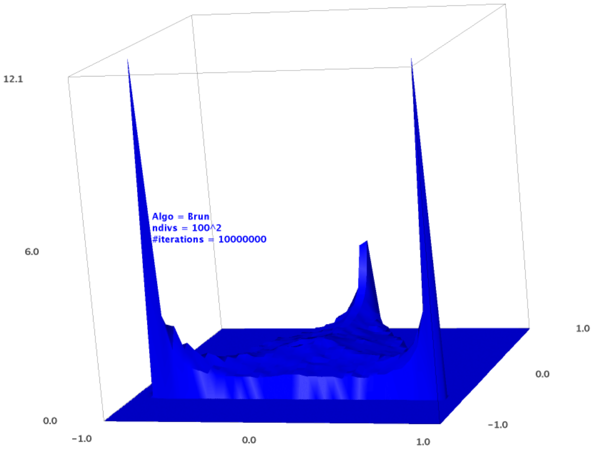
$$p_0 \in \Delta = \{(x_1, x_2, x_3) \in \mathbb{R}^3_+ : x_1 + x_2 + x_3 = 1\},$$

compute the sequence $(p_n)_{n\in\mathbb{N}}$ such that

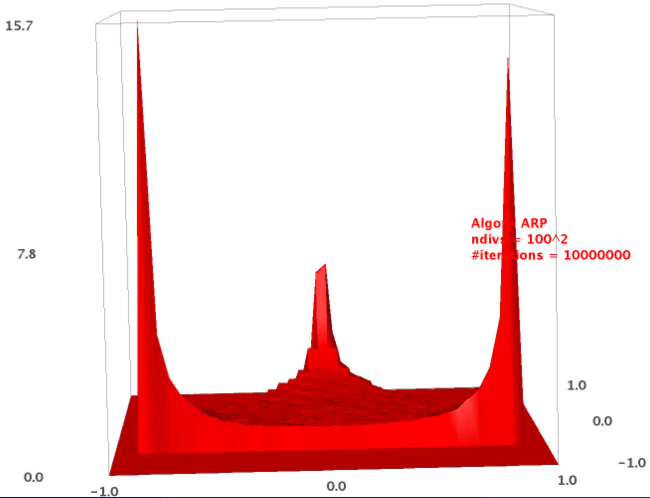
$$p_{n+1} = T(p_n)$$

where T is a Multidimensional continued fraction algorithm.

Brun, 10M iterations



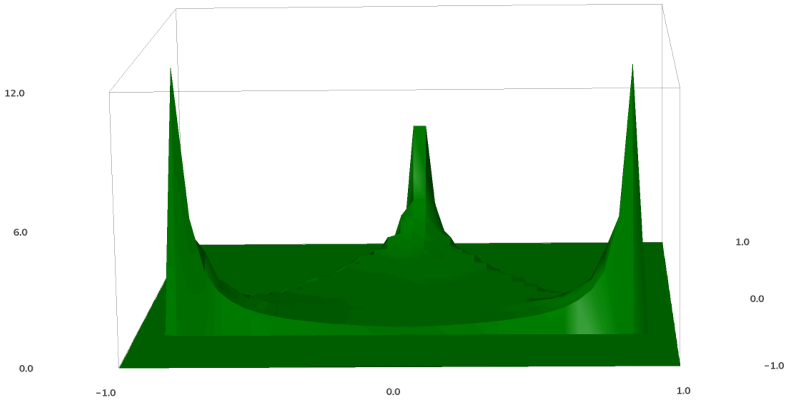
Arnoux-Rauzy-Poincaré, 10M iterations



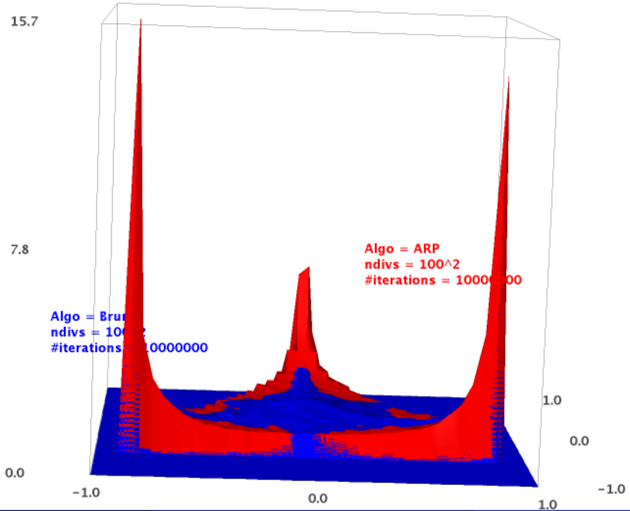
Brun theoretical

On $\{(\alpha,\beta) : 0 < \alpha < \beta < 1\}$, the absolutely continuous invariant measure with respect to Lebesgue is $\frac{12}{\pi}\frac{1}{\beta(1+\alpha)}$. Below, we show it for the simplex

