# Sur les tuiles doublement carrées

Sébastien Labbé

Laboratoire d'Informatique Algorithmique : Fondements et Applications
Université Paris Diderot Paris 7

Séminaire de Combinatoire et Théorie des Nombres 2012-2013
Institut Camille Jordan, Lyon
16 avril 2013

travail en commun avec Alexandre Blondin Massé, Ariane Garon et Srečko Brlek

# Plan

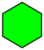
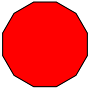

1. Pavages
2. Mots de contour
3. Nombre de pavages réguliers
4. Au plus deux pavages réguliers carrés
5. Les tuiles de Fibonacci et Christoffel sont des tuiles double carrées
6. Réduction (et reconstruction) des tuiles doubles carrées
7. Questions ouvertes

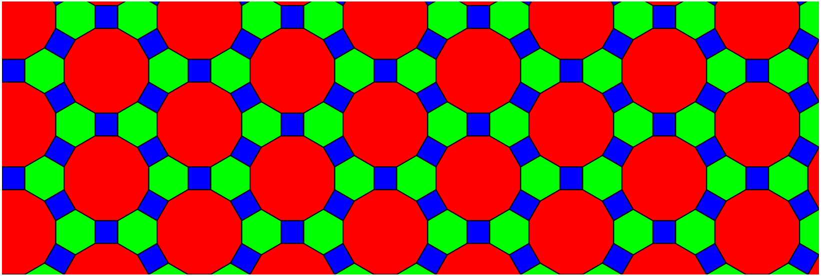
# Plan


1. Pavages
2. Mots de contour
3. Nombre de pavages réguliers
4. Au plus deux pavages réguliers carrés
5. Les tuiles de Fibonacci et Christoffel sont des tuiles double carrées
6. Réduction (et reconstruction) des tuiles doubles carrées
7. Questions ouvertes

## Pavage

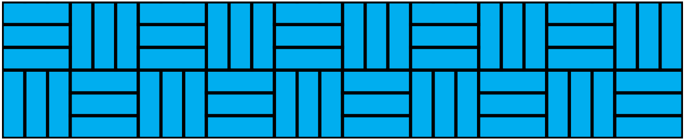
Soit un ensemble $S = \{P_1, P_2, \ldots, P_k\}$ de polygones. Un pavage du plan est une partition du plan $\mathbb{R}^2$ par des copies isométriques des polygones $P_i$. On dit que $S$ pave le plan.

Par exemple, l'ensemble $S = \{$, , $\}$ pave le plan :

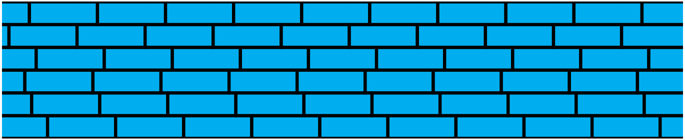


# Types de pavages

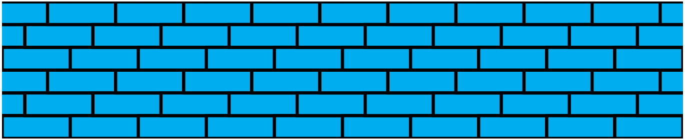
Un pavage périodique où les rotations sont permises :



Un pavage par translation :



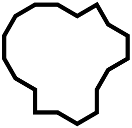
Un pavage régulier :

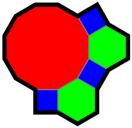


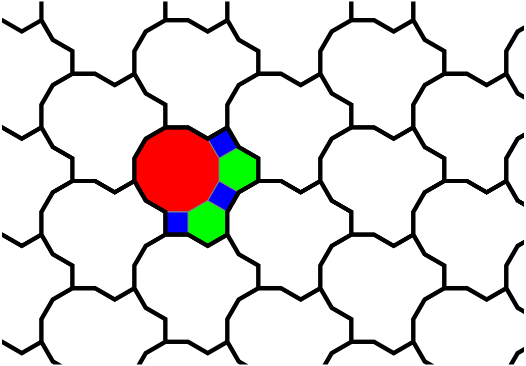
## Problème du pavage

*Étant donné un ensemble S de polygones,*
*existe-t-il un pavage du plan par S ?*

Une façon de répondre est de trouver un pavage périodique du plan.



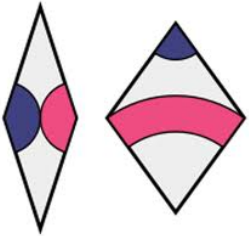


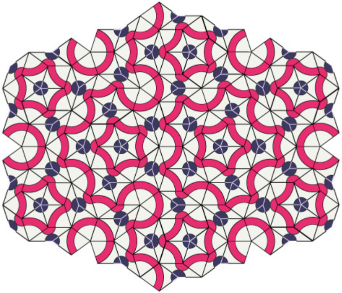


**Theorem (Berger, 1961)**

*Il existe un ensemble S qui pave le plan, mais pas de façon périodique.*

Le premier exemple construit par Berger contenait $|S| = 20426$ tuiles.
En 1974, Penrose a fourni un exemple contenant deux polygones :

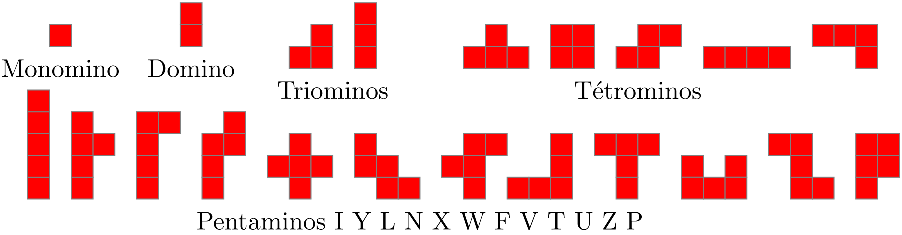




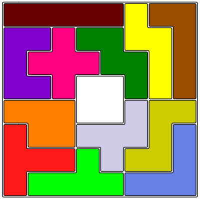
**Theorem (Berger, 1961)**

*Le Problème du pavage est indécidable.*

Le mot polyomino (Golomb, 1952) provient de domino. Alors qu'un domino est fait de deux carrés, un polyomino est fait de plusieurs.

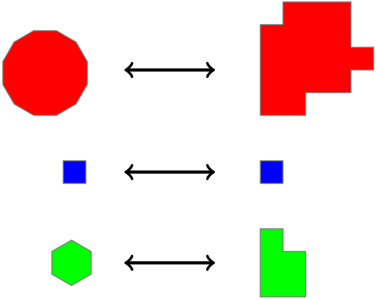



Donald Knuth (Dancing links, 2000) s'est intéressé au pavage d'un espace par des polyominos ou de façon plus générale au problème de couverture exacte. Cette méthode permet aussi de résoudre un sudoku.



# Pavage par polyominos est indécidable

En associant un ensemble de polyominos à un ensemble de polygones,



,

Golomb obtient le résultat suivant :

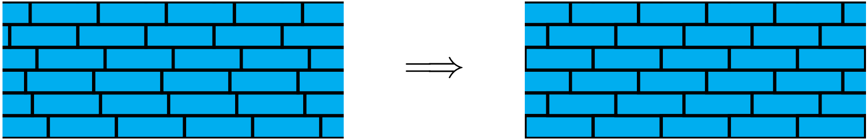
**Theorem (Golomb, 1970)**

*Le Problème du pavage par un ensemble de polyominos est aussi indécidable.*

# Pavage par translation d'un polyomino est décidable

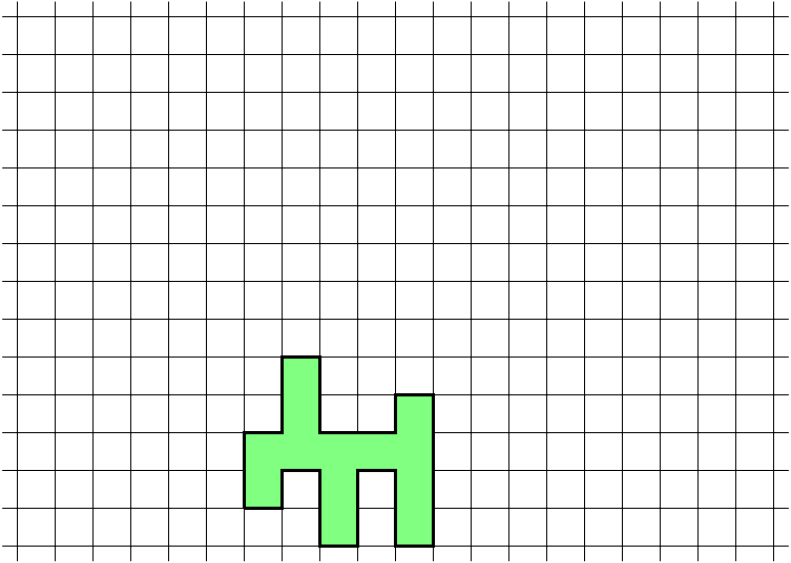
**Theorem (Wijshoff, van Leuveen, 1984)**

*Si un polyomino pave le plan par translation, alors il peut également le faire de manière régulière.*

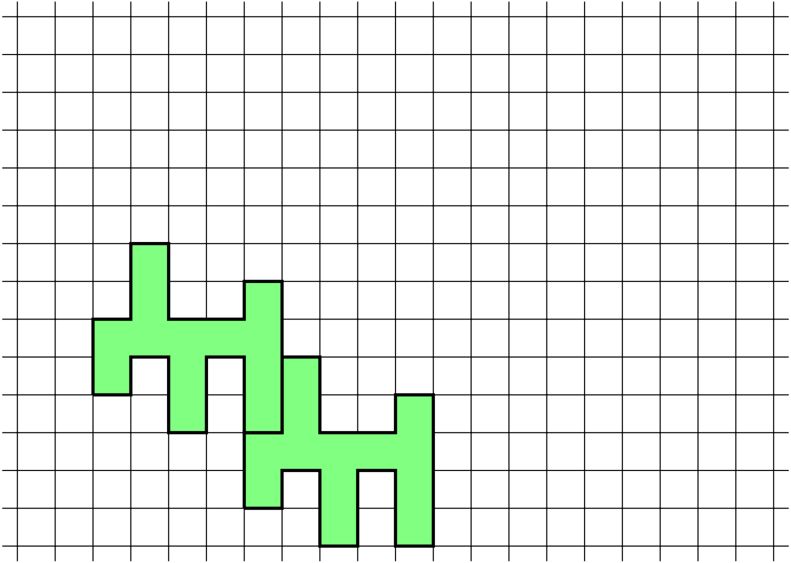


Donc, le problème du pavage où l'ensemble $S$ contient un seul polyomino est décidable.

