On Double Square tiles

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April 4th 2013

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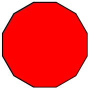
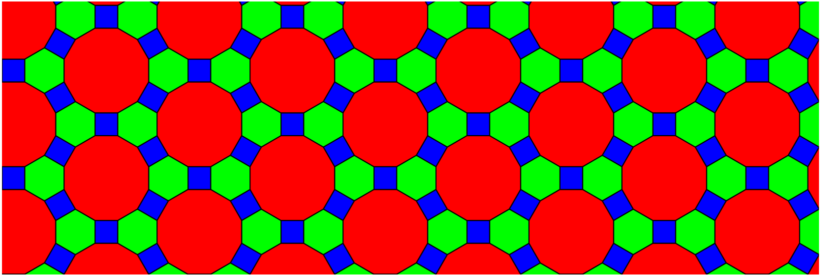
Outline

Plan

Tilings

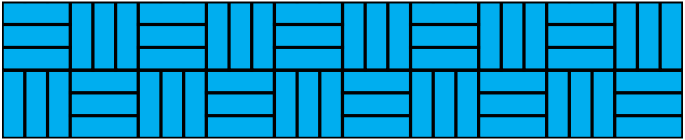
A set $S = \{P_1, P_2, \ldots, P_k\}$ of polyominoes tiles the plane if there exists a partition of $\mathbb{Z}^2$ into translated copies of P_i.

For example, the set $S = \{$, , $\}$ tiles the plane :

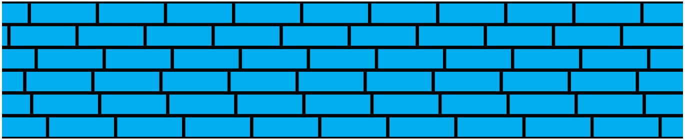


Types of tilings

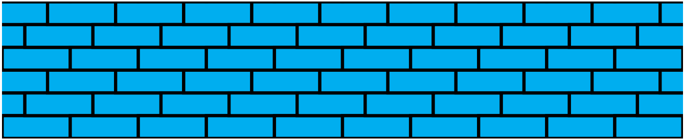
A tiling periodic where the rotations are allowed :



A tiling by translation :



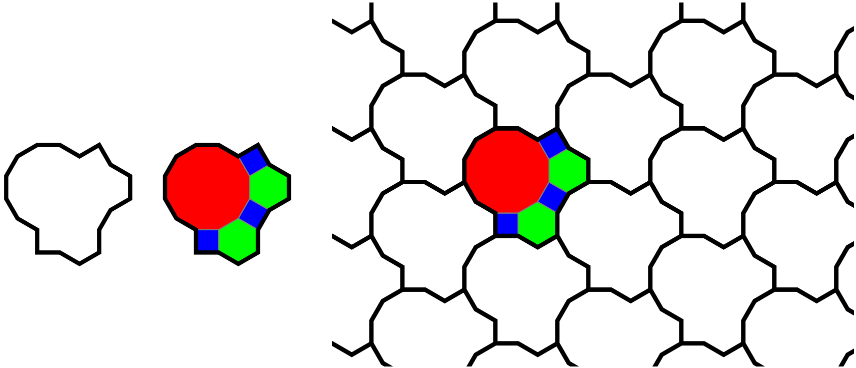
A regular tiling :



Problème du pavage

Given a set S of polygons,
is there a tiling of the plane by S ?

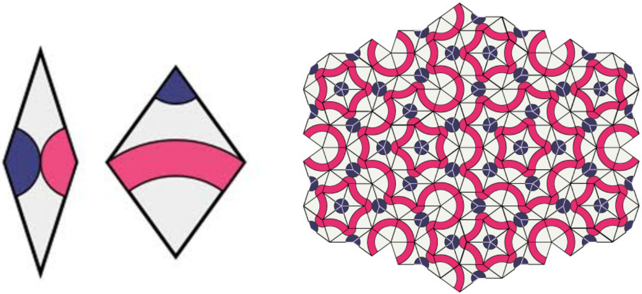
One way to anwer is to find a periodic tiling of the plane.



Theorem (Berger, 1961)

There is a set S which tiles the plane, but not in a periodic way.

The first example found by Berger contains $|S| = 20426$ tiles.
In 1974, Penrose provided an example made of two polygons :

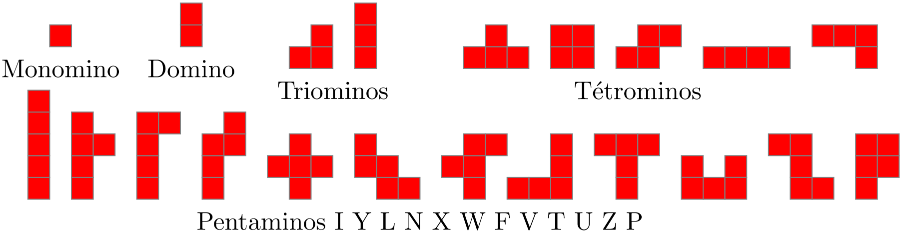


Theorem (Berger, 1961)

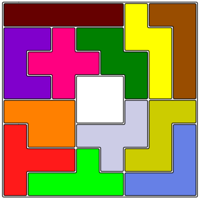
The Tiling Problem is not decidable.

Polyomino

The word polyomino (Golomb, 1952) comes from domino. The domino is made of two squares, a polyomino is made of many.

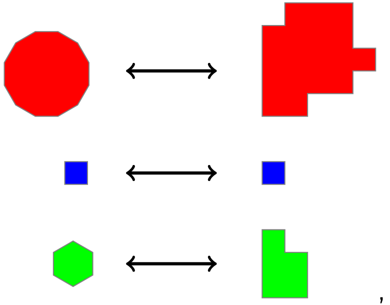


Donald Knuth (Dancing links, 2000) was interested by the tiling problem by polyominoes and more generally by the exact cover problem. This method allows to solve a sudoku.



The Tiling Problem by polyominoes is not decidable

By association of a set of polyominoes with a set of polygons,



Golomb obtains the following result :

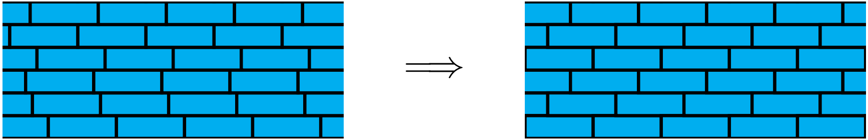
Theorem (Golomb, 1970)

The Tiling Problem by a set of polyominoes is also not decidable.