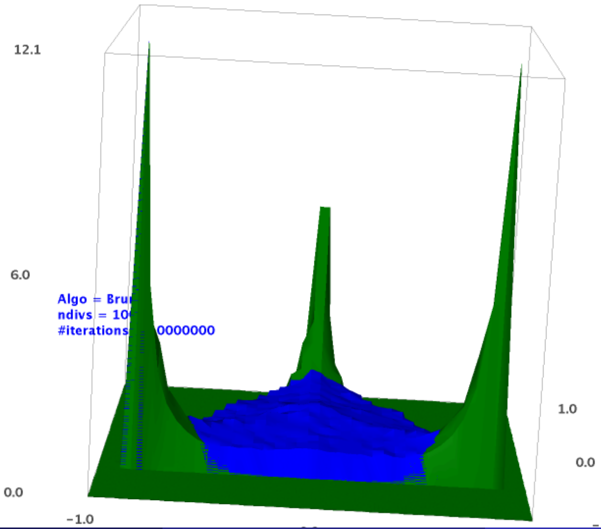
$$\Delta = \{(x_1, x_2, x_3) \in \mathbb{R}^3_+ : x_1 + x_2 + x_3 = 1\}.$$



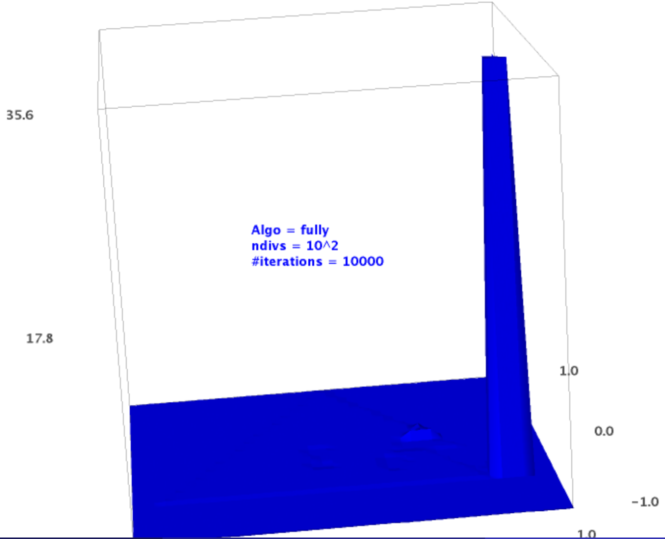
Brun vs ARP, 10M iterations



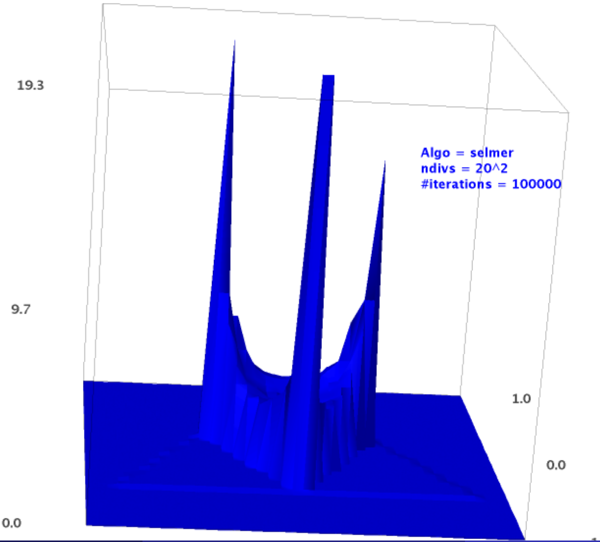
Brun 10M iterations vs Brun theretical



Fully substractive, 10K iterations



Selmer, 100K iterations



Natural extension

From a pair of starting points

$$p_0 \in \Delta, \quad q_0 \in \Delta$$

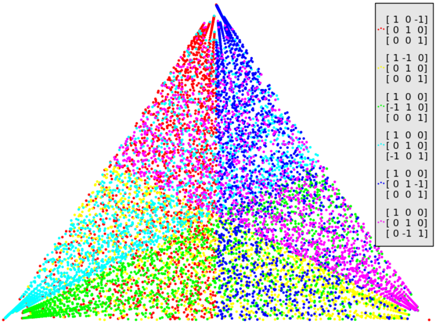
compute the sequences $(p_n)_{n\in\mathbb{N}}$ and $(q_n)_{n\in\mathbb{N}}$ such that