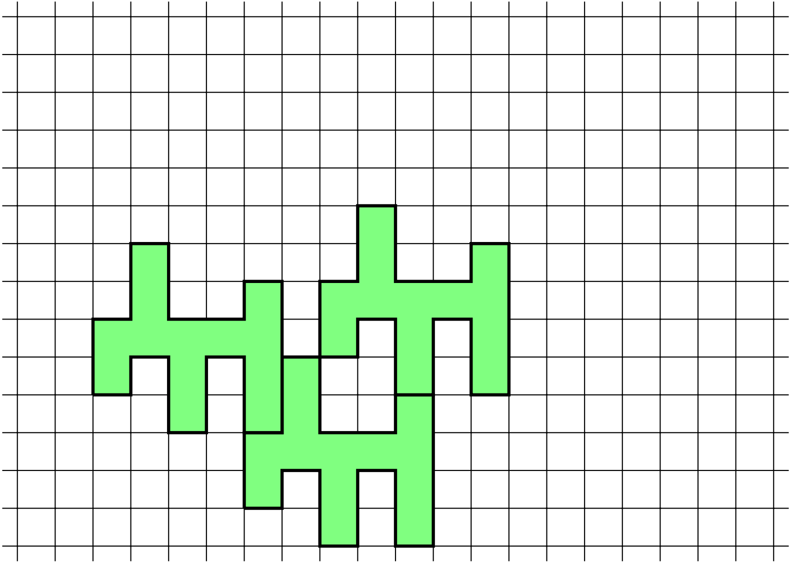
# Does a polyomino tile the plane by translation ?



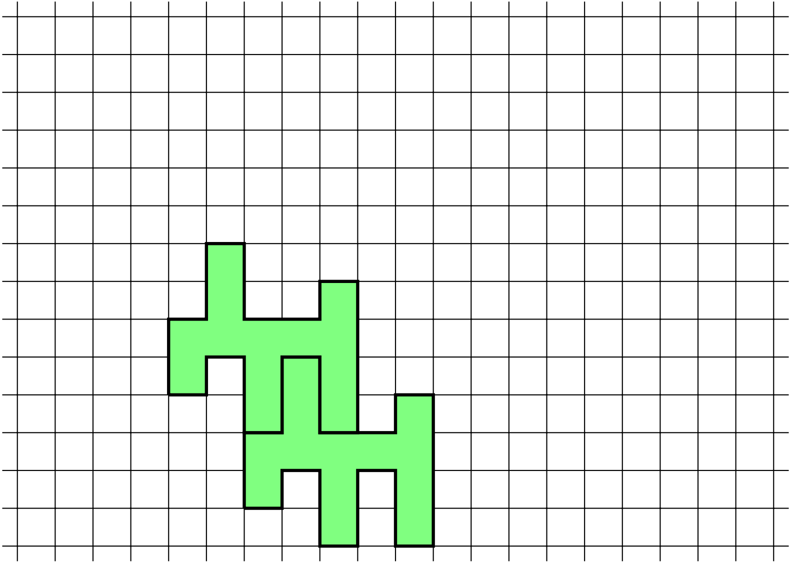
# Does a polyomino tile the plane by translation ?



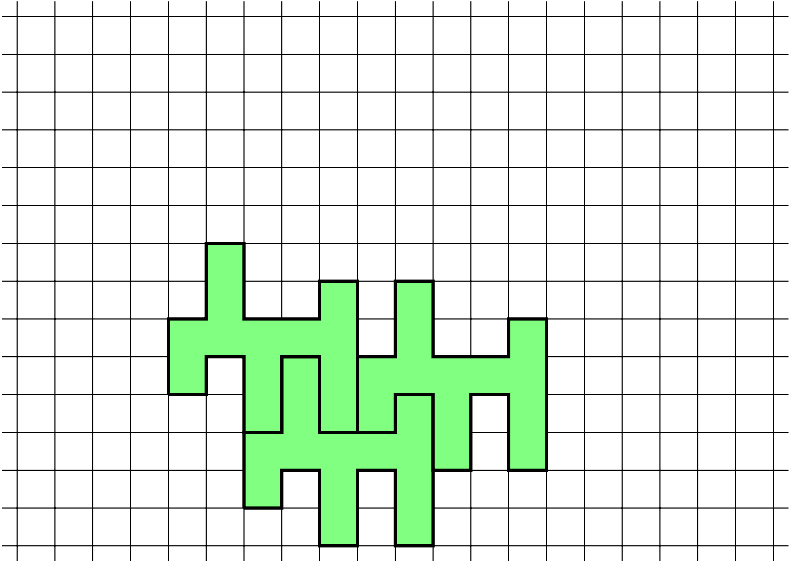
# Does a polyomino tile the plane by translation ?



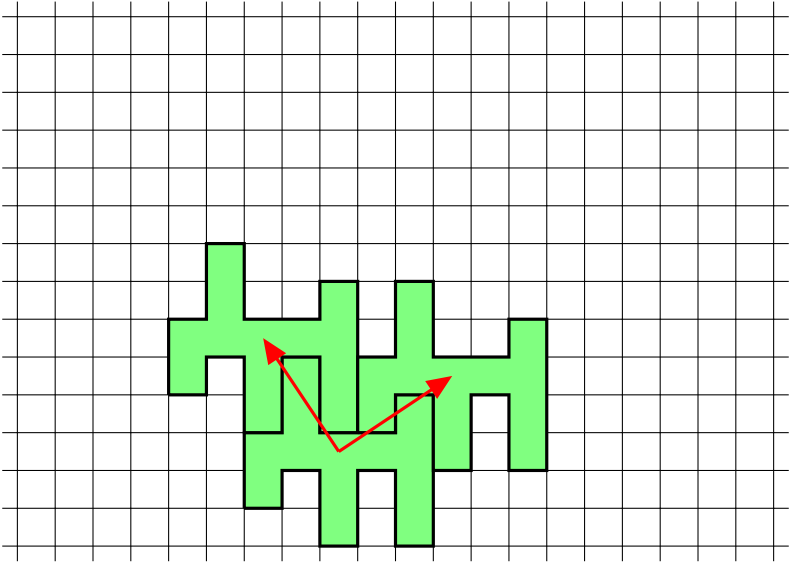
# Does a polyomino tile the plane by translation ?



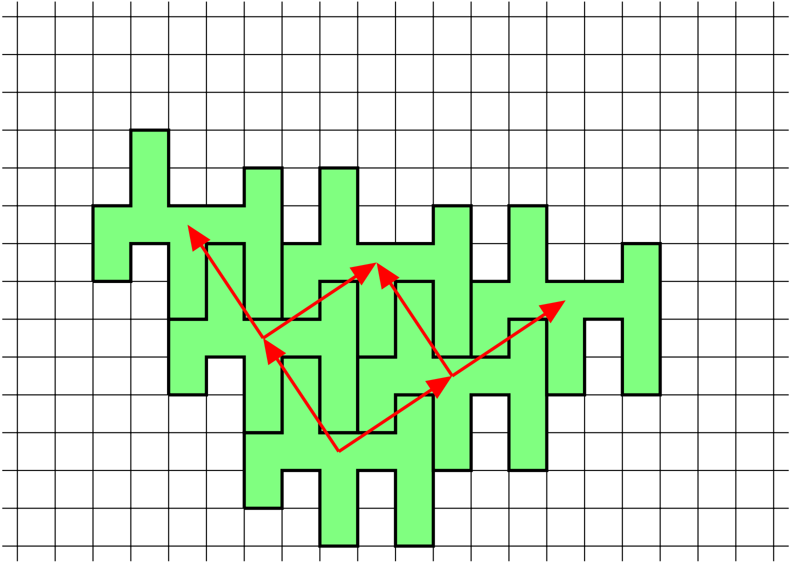
# Does a polyomino tile the plane by translation ?



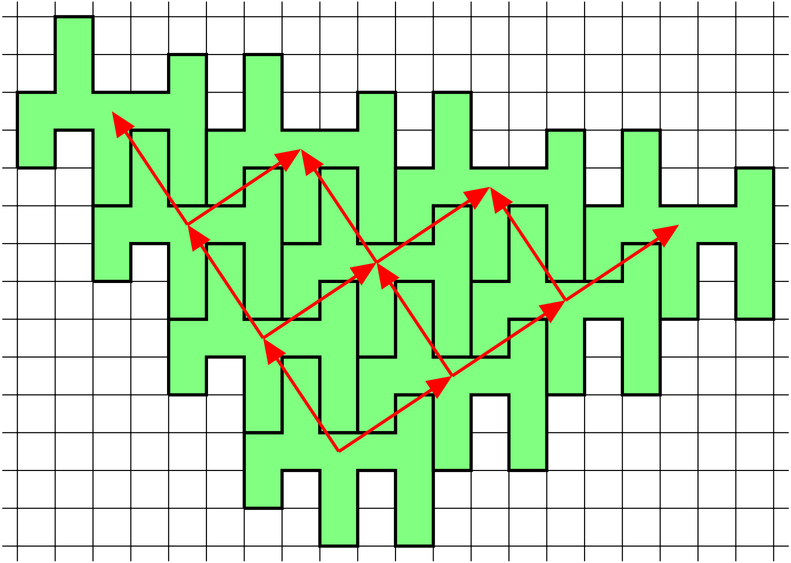
# Does a polyomino tile the plane by translation ?



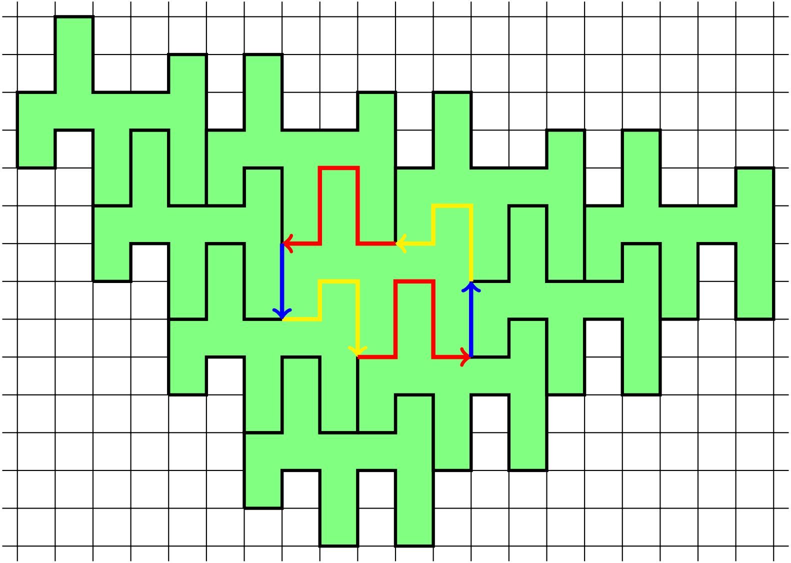
# Does a polyomino tile the plane by translation ?



# Does a polyomino tile the plane by translation ?



# Does a polyomino tile the plane by translation ?

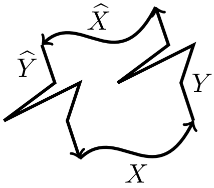


# Beauquier-Nivat

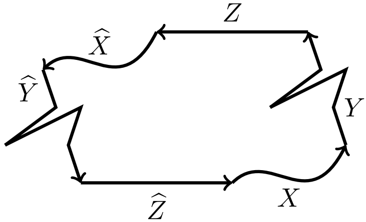
Critère de Conway, 1980 : une condition suffisante pour qu'un polygone pave le plan.