Tiling by translation by one polyomino is decidable

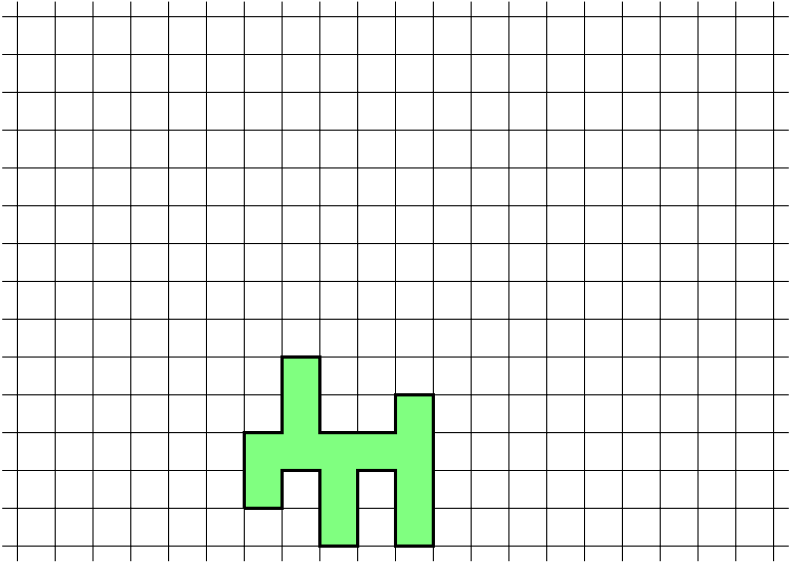
Theorem (Wijshoff, van Leuveen, 1984)

If a polyomino tiles the plane by translation, then it tiles the plane regularly.



Then, the tiling problem where the set S contains only one polyomino is decidable.

Does a polyomino tile the plane by translation ?



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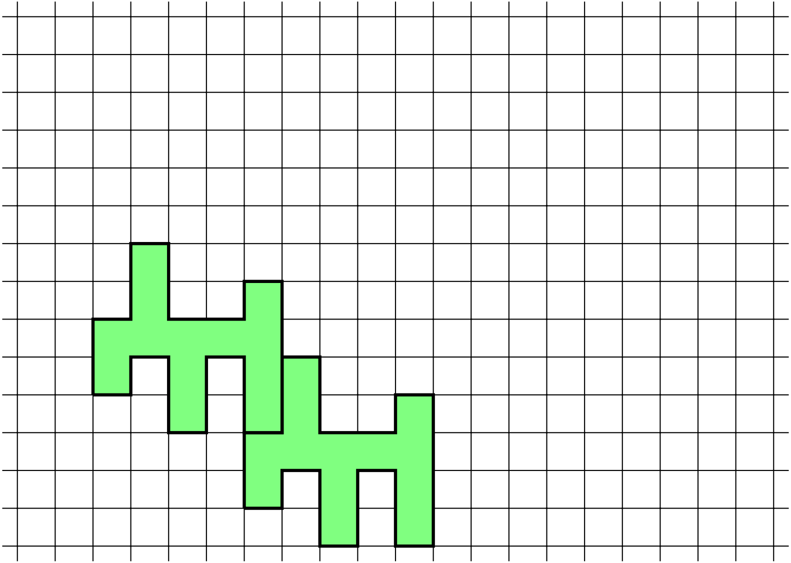

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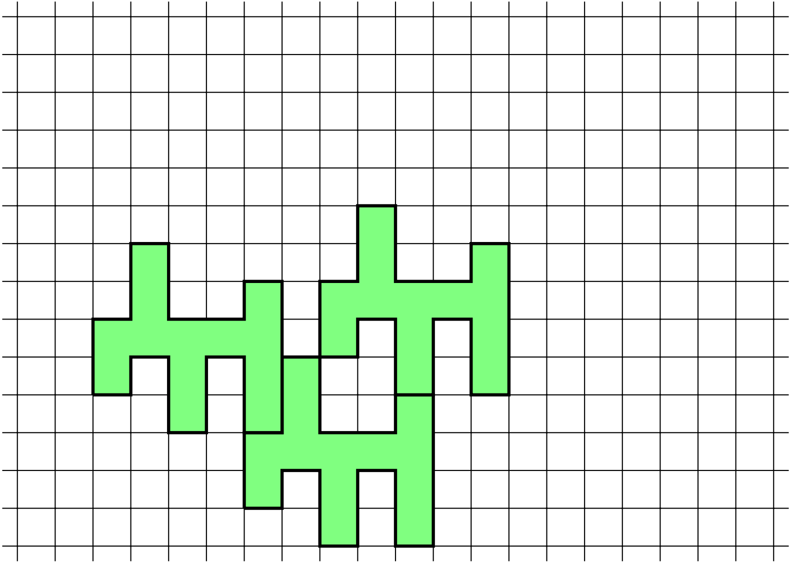

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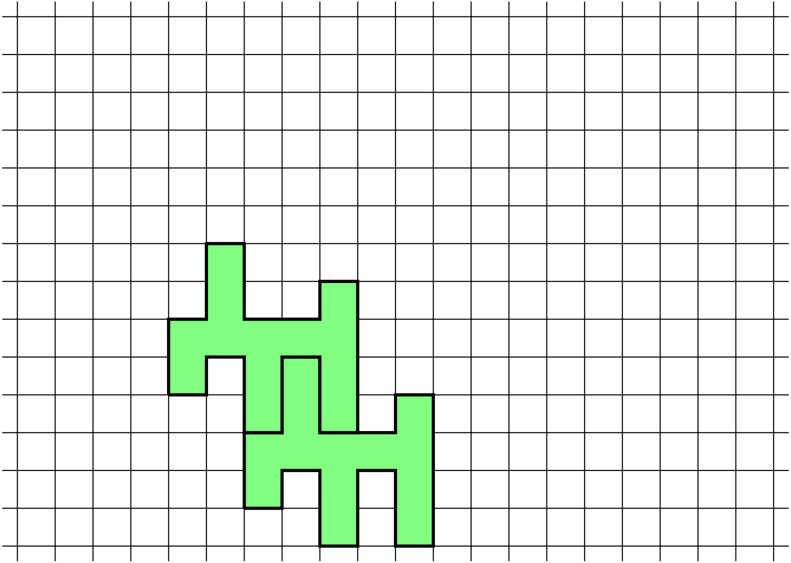