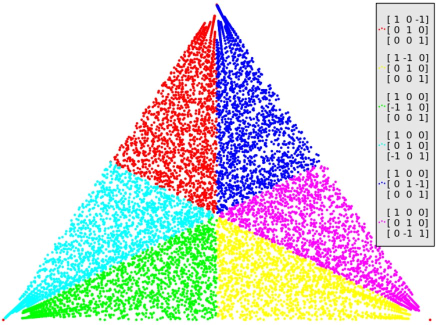
$$\begin{array}{rclcl} p_{n+1} & = & T(p_n) & = & A^{-1}p_n \\ q_{n+1} & = & & = & A^T q_n \end{array}$$

where T is a Multidimensional continued fraction algorithm.

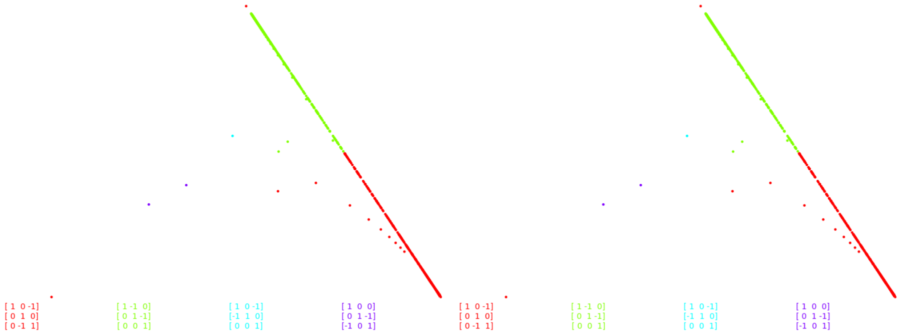
Pierre Arnoux and Arnaldo Nogueira. Mesures de gauss pour des algorithmes de fractions continues multidimensionnelles. *Annales Scientifiques de l'École Normale Supérieure. Quatrième Série*, 26(6) :645–664, 1993.

Natural extension of Brun, 10K iterations

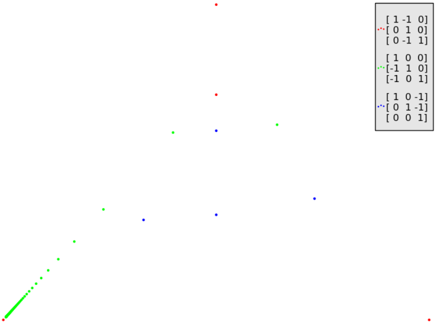


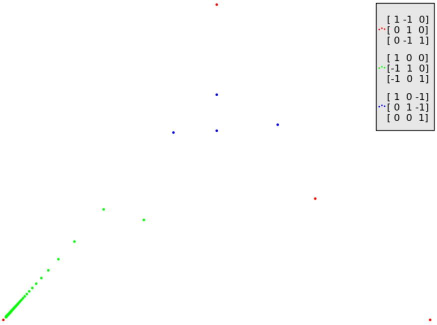


Natural extension of Poincaré, 1K iterations

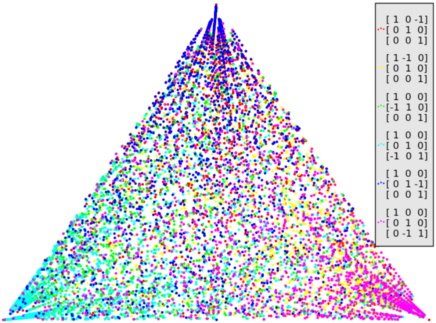


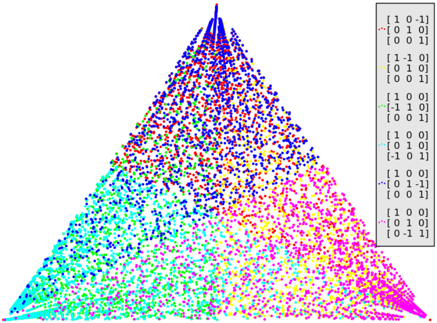
Natural extension of Fully sub., 100 iterations