**Theorem (Beauquier, Nivat, 1991)**

*Un polyomino pave le plan par translation si et seulement si son contour se factorise en $XY\widehat{X}\widehat{Y}$ ou $XYZ\widehat{X}\widehat{Y}\widehat{Z}$.*




tuile carrée




tuile hexagonale

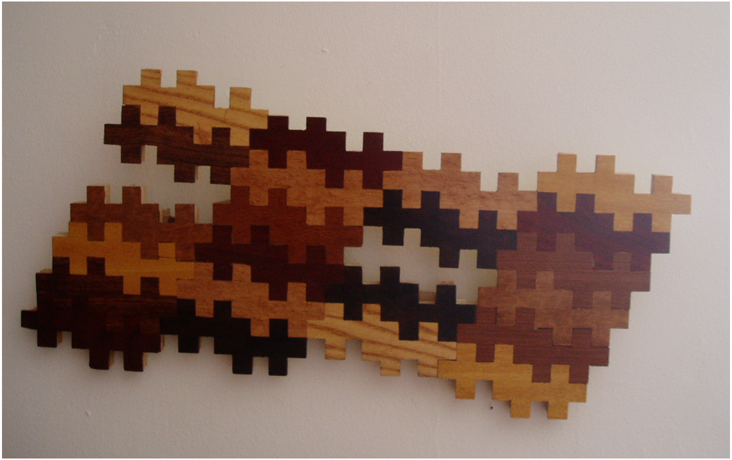




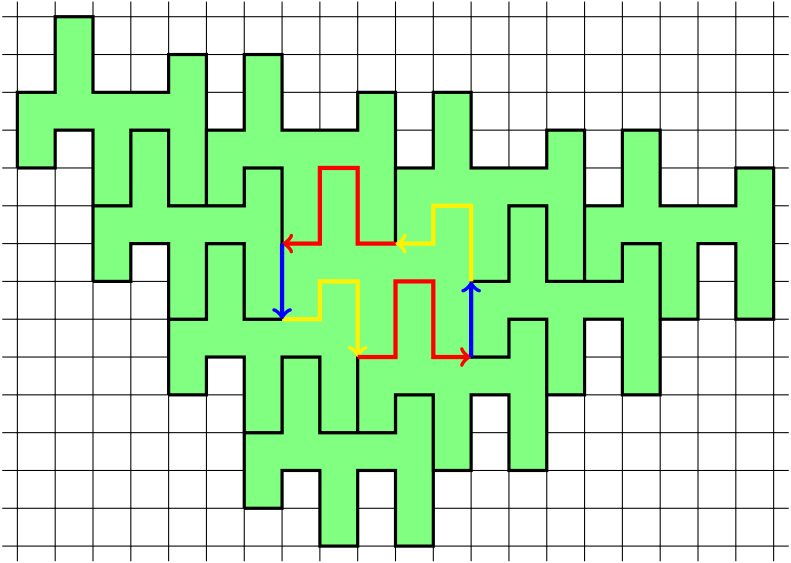
Maurits Cornelis Escher (1898-1972). Pavage hexagonal. Pavage carré.

# Tuile carrée

Oeuvre récente de l'artiste Marc Dumont.



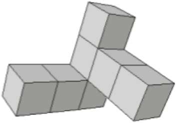
# Tuile hexagonale



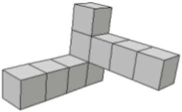
# Unbounded number of surrounding in 3D

**Theorem (Vuillon, Gambini, 2011)**

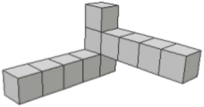
*There exists a class of polycubes such that each polycube tiles $\mathbb{Z}^3$ in a unique regular way and such that the number of faces is $4k+8$ where $2k+1$ is the volume of the polycube.*



volume 7, 20 faces



volume 9, 24 faces



volume 11, 28 faces

$\Longrightarrow$ the number of tiles surrounding the surface of a space-filler cannot be bounded in 3D.

Image credit : Gambini, I., et L. Vuillon. How many faces can polycubes of lattice tilings by translation of $R^3$ have ? *Electronic Journal of Combinatorics* 18 (2011).

# Non regular tilings of $\mathbb{Z}^3$ by single polycubes

**Theorem (Vuillon, Gambini, 2012)**

*There exist a polycube $T_2$ with the property that 2 copies of $T_2$ are assembled by translation in order to form a metatile that has a regular tiling of $\mathbb{Z}^3$ but $T_2$ itself do not have a regular tiling of $\mathbb{Z}^3$.*

(Their result is more general : polycubes $T_k$ with $k \geq 2$.)

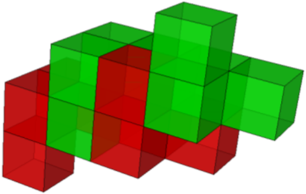


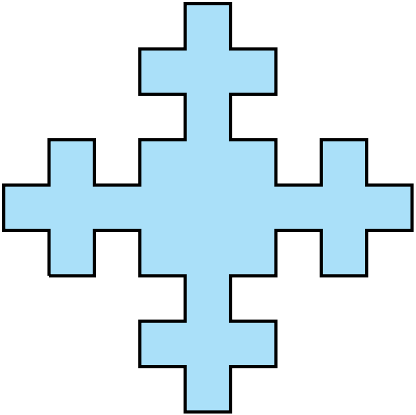
Figure 2: The polycube $T_2 \cup (T_2 + (0,1,1))$ .

Image credit : Gambini, I., L. Vuillon. Non-lattice-periodic tilings of $R^3$ by single polycubes. *Theoret. Comp. Sci.* 432 (2012) 52–57.

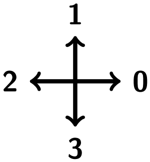
# Plan

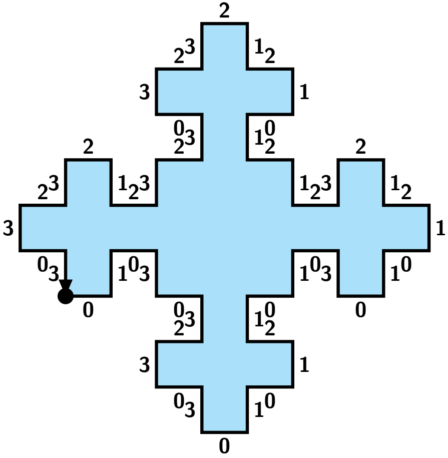

1. Pavages
2. Mots de contour
3. Nombre de pavages réguliers
4. Au plus deux pavages réguliers carrés
5. Les tuiles de Fibonacci et Christoffel sont des tuiles double carrées
6. Réduction (et reconstruction) des tuiles doubles carrées
7. Questions ouvertes

# Representation of a polyomino by its boundary

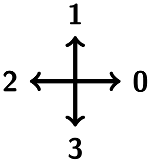


$$\Sigma = \mathbb{Z}_4 = \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}\}$$



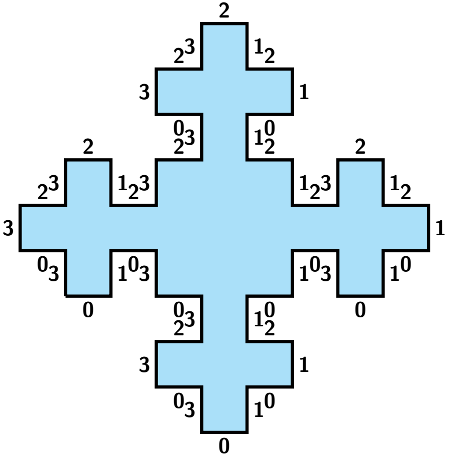



$\Sigma = \mathbb{Z}_4 = \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}\}$

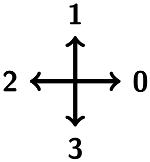



$w = \mathbf{0103032303010121010301012123212101212323032321232303}$

# Representation of a polyomino by its boundary

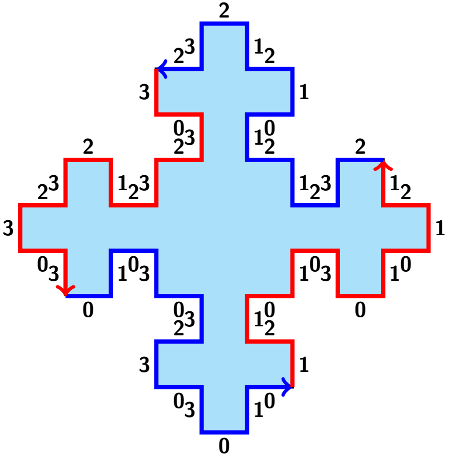



$$\Sigma = \mathbb{Z}_4 = \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}\}$$

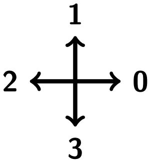



$[w] \equiv \mathbf{0103032303010121010301012123212101212323032321232303}$

# Representation of a polyomino by its boundary

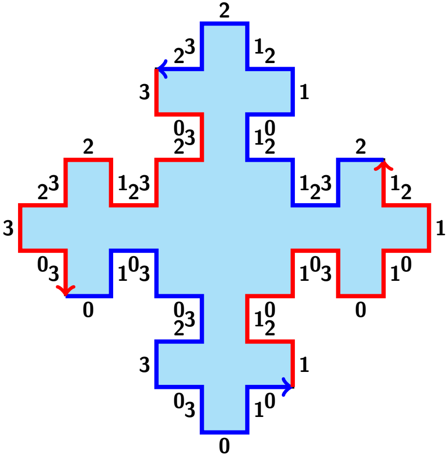


$$\Sigma = \mathbb{Z}_4 = \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}\}$$

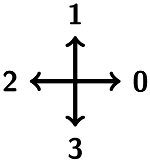



$[w] \equiv \mathbf{0103032303010121010301012123212101212323032321232303}$

# Representation of a polyomino by its boundary




$$\Sigma = \mathbb{Z}_4 = \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}\}$$




$[w] \equiv$

| | | | |
|---|---|---|---|
| **0103032303010** | **1210103010121** | **2321210121232** | **3032321232303** |
| $X$ | $Y$ | $\widehat{X}$ | $\widehat{Y}$ |

# Results on polyominoes using boundary word

Many statistics on polyominoes can be computed efficiently from the boundary word including :

- area,
- moment of inertia (thus center of gravity),
- size of projection,
- intersection,
- digital convexity,
- whether it tiles the plane by translations.

See publications of S. Brlek, A. Lacasse and X. Provençal and their coauthors.

**Theorem (Brlek, Koskas, Provençal, 2011)**

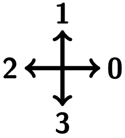
*There exists a linear time and space algorithm for detecting path intersection in* $\mathbb{Z}^d$.

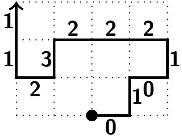
# Why $\{0, 1, 2, 3\}$ is the best alphabet for paths ?

The first differences sequence of $w \in (\mathbb{Z}_4)^*$

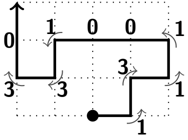
$$\Delta w = (w_2 - w_1) \cdot (w_3 - w_2) \cdots (w_n - w_{n-1}).$$

represents the sequence of turns of the path.




$w = \mathbf{01012223211}$



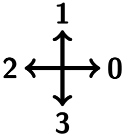

$\Delta w = \mathbf{1311001330}$

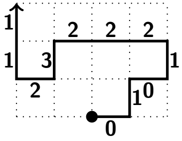
# Why $\{0,1,2,3\}$ is the best alphabet for paths ?