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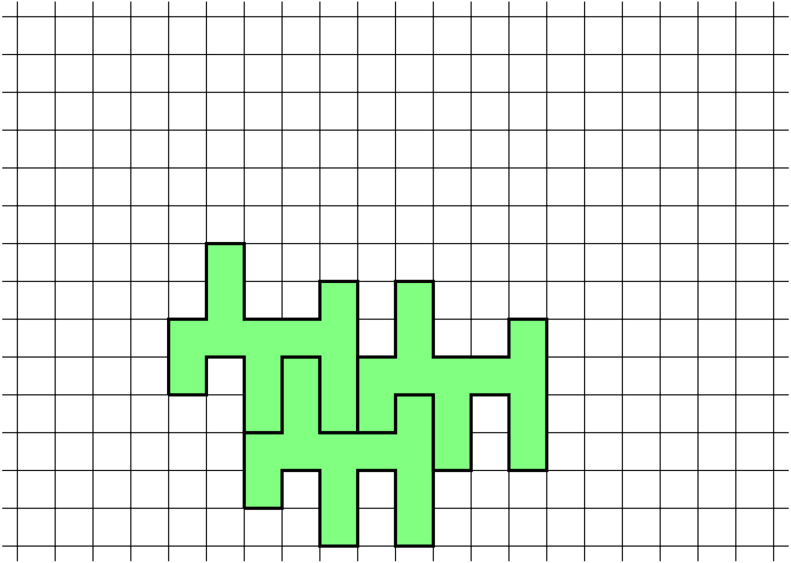

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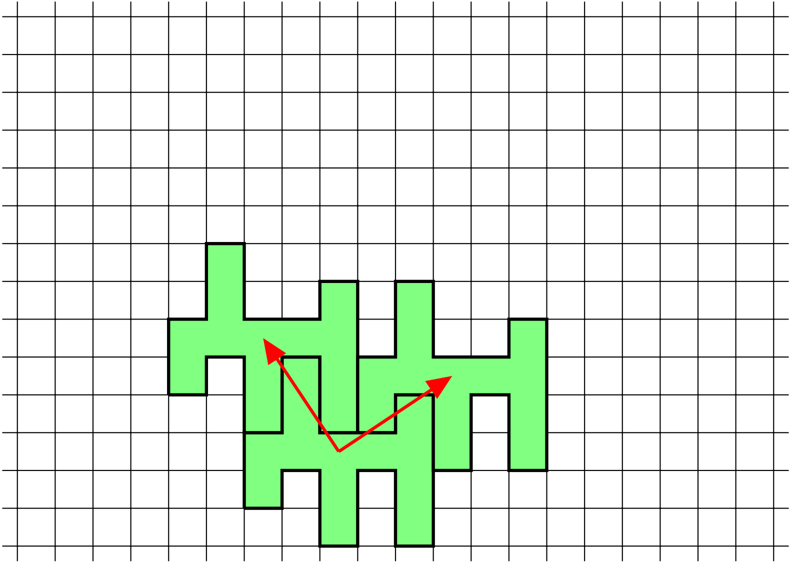

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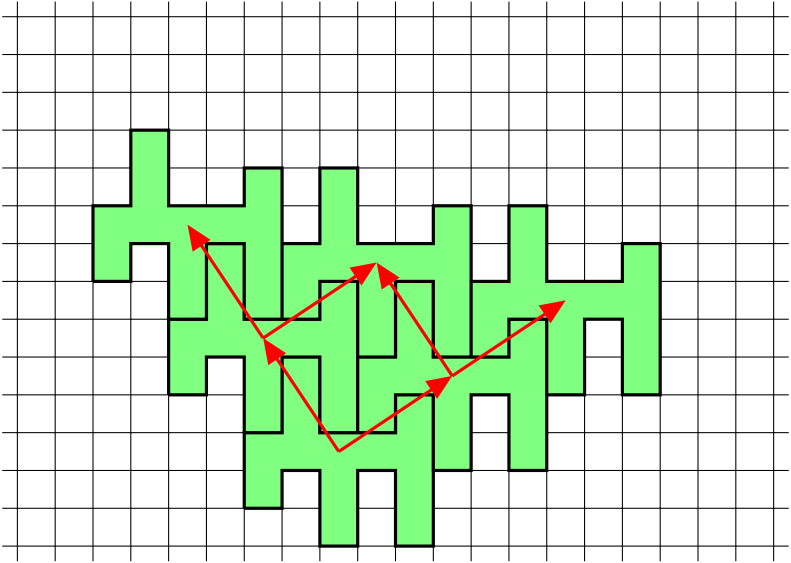

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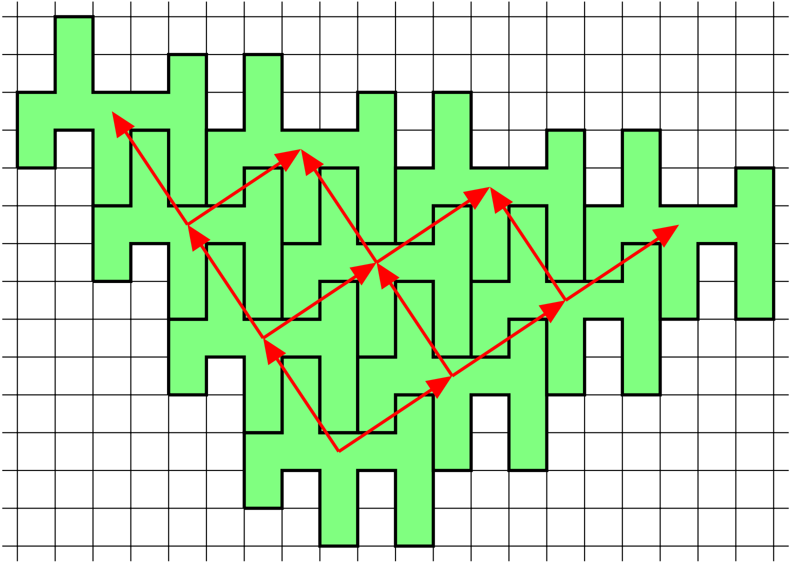