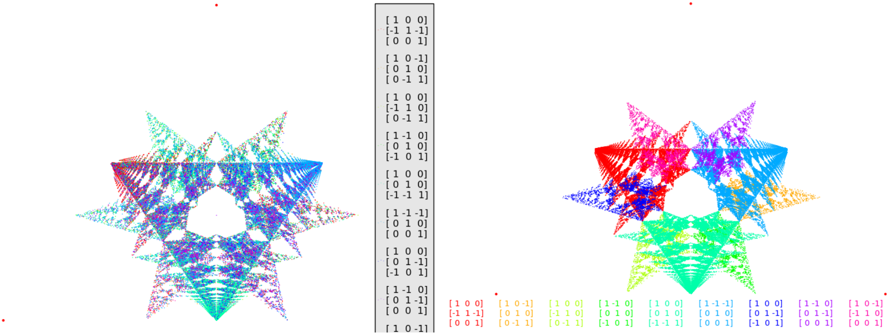


Natural extension of Selmer, 10K iterations

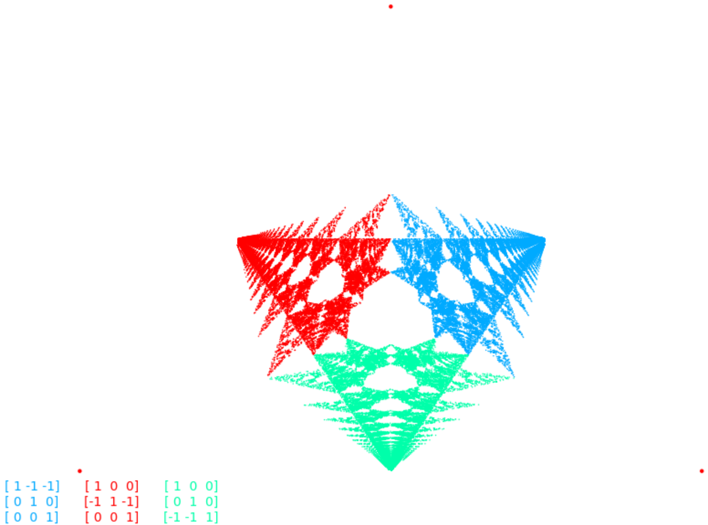




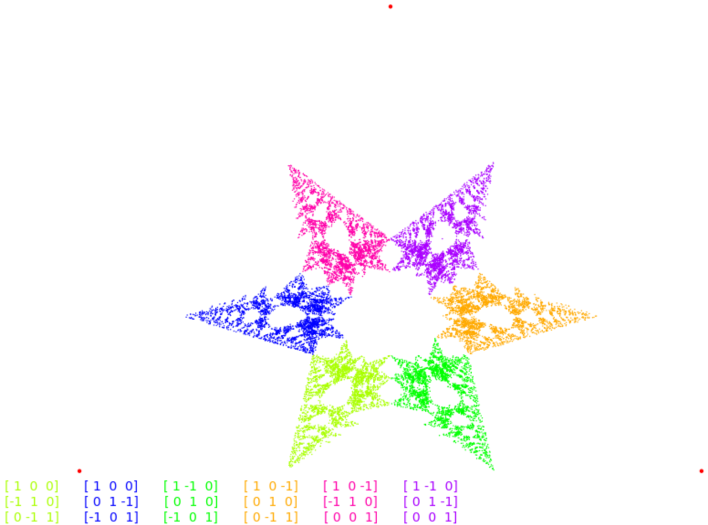
Natural extension of ARP, 100K iterations



Arnoux-Rauzy part in last picture



Poincaré part in last picture



Plan

Questions

- Compute, if it exists, the invariant measure associated to the **Arnoux-Rauzy-Poincaré** algorithm.
- Find under which conditions the **Brun** and **Arnoux-Rauzy-Poincaré** algorithms lead to bounded balance sequences.
- Compute the factor complexity of S-adic sequences obtain using **Brun** algorithm.
- Study ergodic properties of those previous fusion algorithms.