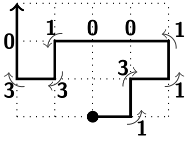
The first differences sequence of $w \in (\mathbb{Z}_4)^*$

$$\Delta w = (w_2 - w_1) \cdot (w_3 - w_2) \cdots (w_n - w_{n-1}).$$

represents the sequence of turns of the path.




$w = \mathbf{01012223211}$ $\Delta w = \mathbf{1311001330}$

We also consider $\Delta[w]$ well defined on the conjugacy classes :

$$\Delta[w] = (w_2 - w_1) \cdot (w_3 - w_2) \cdots (w_n - w_{n-1}) \cdot (w_1 - w_n) = \Delta w \cdot (w_1 - w_n).$$

# Turning number

The turning number of a path $w$ is $\mathcal{T}(w) = \frac{|\Delta w|_{\mathbf{1}} - |\Delta w|_{\mathbf{3}}}{4}$ and corresponds to its total curvature divided by $2\pi$ (Wikipedia). We have that

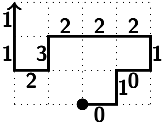
- $\mathcal{T}(w) = -\mathcal{T}(\widehat{w})$ for all path $w \in \Sigma^*$
- $\mathcal{T}([w]) = \pm 1$ for all simple and closed path $w$.

# Turning number

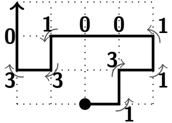
The turning number of a path $w$ is $\mathcal{T}(w) = \frac{|\Delta w|_1 - |\Delta w|_3}{4}$ and corresponds to its total curvature divided by $2\pi$ (Wikipedia). We have that

- $\mathcal{T}(w) = -\mathcal{T}(\widehat{w})$ for all path $w \in \Sigma^*$
- $\mathcal{T}([w]) = \pm 1$ for all simple and closed path $w$.

For example,

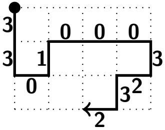



$w = \mathbf{01012223211}$

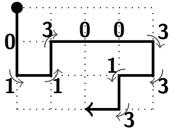



$\Delta w = \mathbf{1311001330}$

$\mathcal{T}(w) = 1/4$




$\widehat{w} = \mathbf{33010003232}$




$\Delta\widehat{w} = \mathbf{0113003313}$

$\mathcal{T}(\widehat{w}) = -1/4$

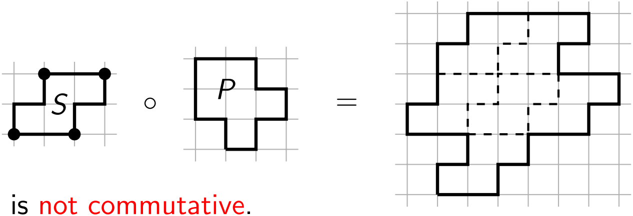
# Composition of tiles

The factorization $AB\widehat{A}\widehat{B}$ of a square $S$ allows to define the substitution

$$\varphi_S : \mathbf{0} \mapsto A, \mathbf{1} \mapsto B, \mathbf{2} \mapsto \widehat{A}, \mathbf{3} \mapsto \widehat{B}.$$

For any polyomino $P$ having boundary $w$ we define the composition

$$S \circ P := \varphi_S(w).$$




Note : This is not commutative.

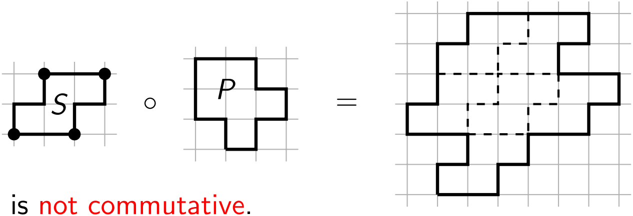
# Composition of tiles

The factorization $AB\widehat{A}\widehat{B}$ of a square $S$ allows to define the substitution

$$\varphi_S : \mathbf{0} \mapsto A, \mathbf{1} \mapsto B, \mathbf{2} \mapsto \widehat{A}, \mathbf{3} \mapsto \widehat{B}.$$

For any polyomino $P$ having boundary $w$ we define the composition

$$S \circ P := \varphi_S(w).$$



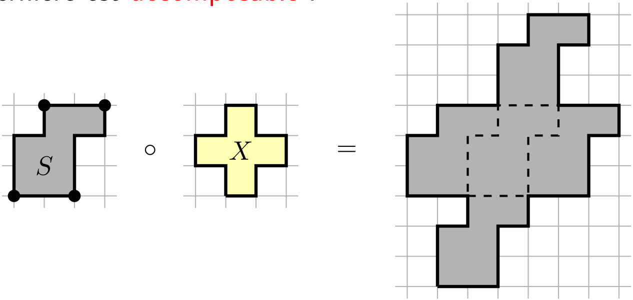

Note : This is not commutative.

**Definition**

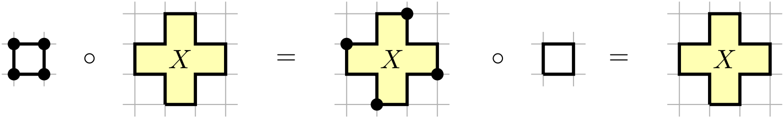
A polyomino $Q$ is prime if $Q = S \circ P$ implies that $S$ or $P$ is the unit square.

# Tuile décomposable

Or, cette dernière est décomposable :




Le pentamino $X$ est indécomposable :



# Why prime polyominoes are interesting ?

Prime polyominoes are a subset of polyominoes that allows to reconstruct every polyominoes with the composition rule.

Some questions are open :

- Detect whether a polyomino is prime.
- Find an algorithm that enumerate prime polyominoes.
- Count prime polyominoes.
- Is the growth rate the same or less than the growth rate of polyominoes ?

# Plan


1. Pavages
2. Mots de contour
3. Nombre de pavages réguliers
4. Au plus deux pavages réguliers carrés
5. Les tuiles de Fibonacci et Christoffel sont des tuiles double carrées
6. Réduction (et reconstruction) des tuiles doubles carrées
7. Questions ouvertes

# Does a polyomino tiles the plane by translation ?

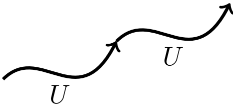
**Theorem (Brlek, Fédou, Provençal, 2009)**

*Déterminer si un polyomino $P$ est une tuile carrée est décidable en temps linéaire (en la taille du périmètre).*

**Theorem (Brlek, Fédou, Provençal, 2009)**

*Soit $P$ un polyomino dont la taille du plus grand motif répété $UU$ est bornée par la racine carrée du périmètre. Déterminer si $P$ est une tuile hexagonale est décidable en temps linéaire (en la taille du périmètre).*

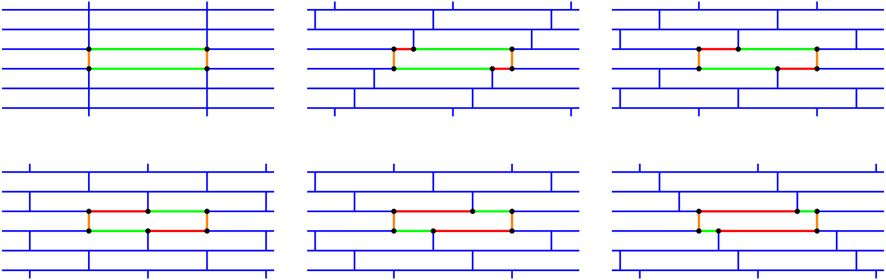
Un motif répété $UU$ est la concaténation de deux chemins identiques :



# Nombre de pavages réguliers d'une tuile hexagone

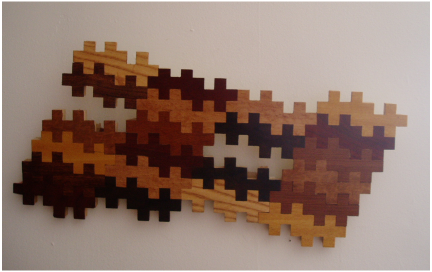
A polyomino may have many regular tilings of the plane.

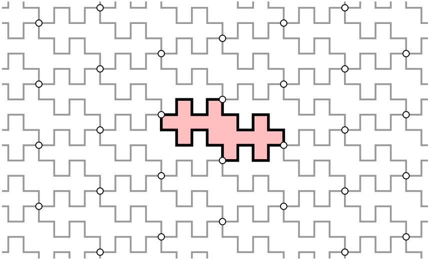
Exemple : Un rectangle $1 \times 6$ pave le plan $\mathbb{Z}^2$ comme un hexagone en 5 façons et comme un carré en une seule façon.



# Nombre de pavages réguliers d'une tuile carrée

La tuile de Dumont possède un deuxième pavage carré.

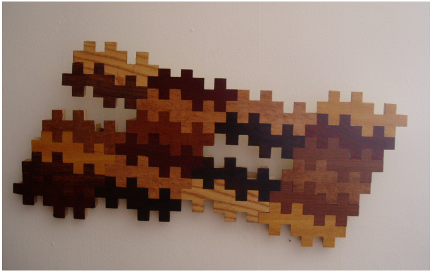


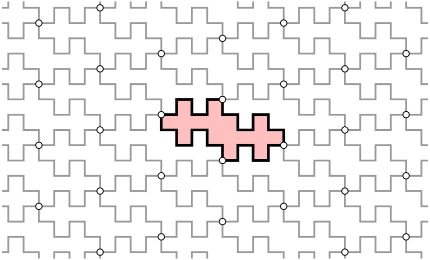


Pourrait-il en contenir d'autres ?

# Nombre de pavages réguliers d'une tuile carrée

La tuile de Dumont possède un deuxième pavage carré.



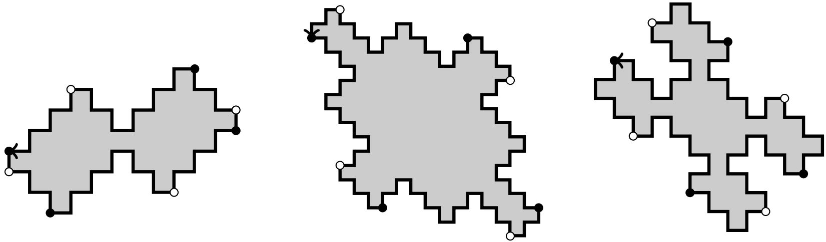


Pourrait-il en contenir d'autres ?

Brlek, Dulucq, Fédou, Provençal conjectured in 2007 that a tile has at most 2 square factorizations.

# Some double square tiles

Une tuile possédant deux factorisations carrées est appelée double carrée.