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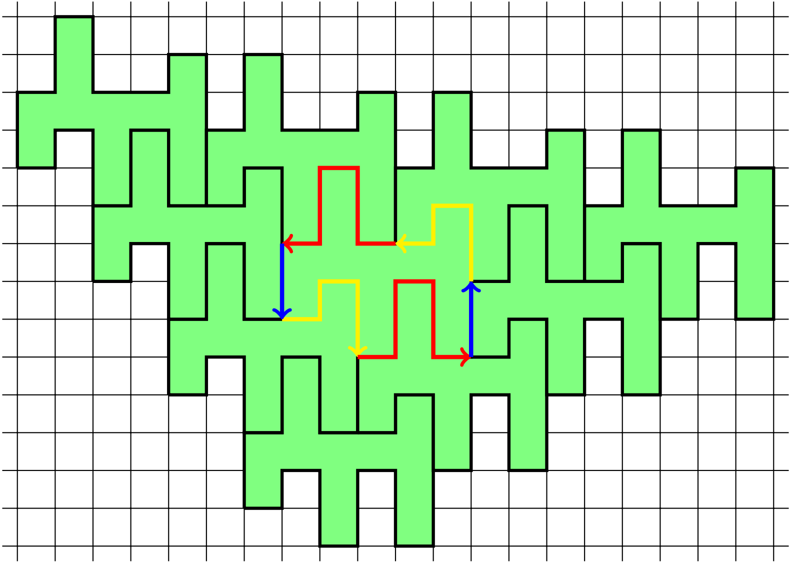
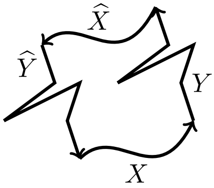


Beauquier-Nivat

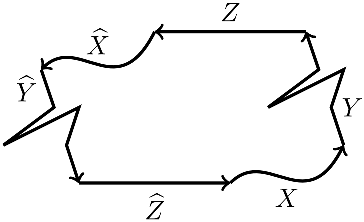
Conway criterion, 1980 : a sufficient condition for a polyomino to tile the plane.

Theorem (Beauquier, Nivat, 1991)

A polyomino tiles the plane by translation if and only if its boundary word factorize into $XY\widehat{X}\widehat{Y}$ *or* $XYZ\widehat{X}\widehat{Y}\widehat{Z}$.



tuile carrée

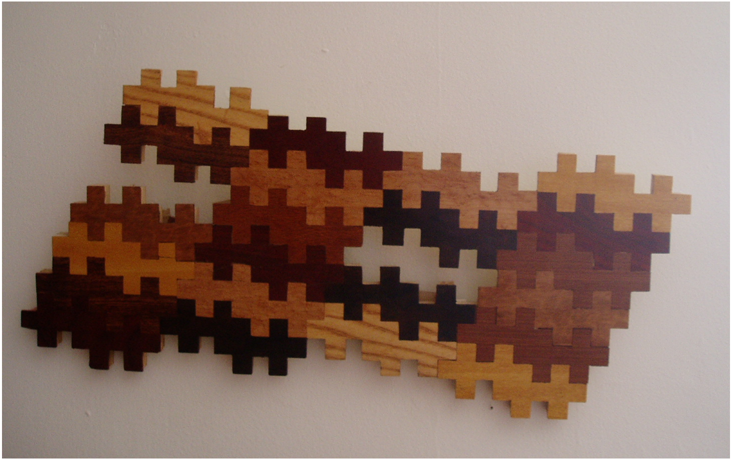


tuile hexagonale

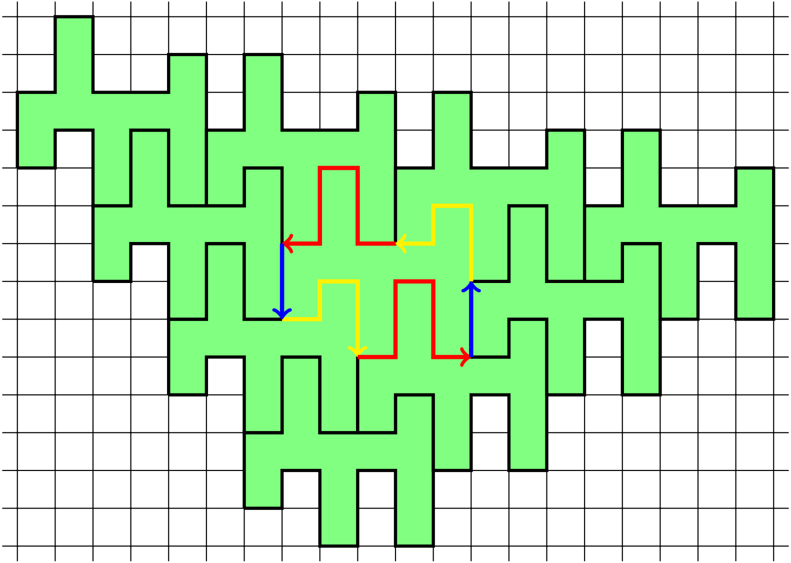
Maurits Cornelis Escher (1898-1972). Hexagonal tiling. Square tiling.

Tuile carrée

Recent artwork of Marc Dumont.

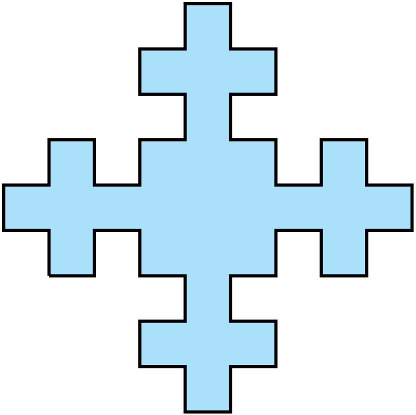


Hexagonal tile

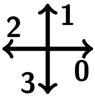


Plan

Representation of a polyomino by its boundary



$$\Sigma = \mathbb{Z}_4 = \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}\}$$



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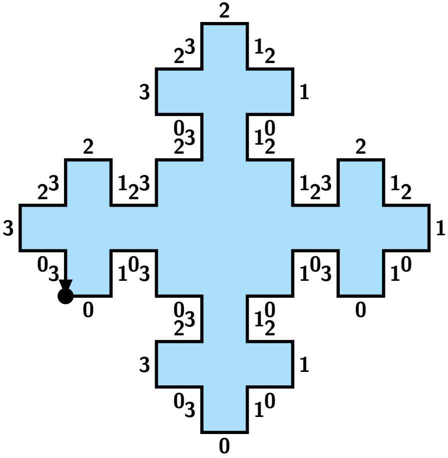