# Double Square in Sage free software

Creation of a double square tile in Sage from the boundary word of a known double square :

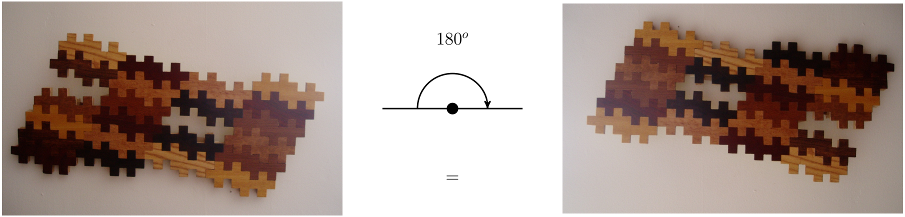
```
sage: from sage.combinat.double_square_tile import DoubleSquare
sage: DoubleSquare(words.christoffel_tile(4,7))
Double Square Tile
  w0 = 03                                 w4 = 21
  w1 = 0103010103010301010301030          w5 = 2321232321232123232123212
  w2 = 10103010                           w6 = 32321232
  w3 = 1                                  w7 = 3
(|w0|, |w1|, |w2|, |w3|) = (2, 25, 8, 1)
(d0, d1, d2, d3)         = (26, 10, 26, 10)
(n0, n1, n2, n3)         = (0, 2, 0, 0)
```

DoubleSquare will be available in Sage soon :

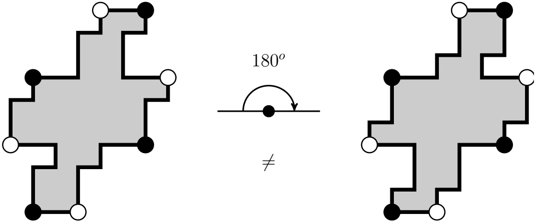
http://trac.sagemath.org/sage_trac/ticket/13069

# Invariance sous une rotation de 180 degrés

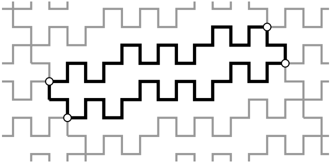
On remarque que la tuile double carrée de Dumont est invariante sous une rotation de 180 degrés :




mais pas la tuile double carrée suivante :



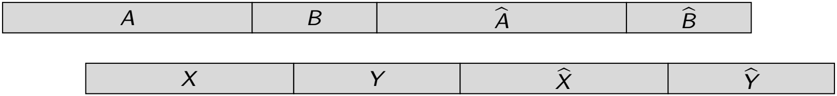
# Invariance sous une rotation de 180 degrés



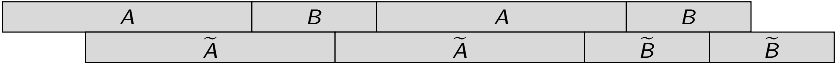
X. Provençal et L. Vuillon ont conjecturé en 2008 que si un polyomino est une tuile double carrée indécomposable, alors elle est invariante sous une rotation de 180 degrés.

# Motivation to study double squares

Let $AB\widehat{A}\widehat{B} \equiv XY\widehat{X}\widehat{Y}$ be the factorizations of a double square tile. We need to understand equations on words of the following form :

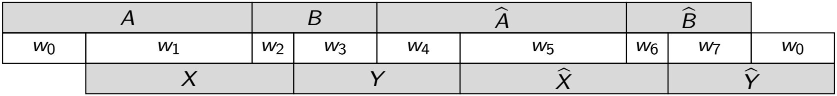



To study Hof, Knill, Simon Conjecture (1995), one need to study equations on words of the form :

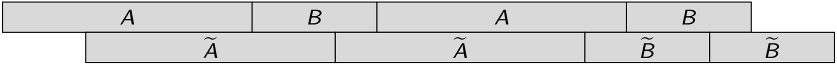


# Motivation to study double squares

Let $AB\widehat{A}\widehat{B} \equiv XY\widehat{X}\widehat{Y}$ be the factorizations of a double square tile. We need to understand equations on words of the following form :

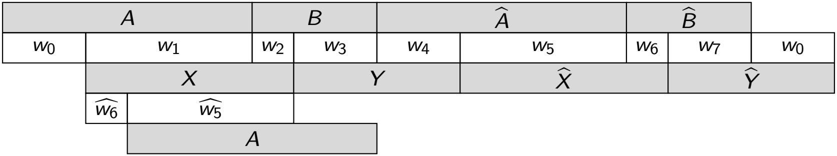



To study Hof, Knill, Simon Conjecture (1995), one need to study equations on words of the form :

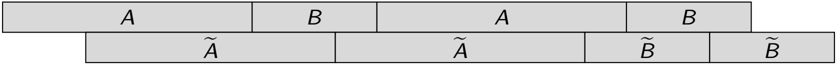


# Motivation to study double squares

Let $AB\widehat{A}\widehat{B} \equiv XY\widehat{X}\widehat{Y}$ be the factorizations of a double square tile. We need to understand equations on words of the following form :




To study Hof, Knill, Simon Conjecture (1995), one need to study equations on words of the form :



# Hof, Knill, Simon Conjecture

We say that a morphism $\varphi$ is in class $P$ if there exists a palindrome $p$ and for every $\alpha \in \Sigma$ there exists a palindrome $q_\alpha$ such that $\varphi(\alpha) = pq_\alpha$.

**Conjecture (Hof, Knill, Simon, 1995, rephrased by us in 2008)**

*Let $\varphi$ be a primitive morphism such that $\mathbf{u} = \varphi(\mathbf{u})$ is a fixed point. Then, the palindromic complexity of $\mathbf{u}$ is infinite if and only if there exists a morphism $\varphi'$ in class $P$ such that $\varphi'(\mathbf{u}) = \mathbf{u}$.*

- Proved in my master thesis (2008) for the binary alphabet and uniform morphisms.
- Proved by B. Tan in 2008 for the binary alphabet.
- AFAIK, still open for larger alphabet.

# Resultats

**Theorem (Blondin Massé, Brlek, L., 2012)**

*Un polyomino possède au plus deux factorisations carrées.*

**Theorem (Blondin Massé, Brlek, Garon, L., 2011)**

*Les tuiles de Christoffel et de Fibonacci sont des tuiles doublement carrées.*

**Theorem (Blondin Massé, Garon, L., 2012)**

*Toute tuile doublement carrée peut être construite à partir de deux règles combinatoires simples et inversibles :* SWAP *et* TRIM.

**Theorem (Blondin Massé, Garon, L., 2012)**

*Soit $P$ un polyomino. Si $P$ est une tuile double carrée indécomposable, alors elle est invariante sous une rotation de 180 degrés.*

# Plan


1. Pavages
2. Mots de contour
3. Nombre de pavages réguliers
4. Au plus deux pavages réguliers carrés
5. Les tuiles de Fibonacci et Christoffel sont des tuiles double carrées
6. Réduction (et reconstruction) des tuiles doubles carrées
7. Questions ouvertes

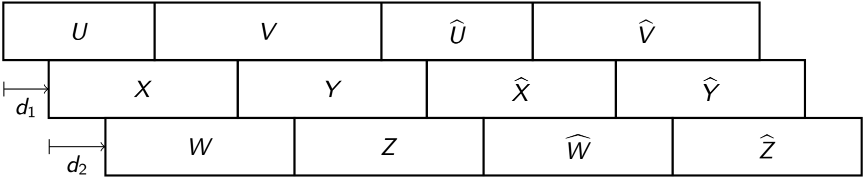
# Idée de la preuve : au plus 2 factorisations carrées

**Lemma (Brlek, Fédou, Provençal, 2008)**

*The factorizations $UV\widehat{U}\widehat{V} \equiv_{d_1} XY\widehat{X}\widehat{Y}$ of a double square tile must alternate, that is $0 < d_1 < |U| < d_1 + |X|$.*

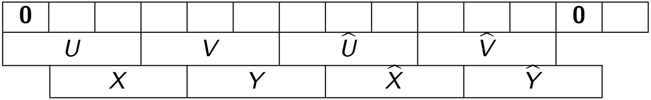
Suppose that there is a triple square tile having the following boundary :

$$UV\widehat{U}\widehat{V} \equiv_{d_1} XY\widehat{X}\widehat{Y} \equiv_{d_2} WZ\widehat{W}\widehat{Z}.$$



# Examples

Suppose that $|U| = |V| = |X| = |Y| = |W| = |Z| = 3$.



# Examples

Suppose that $|U| = |V| = |X| = |Y| = |W| = |Z| = 3$.

<table>
<tr><td>0</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td>2</td><td></td><td></td><td></td><td>0</td><td></td></tr>
<tr><td colspan="3">$U$</td><td colspan="3">$V$</td><td colspan="3">$\widehat{U}$</td><td colspan="3">$\widehat{V}$</td><td colspan="2"></td></tr>
<tr><td></td><td colspan="3">$X$</td><td colspan="3">$Y$</td><td colspan="3">$\widehat{X}$</td><td colspan="3">$\widehat{Y}$</td><td></td></tr>
</table>

# Examples

Suppose that $|U| = |V| = |X| = |Y| = |W| = |Z| = 3$.

<table>
<tr><td>0</td><td></td><td>0</td><td></td><td></td><td></td><td></td><td></td><td>2</td><td></td><td></td><td></td><td>0</td><td></td></tr>
<tr><td colspan="3">$U$</td><td colspan="3">$V$</td><td colspan="3">$\widehat{U}$</td><td colspan="3">$\widehat{V}$</td><td colspan="2"></td></tr>
<tr><td></td><td colspan="3">$X$</td><td colspan="3">$Y$</td><td colspan="3">$\widehat{X}$</td><td colspan="3">$\widehat{Y}$</td><td></td></tr>
</table>

# Examples

Suppose that $|U| = |V| = |X| = |Y| = |W| = |Z| = 3$.

<table>
<tr><td>0</td><td></td><td>0</td><td></td><td></td><td></td><td>2</td><td></td><td>2</td><td></td><td></td><td></td><td>0</td><td></td></tr>
<tr><td colspan="3">$U$</td><td colspan="3">$V$</td><td colspan="3">$\widehat{U}$</td><td colspan="3">$\widehat{V}$</td><td colspan="2"></td></tr>
<tr><td></td><td colspan="3">$X$</td><td colspan="3">$Y$</td><td colspan="3">$\widehat{X}$</td><td colspan="3">$\widehat{Y}$</td><td></td></tr>
</table>

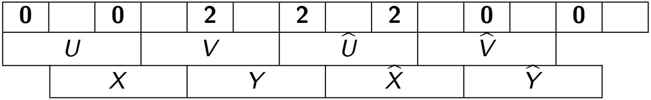
# Examples

Suppose that $|U| = |V| = |X| = |Y| = |W| = |Z| = 3$.

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **0** | | **0** | | | | **2** | | **2** | | **0** | | **0** | |
| | $U$ | | | $V$ | | | $\widehat{U}$ | | | $\widehat{V}$ | | | |
| | | $X$ | | | $Y$ | | | $\widehat{X}$ | | | $\widehat{Y}$ | | |

# Examples

Suppose that $|U| = |V| = |X| = |Y| = |W| = |Z| = 3$.



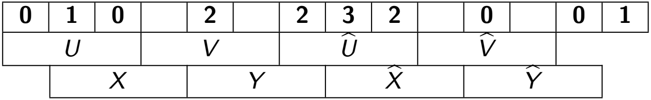
# Examples

Suppose that $|U| = |V| = |X| = |Y| = |W| = |Z| = 3$.