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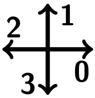
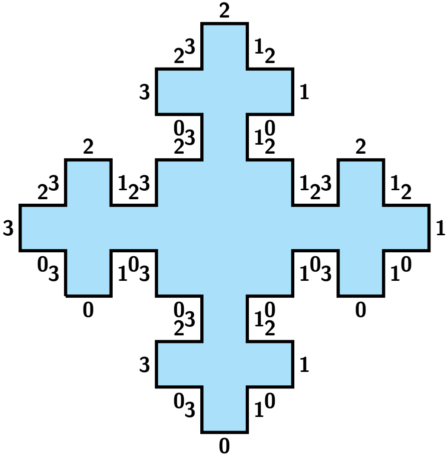


$w = \mathbf{0103032303010121010301012123212101212323032321232303}$



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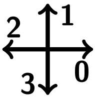

$[w] \equiv \mathbf{0103032303010121010301012123212101212323032321232303}$

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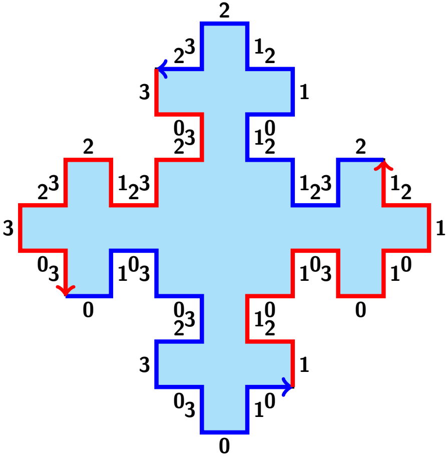

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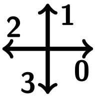

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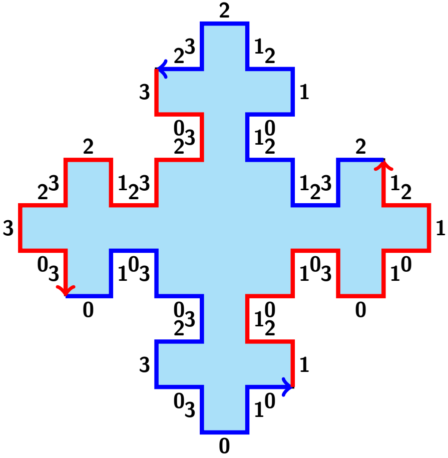

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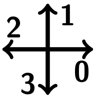


$[w] \equiv$

0103032303010	1210103010121	2321210121232	3032321232303
X	Y	$\widehat{X}$	$\widehat{Y}$

Results on polyominoes using boundary word

Many statistics on polyominoes can be computed efficiently from the boundary word including :

- area,
- moment of inertia (thus center of gravity),
- size of projection,
- intersection,
- digital convexity,
- whether it tiles the plane by translations.

See publications of S. Brlek, A. Lacasse and X. Provençal and their coauthors.

Theorem (Brlek, Koskas, Provençal, 2011)

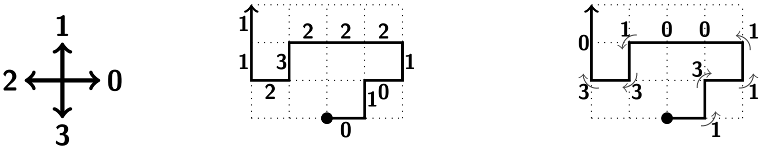
There exists a linear time and space algorithm for detecting path intersection in $\mathbb{Z}^d$.

Why $\{0, 1, 2, 3\}$ is the best alphabet for paths ?

The first differences sequence of $w \in (\mathbb{Z}_4)^*$

$$\Delta w = (w_2 - w_1) \cdot (w_3 - w_2) \cdots (w_n - w_{n-1}).$$

represents the sequence of turns of the path.



$w =$ **01012223211** $\qquad \Delta w =$ **1311001330**

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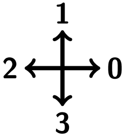

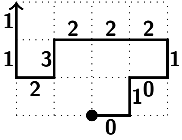

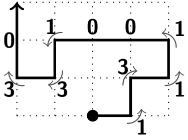

$w = \mathbf{01012223211}$ $\qquad$ $\Delta w = \mathbf{1311001330}$

We also consider $\Delta[w]$ well defined on the conjugacy classes :

$$\Delta[w] = (w_2 - w_1) \cdot (w_3 - w_2) \cdots (w_n - w_{n-1}) \cdot (w_1 - w_n) = \Delta w \cdot (w_1 - w_n).$$

Turning number

The turning number of a path w is $\mathcal{T}(w) = \frac{|\Delta w|_{\mathbf{1}} - |\Delta w|_{\mathbf{3}}}{4}$ and corresponds to its total curvature divided by 2π (Wikipedia). We have that

- $\mathcal{T}(w) = -\mathcal{T}(\widehat{w})$ for all path $w \in \Sigma^*$
- $\mathcal{T}([w]) = \pm 1$ for all simple and closed path w.

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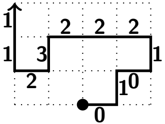
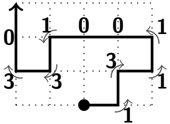
For example,

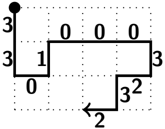


$w = \mathbf{01012223211}$

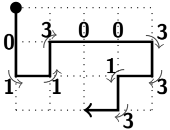


$\Delta w = \mathbf{1311001330}$

$\mathcal{T}(w) = 1/4$



$\widehat{w} = \mathbf{33010003232}$



$\Delta\widehat{w} = \mathbf{0113003313}$

$\mathcal{T}(\widehat{w}) = -1/4$

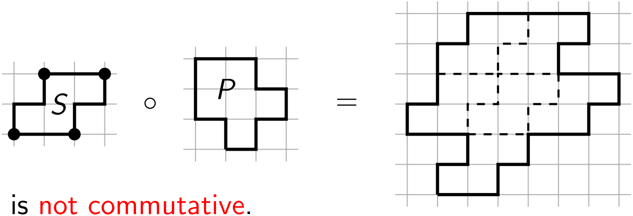
Composition of tiles

The factorization $AB\widehat{A}\widehat{B}$ of a square S allows to define the substitution

$$\varphi_S : \mathbf{0} \mapsto A, \mathbf{1} \mapsto B, \mathbf{2} \mapsto \widehat{A}, \mathbf{3} \mapsto \widehat{B}.$$

For any polyomino P having boundary w we define the composition

$$S \circ P := \varphi_S(w).$$



Note : This is not commutative.

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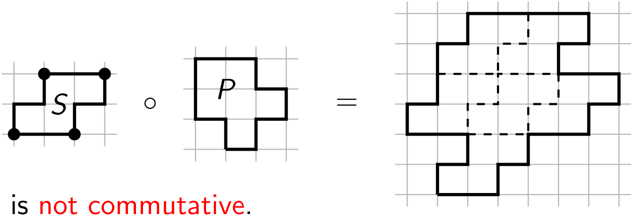

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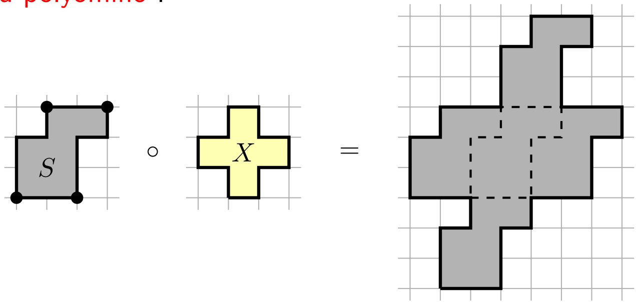
Definition

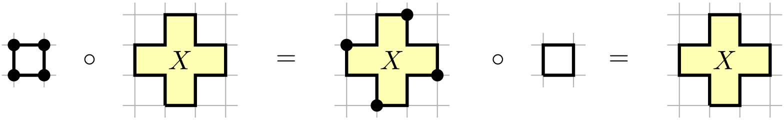
A polyomino Q is prime if $Q = S \circ P$ implies that S or P is the unit square.

Prime polyomino

A composed polyomino :



The X pentomino is prime :



Why prime polyominoes are interesting ?

Prime polyominoes are a subset of polyominoes that allows to reconstruct every polyominoes with the composition rule.

Some questions are open :

- Detect whether a polyomino is prime.
- Find an algorithm that enumerate prime polyominoes.
- Count prime polyominoes.
- Is the growth rate the same or less than the growth rate of polyominoes ?

Plan

Does a polyomino tiles the plane by translation ?

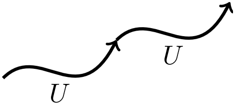
Theorem (Brlek, Fédou, Provençal, 2009)

Determining if a polyomino is a square tile is decidable in linear time.

Theorem (Brlek, Fédou, Provençal, 2009)

Let P be a polyomino such that the length of the largest repeated pattern UU is bounded by the square root of the perimeter. Determining if P is a hexagonal tile is decidable in linear time from the boundary word.

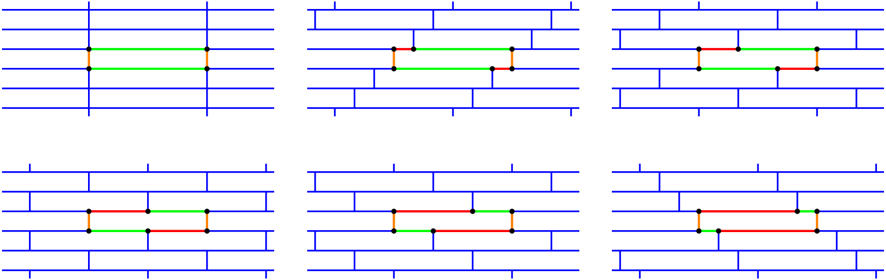
A repeated pattern UU is the concatenation of two identical paths :



Number of regular tilings of an hexagonal tile

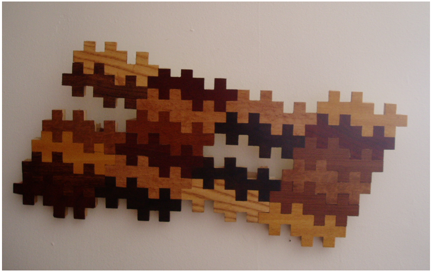
A polyomino may have many regular tilings of the plane.

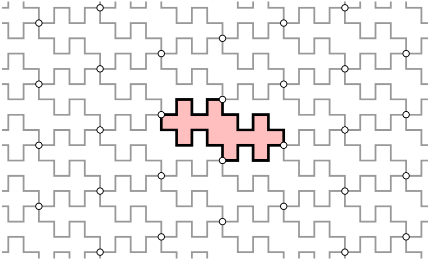
Example : A rectangle 1×6 tiles the plane $\mathbb{Z}^2$ as an hexagon in 5 ways and as a square in only one way.



Number of regular tilings of a square tile

The Dumont tile have a second regular square tiling.





Could it contain more ?

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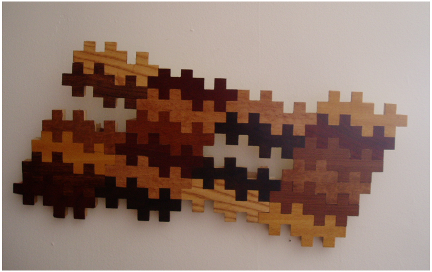
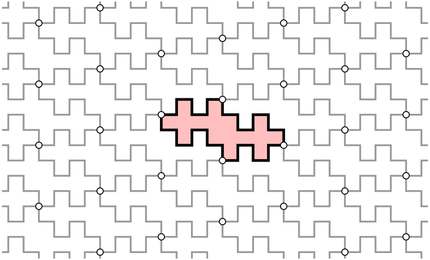


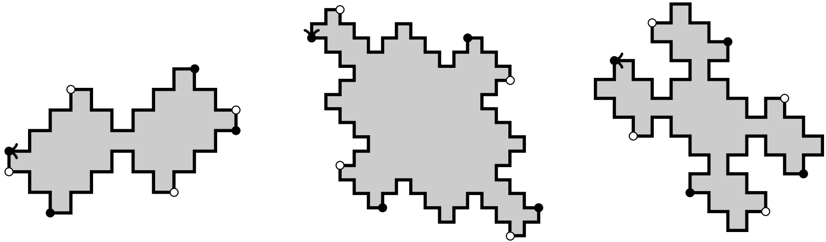


Could it contain more ?

Brlek, Dulucq, Fédou, Provençal conjectured in 2007 that a tile has at most 2 square factorizations.

Some double square tiles

A tile having two regular square tiling is called a double square.