<table>
<tr><td>0</td><td>1</td><td>0</td><td></td><td>2</td><td></td><td>2</td><td></td><td>2</td><td></td><td>0</td><td></td><td>0</td><td>1</td></tr>
<tr><td colspan="3">$U$</td><td colspan="3">$V$</td><td colspan="3">$\widehat{U}$</td><td colspan="3">$\widehat{V}$</td><td colspan="2"></td></tr>
<tr><td></td><td colspan="3">$X$</td><td colspan="3">$Y$</td><td colspan="3">$\widehat{X}$</td><td colspan="3">$\widehat{Y}$</td><td></td></tr>
</table>

# Examples

Suppose that $|U| = |V| = |X| = |Y| = |W| = |Z| = 3$.



# Examples

Suppose that $|U| = |V| = |X| = |Y| = |W| = |Z| = 3$.

<table>
<tr><td>0</td><td>1</td><td>0</td><td>1</td><td>2</td><td></td><td>2</td><td>3</td><td>2</td><td></td><td>0</td><td></td><td>0</td><td>1</td></tr>
<tr><td colspan="3">$U$</td><td colspan="3">$V$</td><td colspan="3">$\widehat{U}$</td><td colspan="3">$\widehat{V}$</td><td colspan="2"></td></tr>
<tr><td></td><td colspan="3">$X$</td><td colspan="3">$Y$</td><td colspan="3">$\widehat{X}$</td><td colspan="3">$\widehat{Y}$</td><td></td></tr>
</table>

# Examples

Suppose that $|U| = |V| = |X| = |Y| = |W| = |Z| = 3$.

<table>
<tr><td>0</td><td>1</td><td>0</td><td>1</td><td>2</td><td></td><td>2</td><td>3</td><td>2</td><td></td><td>0</td><td>3</td><td>0</td><td>1</td></tr>
<tr><td colspan="3">$U$</td><td colspan="3">$V$</td><td colspan="3">$\widehat{U}$</td><td colspan="3">$\widehat{V}$</td><td colspan="2"></td></tr>
<tr><td></td><td colspan="3">$X$</td><td colspan="3">$Y$</td><td colspan="3">$\widehat{X}$</td><td colspan="3">$\widehat{Y}$</td><td></td></tr>
</table>

# Examples

Suppose that $|U| = |V| = |X| = |Y| = |W| = |Z| = 3$.

<table>
<tr><td>0</td><td>1</td><td>0</td><td>1</td><td>2</td><td>1</td><td>2</td><td>3</td><td>2</td><td></td><td>0</td><td>3</td><td>0</td><td>1</td></tr>
<tr><td colspan="3">$U$</td><td colspan="3">$V$</td><td colspan="3">$\widehat{U}$</td><td colspan="3">$\widehat{V}$</td><td></td><td></td></tr>
<tr><td></td><td colspan="3">$X$</td><td colspan="3">$Y$</td><td colspan="3">$\widehat{X}$</td><td colspan="3">$\widehat{Y}$</td><td></td></tr>
</table>

# Examples

Suppose that $|U| = |V| = |X| = |Y| = |W| = |Z| = 3$.

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **0** | **1** | **0** | **1** | **2** | **1** | **2** | **3** | **2** | **3** | **0** | **3** | **0** | **1** |
| | $U$ | | | $V$ | | | $\widehat{U}$ | | | $\widehat{V}$ | | | |
| | | $X$ | | | $Y$ | | | $\widehat{X}$ | | | $\widehat{Y}$ | | |

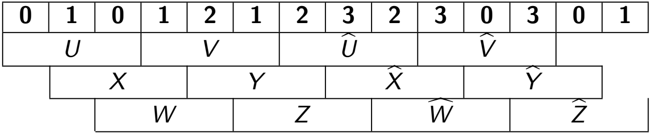
# Examples

Suppose that $|U| = |V| = |X| = |Y| = |W| = |Z| = 3$.

<table>
<tr><td>0</td><td>1</td><td>0</td><td>1</td><td>2</td><td>1</td><td>2</td><td>3</td><td>2</td><td>3</td><td>0</td><td>3</td><td>0</td><td>1</td></tr>
<tr><td colspan="3">$U$</td><td colspan="3">$V$</td><td colspan="3">$\widehat{U}$</td><td colspan="3">$\widehat{V}$</td><td colspan="2"></td></tr>
<tr><td></td><td colspan="3">$X$</td><td colspan="3">$Y$</td><td colspan="3">$\widehat{X}$</td><td colspan="3">$\widehat{Y}$</td><td></td></tr>
<tr><td colspan="2"></td><td colspan="3">$W$</td><td colspan="3">$Z$</td><td colspan="3">$\widehat{W}$</td><td colspan="3">$\widehat{Z}$</td></tr>
</table>

# Examples

Suppose that $|U| = |V| = |X| = |Y| = |W| = |Z| = 3$.




If a third factorization $WZ\widehat{W}\widehat{Z}$ exists, then, $\mathbf{0} = \mathbf{2}$ and $\mathbf{1} = \mathbf{3}$ which is a contradiction. Hence, there is no triple square tile of perimeter 12.

# Examples

Suppose that $|U| = |V| = |X| = |Y| = |W| = |Z| = 3$.

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **0** | **1** | **0** | **1** | **2** | **1** | **2** | **3** | **2** | **3** | **0** | **3** | **0** | **1** |
| $U$ | | | $V$ | | | $\widehat{U}$ | | | $\widehat{V}$ | | | | |
| | $X$ | | | $Y$ | | | $\widehat{X}$ | | | $\widehat{Y}$ | | | |
| | | $W$ | | | $Z$ | | | $\widehat{W}$ | | | $\widehat{Z}$ | | |

If a third factorization $WZ\widehat{W}\widehat{Z}$ exists, then, $\mathbf{0} = \mathbf{2}$ and $\mathbf{1} = \mathbf{3}$ which is a contradiction. Hence, there is no triple square tile of perimeter 12.

Although, there are words having more than two square factorizations. An example of length 36 was provided by X. Provençal :

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **0** | **0** | **122** | **10012** | **21001** | **221** | **0** | **0** | **322** | **30032** | **23003** | **223** |
| $U$ | | | $V$ | | | $\widehat{U}$ | | | $\widehat{V}$ | | |
| | $X$ | | | $Y$ | | | $\widehat{X}$ | | | $\widehat{Y}$ | |
| | | $W$ | | | $Z$ | | | $\widehat{W}$ | | | $\widehat{Z}$ |

# Examples

Suppose that $|U| = |V| = |X| = |Y| = |W| = |Z| = 3$.

<table>
<tr><td>0</td><td>1</td><td>0</td><td>1</td><td>2</td><td>1</td><td>2</td><td>3</td><td>2</td><td>3</td><td>0</td><td>3</td><td>0</td><td>1</td></tr>
<tr><td colspan="3">$U$</td><td colspan="3">$V$</td><td colspan="3">$\widehat{U}$</td><td colspan="3">$\widehat{V}$</td><td colspan="2"></td></tr>
<tr><td></td><td colspan="3">$X$</td><td colspan="3">$Y$</td><td colspan="3">$\widehat{X}$</td><td colspan="3">$\widehat{Y}$</td><td></td></tr>
<tr><td colspan="2"></td><td colspan="3">$W$</td><td colspan="3">$Z$</td><td colspan="3">$\widehat{W}$</td><td colspan="3">$\widehat{Z}$</td></tr>
</table>

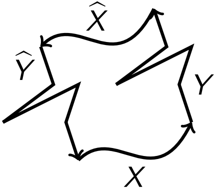
If a third factorization $WZ\widehat{W}\widehat{Z}$ exists, then, $\mathbf{0} = \mathbf{2}$ and $\mathbf{1} = \mathbf{3}$ which is a contradiction. Hence, there is no triple square tile of perimeter 12.

Although, there are words having more than two square factorizations. An example of length 36 was provided by X. Provençal :

<table>
<tr><td>0</td><td>0</td><td>122</td><td>10012</td><td>21001</td><td>221</td><td>0</td><td>0</td><td>322</td><td>30032</td><td>23003</td><td>223</td><td></td></tr>
<tr><td colspan="3">$U$</td><td colspan="3">$V$</td><td colspan="3">$\widehat{U}$</td><td colspan="3">$\widehat{V}$</td><td></td></tr>
<tr><td></td><td colspan="3">$X$</td><td colspan="3">$Y$</td><td colspan="3">$\widehat{X}$</td><td colspan="3">$\widehat{Y}$</td></tr>
<tr><td colspan="2"></td><td colspan="3">$W$</td><td colspan="3">$Z$</td><td colspan="3">$\widehat{W}$</td><td colspan="2">$\widehat{Z}$</td></tr>
</table>

Note that the factor **221003** is a closed path...

# Turning number




Since a square tile determined a closed and simple boundary, the turning number of $XY\widehat{X}\widehat{Y}$ must be $\pm 1$.

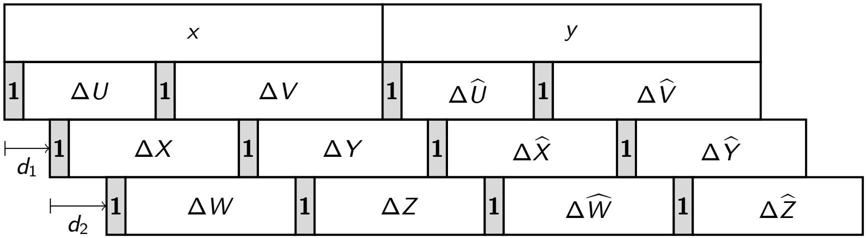
**Lemma (Blondin-Massé, Brlek, Garon, L. 2010)**

*Si $XY\widehat{X}\widehat{Y}$ est la frontière orientée positivement d'une tuile carrée, alors*

$$\Delta[XY\widehat{X}\widehat{Y}] = \Delta X \cdot \mathbf{1} \cdot \Delta Y \cdot \mathbf{1} \cdot \Delta \widehat{X} \cdot \mathbf{1} \cdot \Delta \widehat{Y} \cdot \mathbf{1}.$$

# Idée de la preuve : au plus 2 factorisations carrées

We get 12 positions where there must be a **1** in the first differences of the boundary word :




We show that there is a $\{\mathbf{1}, \mathbf{3}\}$-alternating deduction path of odd length between two **1** :

$$\mathbf{1} \to \mathbf{3} \to \mathbf{1} \to \mathbf{3} \to \mathbf{1} \to \mathbf{3} = \mathbf{1}$$

and we get the desired contradiction. Hence, if the turning number of a boundary word is $\pm 1$, there can't be a third square factorisation.

# Idée de la preuve : au plus 2 factorisations carrées

**Theorem (Blondin Massé, Brlek, Garon, L. 2010)**

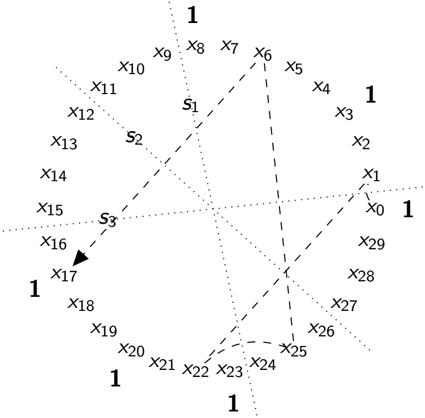
*A tile has at most 2 regular square tilings.*

Reflexions : $s_1$, $s_2$, $s_3$.
We have

$$I = (s_1 s_3 s_2)^2.$$

thus

$$s_1 = s_3 s_2 s_1 s_3 s_2.$$

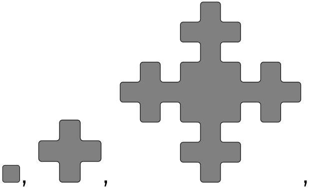


# Plan


1. Pavages
2. Mots de contour
3. Nombre de pavages réguliers
4. Au plus deux pavages réguliers carrés
5. Les tuiles de Fibonacci et Christoffel sont des tuiles double carrées
6. Réduction (et reconstruction) des tuiles doubles carrées
7. Questions ouvertes

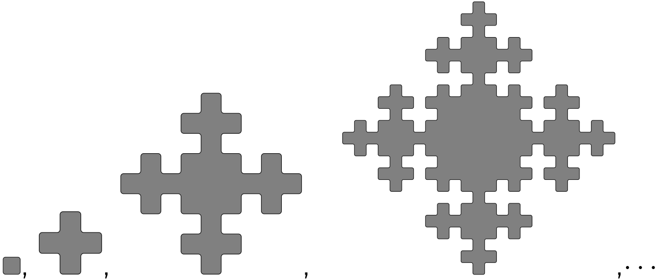
# Fibonacci Tiles are double squares

Complete the sequence :



# Fibonacci Tiles are double squares

Complete the sequence :



```
sage: p = words.fibonacci_tile(2) * Word([3]); print p
32303010303232123230323212101212321210103010121010303
sage: p.finite_differences().primitive()
word: 3113113313313
sage: p.finite_differences().primitive().finite_differences()
word: 202202022022
sage: words.FibonacciWord([2,0])
word: 2022020220220202202022022020220220202202...
```

# Fibonacci Tiles

**Theorem (Blondin Massé, Brlek, Garon, L., 2011)**

*The n-th Fibonacci Tiles is a double square.*

**Theorem (Blondin Massé, Brlek, L., Mendès France, 2011)**

*The limit ratio between the area of the n-th Fibonacci tile $A(n)$ and the area of its convex hull $H(n)$ is*

$$\lim_{n\to\infty} \frac{A(n)}{H(n)} = 2 - \sqrt{2} = 0.58578643\cdots$$

**Theorem (Blondin Massé, Brlek, L., Mendès France, 2012)**

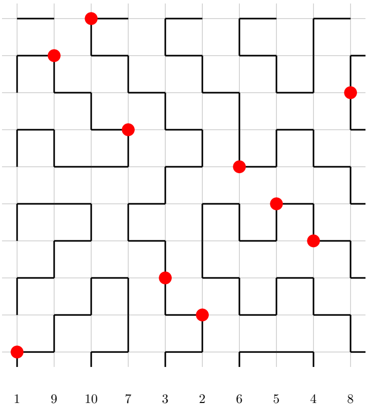
*The fractal dimension of the n-th Fibonacci tile is*

$$d = \frac{\log(2+\sqrt{5})}{\log(1+\sqrt{2})} = 1.637938210\cdots$$

# Fibonacci Tiles in fully packed loop configurations

Fully-packed loop configuration from the permutation

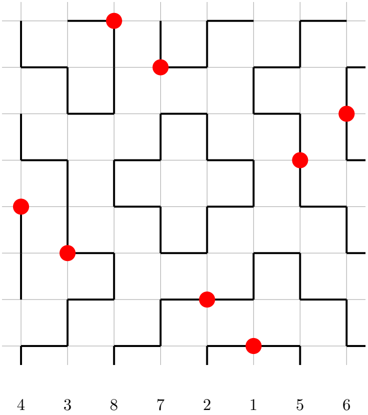
$$[1, 9, 10, 7, 3, 2, 6, 5, 4, 8] :$$



# Fibonacci Tiles in fully packed loop configurations

Fully-packed loop configuration from the permutation

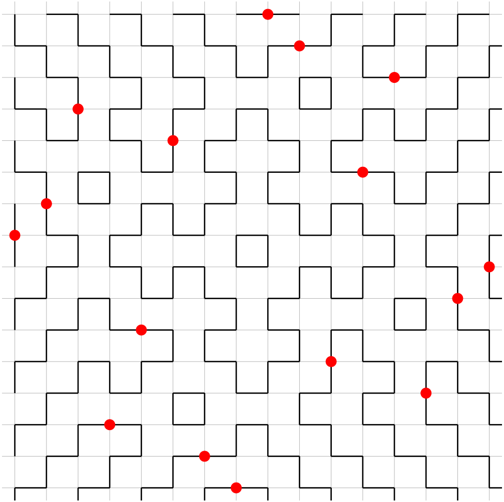
$$[4, 3, 8, 7, 2, 1, 5, 6]$$



# Fibonacci Tiles in fully packed loop configurations

Fully-packed loop configuration from the permutation

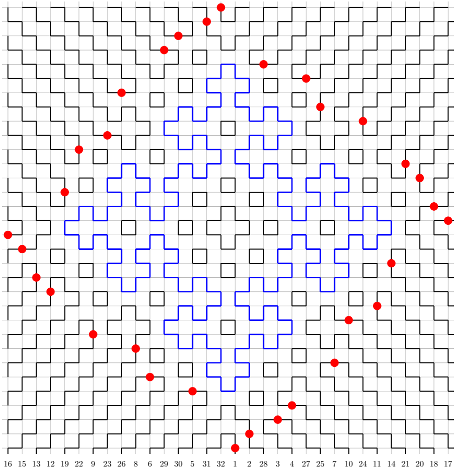
$$[9, 10, 13, 3, 6, 12, 2, 1, 16, 15, 5, 11, 14, 4, 7, 8]$$



# Fibonacci Tiles in fully packed loop configurations

Fully-packed loop configuration from the permutation

$$[16, 15, 13, 12, 19, 22, 9, 23, 26, 8, 6, 29, 30, 5, 31, 32, 1, 2, 28, 3, 4, 27, 25, 7, 10, 24, 11, 14, 21, 20, 18, 17]$$




**Question**

*Which permutations of size $2^{n+1}$ generates the n-th Fibonacci tile ?*

# Christoffel Tiles are double squares

Let $\lambda$ defined by

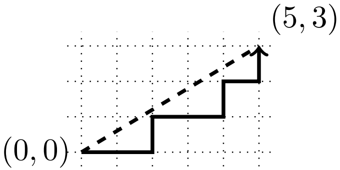
$$\mathbf{0} \mapsto \mathbf{0301}, \mathbf{1} \mapsto \mathbf{01}, \mathbf{2} \mapsto \mathbf{2123}, \mathbf{3} \mapsto \mathbf{23}.$$

Let $\overline{\alpha} = \alpha + 2 \in \mathbb{Z}_4$ for all $\alpha \in \mathbb{Z}_4$.

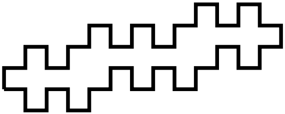
**Theorem (Blondin Massé, Brlek, Garon, L., 2011)**

*Let $w = \mathbf{0}v\mathbf{1} \in \{\mathbf{0}, \mathbf{1}\}^*$.*

- *$w$ is a Christoffel word if and only if $\lambda(w\overline{w})$ is a double square.*




$w = \mathbf{00100101}$



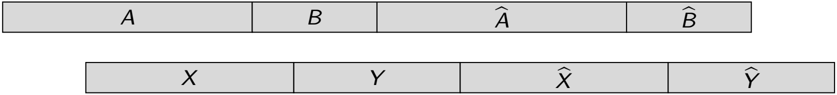
$\lambda(w\overline{w})$

# Plan


1. Pavages
2. Mots de contour
3. Nombre de pavages réguliers
4. Au plus deux pavages réguliers carrés
5. Les tuiles de Fibonacci et Christoffel sont des tuiles double carrées
6. Réduction (et reconstruction) des tuiles doubles carrées
7. Questions ouvertes

# Periods in the boundary of double square tiles

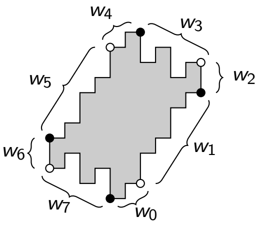
Let $AB\widehat{A}\widehat{B} \equiv XY\widehat{X}\widehat{Y}$ be the factorizations of a double square tile.



# Periods in the boundary of double square tiles

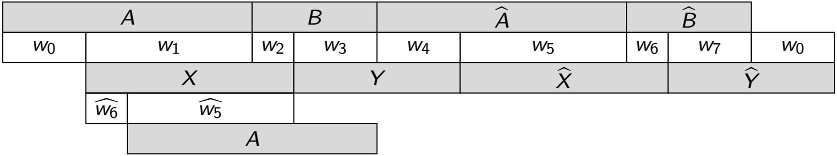
Let $AB\widehat{A}\widehat{B} \equiv XY\widehat{X}\widehat{Y}$ be the factorizations of a double square tile.

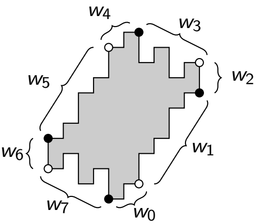
| $A$ | | $B$ | | $\widehat{A}$ | | $\widehat{B}$ | | |
|---|---|---|---|---|---|---|---|---|
| $w_0$ | $w_1$ | $w_2$ | $w_3$ | $w_4$ | $w_5$ | $w_6$ | $w_7$ | $w_0$ |
| | $X$ | | $Y$ | | $\widehat{X}$ | | $\widehat{Y}$ | |



# Periods in the boundary of double square tiles

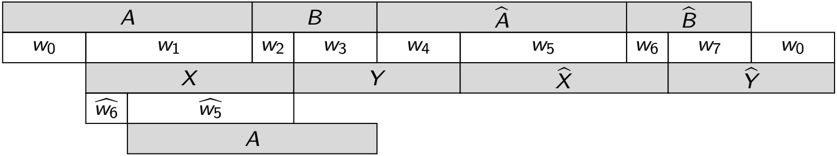
Let $AB\widehat{A}\widehat{B} \equiv XY\widehat{X}\widehat{Y}$ be the factorizations of a double square tile.



# Periods in the boundary of double square tiles

Let $AB\widehat{A}\widehat{B} \equiv XY\widehat{X}\widehat{Y}$ be the factorizations of a double square tile.



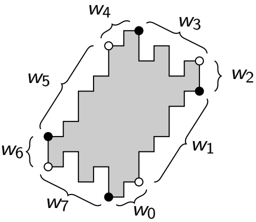

In general

- $d_i = |w_{i-1}| + |w_{i+1}|$ is a period of $w_{i-1}w_iw_{i+1}$.

Hence we write

- $w_i = (u_iv_i)^{n_i}u_i$ where $|u_iv_i| = d_i$.

Remark : $u_i$ and $v_i$ always exist even if $|w_i| < d_i$.