Double Square in Sage free software

Creation of a double square tile in Sage from the boundary word of a known double square :

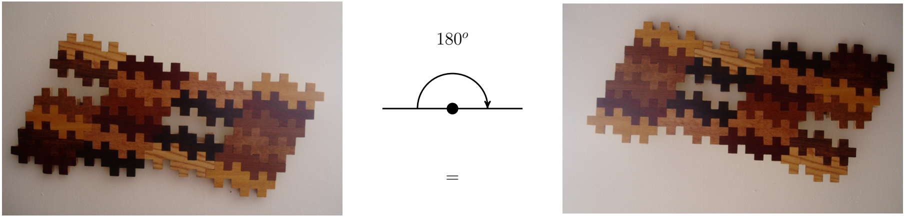
```
sage: from sage.combinat.double_square_tile import DoubleSquare
sage: DoubleSquare(words.christoffel_tile(4,7))
Double Square Tile
  w0 = 03                              w4 = 21
  w1 = 0103010103010301010301030       w5 = 2321232321232123232123212
  w2 = 10103010                        w6 = 32321232
  w3 = 1                               w7 = 3
(|w0|, |w1|, |w2|, |w3|) = (2, 25, 8, 1)
(d0, d1, d2, d3)         = (26, 10, 26, 10)
(n0, n1, n2, n3)         = (0, 2, 0, 0)
```

DoubleSquare will be available in Sage soon :

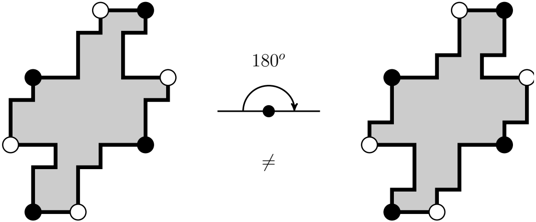
http://trac.sagemath.org/sage_trac/ticket/13069

Invariance under a rotation of 180 degrees

We observe that the Dumont square tile is invariant under a rotation of 180 degrees :



but it is not the case for the following square tile :



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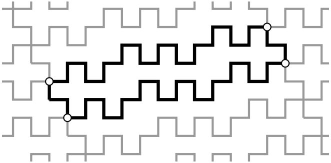
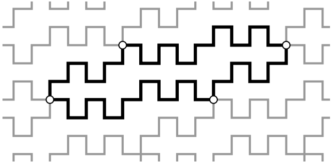
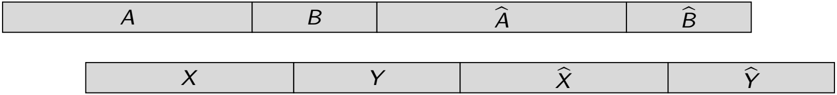
 

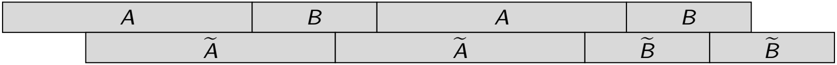
X. Provençal and L. Vuillon conjectured in 2008 that if a polyomino is a prime double square tile, then it is invariant under a rotation of 180 degrees.

Motivation to study double squares

Let $AB\widehat{A}\widehat{B} \equiv XY\widehat{X}\widehat{Y}$ be the factorizations of a double square tile. We need to understand equations on words of the following form :



To study Hof, Knill, Simon Conjecture (1995), one need to study equations on words of the form :



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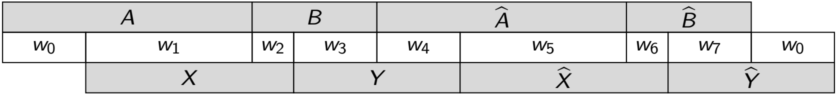

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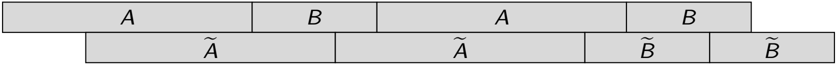

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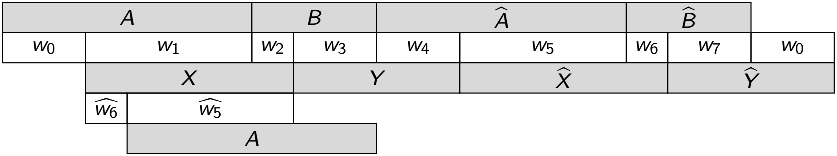

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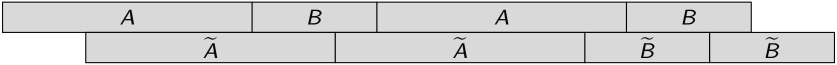


Hof, Knill, Simon Conjecture

We say that a morphism φ is in *class P* if there exists a palindrome p and for every $\alpha \in \Sigma$ there exists a palindrome q_α such that $\varphi(\alpha) = pq_\alpha$.

Conjecture (Hof, Knill, Simon, 1995, rephrased by us in 2008)

Let φ be a primitive morphism such that $\mathbf{u} = \varphi(\mathbf{u})$ is a fixed point. Then, the palindromic complexity of $\mathbf{u}$ is infinite if and only if there exists a morphism φ' in class P such that $\varphi'(\mathbf{u}) = \mathbf{u}$.

- Proved in my master thesis (2008) for the binary alphabet and uniform morphisms.
- Proved by B. Tan in 2008 for the binary alphabet.
- Still open for larger alphabet.

Results

Theorem (Blondin Massé, Brlek, L., 2012)

A tile has at most 2 regular square tilings.

Theorem (Blondin Massé, Brlek, Garon, L., 2011)

Christoffel Tiles and Fibonacci Tiles are double squares.

Theorem (Blondin Massé, Garon, L., 2012)

Any double square tile can be constructed using two simple combinatorial and invertible rules : SWAP *and* TRIM.

Theorem (Blondin Massé, Garon, L., 2012)

Any prime double square tile is invariant under a rotation of 180 degrees.

Plan

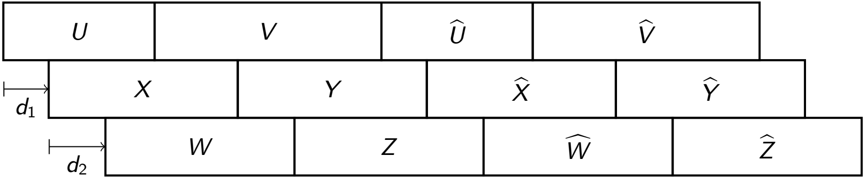
Idea of the proof : at most 2 square factorizations

Lemma (Brlek, Fédou, Provençal, 2008)

The factorizations $UV\widehat{U}\widehat{V} \equiv_{d_1} XY\widehat{X}\widehat{Y}$ of a double square tile must alternate, that is $0 < d_1 < |U| < d_1 + |X|$.