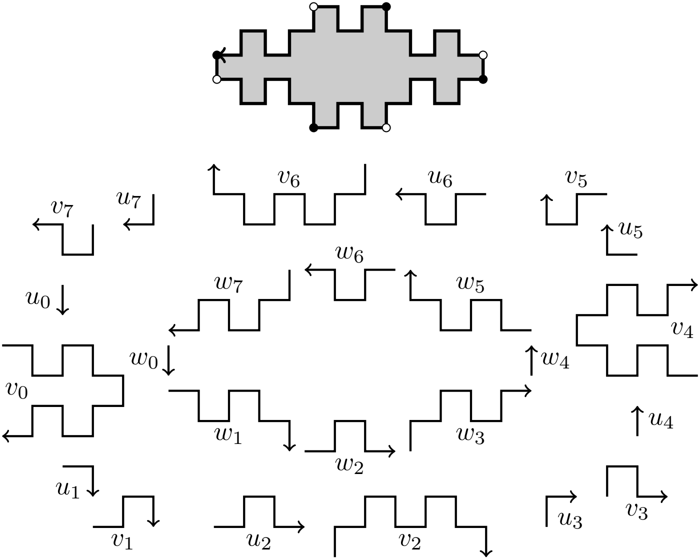


# Double Square in Sage free software

Double Square tile from the words $(w_0, w_1, w_2, w_3)$ :

```
sage: from sage.combinat.double_square_tile import DoubleSquare
sage: DoubleSquare(([3,2], [3], [0,3], [0,1,0,3,0]))
Double Square Tile
  w0 = 32                w4 = 10
  w1 = 3                 w5 = 1
  w2 = 03                w6 = 21
  w3 = 01030             w7 = 23212
(|w0|, |w1|, |w2|, |w3|) = (2, 1, 2, 5)
(d0, d1, d2, d3)         = (6, 4, 6, 4)
(n0, n1, n2, n3)         = (0, 0, 0, 1)
```

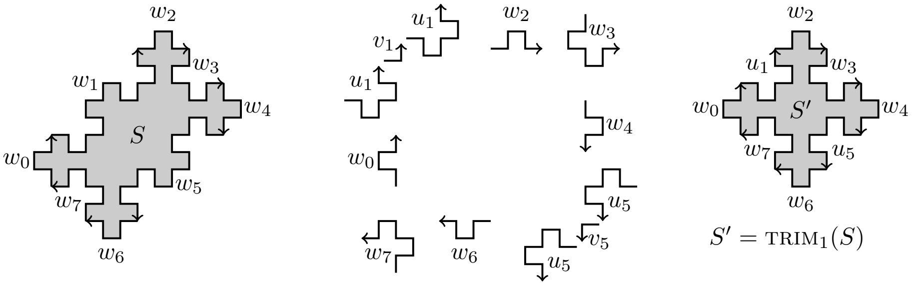
# The factor $u_i$ and $v_i$



# $\text{TRIM}_i$ : removes a period in $w_i$ and $w_{i+4}$

Let $S = (w_0, w_1, w_2, w_3, w_4, w_5, w_6, w_7)$.

$$\text{TRIM}_0(S) = ((u_0 v_0)^{n_i - 1} u_0, w_1, w_2, w_3, (u_4 v_4)^{n_4 - 1} u_4, w_5, w_6, w_7)$$



$S' = \text{TRIM}_1(S)$

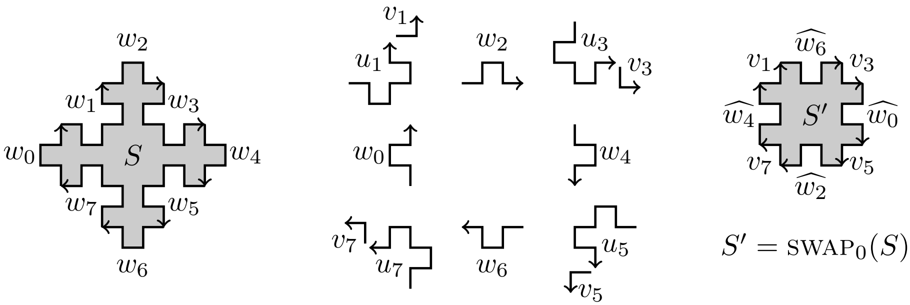
and its conjugates

- $\text{SHIFT}(S) = (w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_0)$,
- $\text{TRIM}_i(S) = \text{SHIFT}^{-i} \circ \text{TRIM}_0 \circ \text{SHIFT}^i(S)$,

# $\text{SWAP}_i$ : the curious involution

Let $S = (w_0, w_1, w_2, w_3, w_4, w_5, w_6, w_7)$.

$\text{SWAP}_0(S) = (\widehat{w_4}, (v_1 u_1)^{n_1} v_1, \widehat{w_6}, (v_3 u_3)^{n_3} v_3, \widehat{w_0}, (v_5 u_5)^{n_5} v_5, \widehat{w_2}, (v_7 u_7)^{n_7} v_7)$



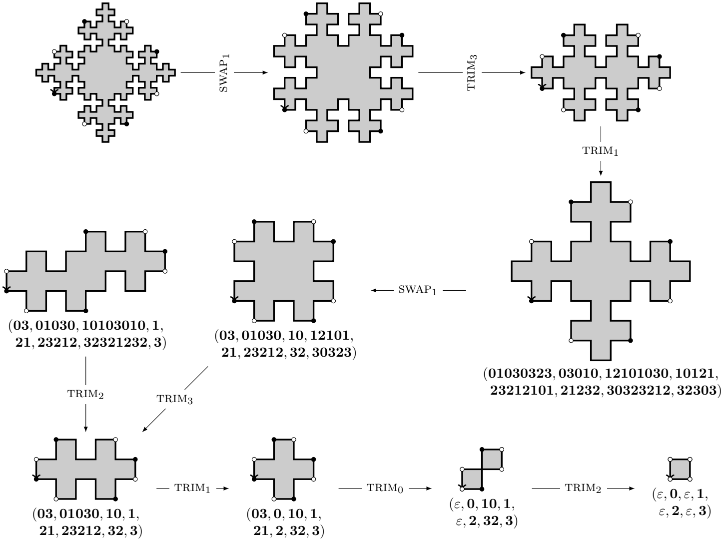

$S' = \text{SWAP}_0(S)$

and its conjugates

- $\text{SHIFT}(S) = (w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_0)$,
- $\text{SWAP}_i(S) = \text{SHIFT}^{-i} \circ \text{SWAP}_0 \circ \text{SHIFT}^i(S)$,

## Theorem (Blondin Massé, Brlek, Garon, L.)

*Every double square tile* *reduces to a square tile* *with* TRIM *and* SWAP.



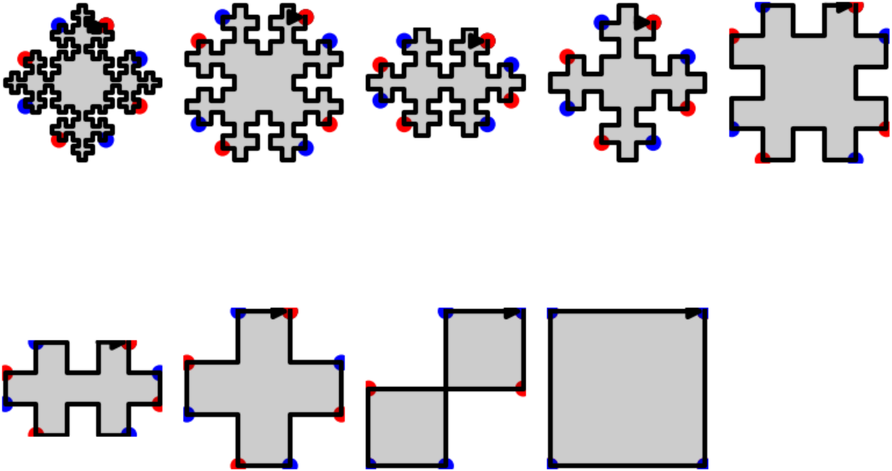
# Double Square in Sage free software

```
sage: D = DoubleSquare(words.christoffel_tile(4,7))
sage: D.reduction()
['TRIM_1', 'TRIM_1', 'TRIM_2', 'TRIM_1', 'TRIM_0', 'TRIM_2']
sage: D.trim(1)
Double Square Tile
  w0 = 03                        w4 = 21
  w1 = 010301010301030           w5 = 232123232123212
  w2 = 10103010                  w6 = 32321232
  w3 = 1                         w7 = 3
(|w0|, |w1|, |w2|, |w3|) = (2, 15, 8, 1)
(d0, d1, d2, d3)         = (16, 10, 16, 10)
(n0, n1, n2, n3)         = (0, 1, 0, 0)
```

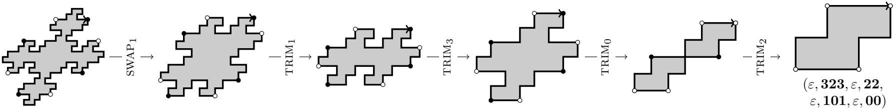
# Double Square in Sage free software

Plot a double square tile and its reduction :

```
sage: D = DoubleSquare((34,21,34,21))
sage: D.plot_reduction(ncols=5)
```



# Reduction of double square tiles




Moreover,

- The transformations $\text{TRIM}_i$ and $\text{SWAP}_i$ are invertible.
- The transformations $\text{TRIM}_i^{-1}$ and $\text{SWAP}_i^{-1}$ preserve palindromes.

## Proposition (Blondin Massé, Brlek, Garon, L.)

*Let $AB\widehat{A}\widehat{B} \equiv XY\widehat{X}\widehat{Y}$ be the boundary of a double square $D$. If $D$ reduces to the unit square tile, then*

- *$A$, $B$, $X$ and $Y$ are palindromes,*
- *$D$ is invariant under a rotation of 180 degrees.*

# Plan


1. Pavages
2. Mots de contour
3. Nombre de pavages réguliers
4. Au plus deux pavages réguliers carrés
5. Les tuiles de Fibonacci et Christoffel sont des tuiles double carrées
6. Réduction (et reconstruction) des tuiles doubles carrées
7. Questions ouvertes

# Problèmes ouverts

Some problems are left open :

- Find an algorithm that decides whether a polyomino is prime.
- If $\alpha\alpha$ appears in the boundary word of a double square tile $D$, where $\alpha \in \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}\}$, then $D$ is not prime.
- Prove that if $S \circ P$ is a square tile, then so is $P$.
- Describe the distribution and the proportion of prime square tiles of half-perimeter $n$ as $n$ goes to infinity.
- Extend the results to 8-connected polyominoes.
- Extend the results to continuous paths and tiles.
- Understand the function $(|w_0|, |w_1|, |w_2|, |w_3|) \mapsto$ double square.
- Understand the tree of double squares under $\textsc{Swap}_i$ and $\textsc{Extend}_i$.

# Double Square in Sage free software

Double Square tile from the lengths of the $w_i$ :

```
sage: from sage.combinat.double_square_tile import DoubleSquare
sage: DoubleSquare((4,7,4,7))
Double Square Tile
  w0 = 3232               w4 = 1010
  w1 = 1212323            w5 = 3030101
  w2 = 2121               w6 = 0303
  w3 = 0101212            w7 = 2323030
(|w0|, |w1|, |w2|, |w3|) = (4, 7, 4, 7)
(d0, d1, d2, d3)         = (14, 8, 14, 8)
(n0, n1, n2, n3)         = (0, 0, 0, 0)
```

# Credits

- This research was driven by computer exploration using the open-source mathematical software `Sage`.
- Les images de ce document ont été produites à l'aide de `pgf/tikz`.