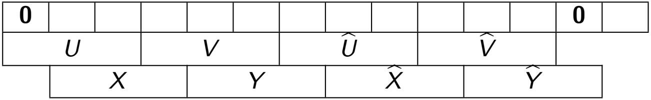
Suppose that there is a triple square tile having the following boundary :

$$UV\widehat{U}\widehat{V} \equiv_{d_1} XY\widehat{X}\widehat{Y} \equiv_{d_2} WZ\widehat{W}\widehat{Z}.$$



Examples

Suppose that $|U| = |V| = |X| = |Y| = |W| = |Z| = 3$.



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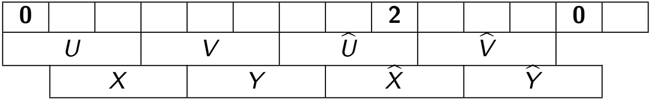

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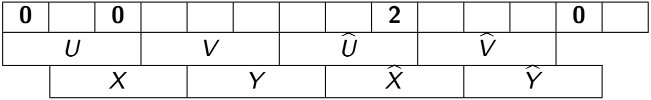

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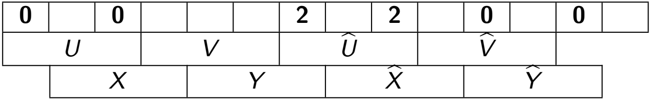

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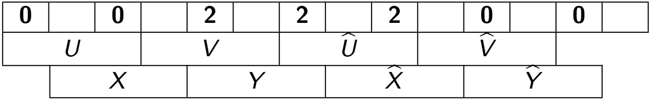

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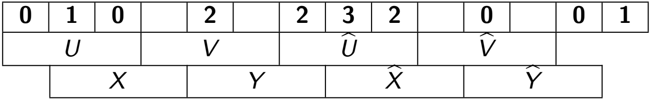

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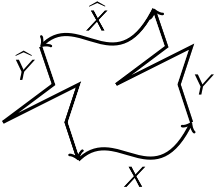
Note that the factor **221003** is a closed path...

Turning number



Since a square tile determined a closed and simple boundary, the turning number of $XY\widehat{X}\widehat{Y}$ must be ± 1.

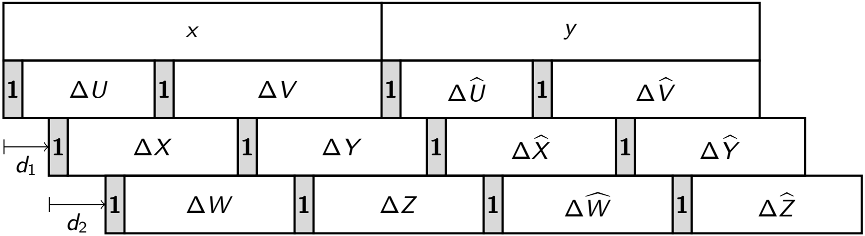
Lemma (Blondin-Massé, Brlek, Garon, L. 2010)

If $XY\widehat{X}\widehat{Y}$ is the boundary word (positively oriented) of a square tile, then

$$\Delta[XY\widehat{X}\widehat{Y}] = \Delta X \cdot \mathbf{1} \cdot \Delta Y \cdot \mathbf{1} \cdot \Delta\widehat{X} \cdot \mathbf{1} \cdot \Delta\widehat{Y} \cdot \mathbf{1}.$$

Idea of the proof : at most 2 square factorizations

We get 12 positions where there must be a $\mathbf{1}$ in the first differences of the boundary word :



We show that there is a $\{\mathbf{1},\mathbf{3}\}$-alternating deduction path of odd length between two $\mathbf{1}$:

$$\mathbf{1} \to \mathbf{3} \to \mathbf{1} \to \mathbf{3} \to \mathbf{1} \to \mathbf{3} = \mathbf{1}$$

and we get the desired contradiction. Hence, if the turning number of a boundary word is ± 1, there can't be a third square factorisation.

Idea of the proof : at most 2 square factorizations

Theorem (Blondin Massé, Brlek, Garon, L. 2010)

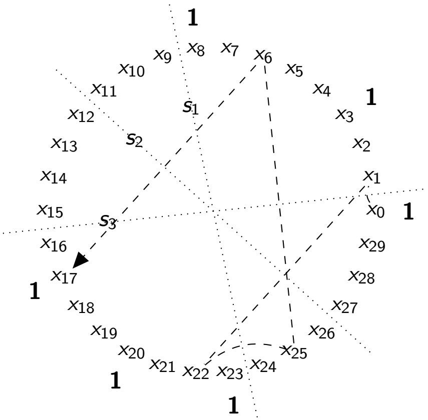
A tile has at most 2 regular square tilings.

Reflexions : s_1, s_2, s_3.
We have

$$I = (s_1 s_3 s_2)^2.$$

thus

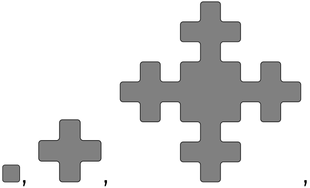
$$s_1 = s_3 s_2 s_1 s_3 s_2.$$



Plan

Fibonacci Tiles are double squares

Complete the sequence :



Fibonacci Tiles are double squares

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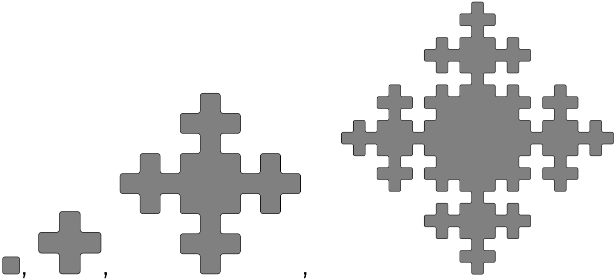


, , , , ...

```
sage: p = words.fibonacci_tile(2); print p
3230301030323212323032321210121232121010301012101030
sage: p.finite_differences().finite_differences()
word: 2022020220220202202022020220220202202202...
sage: words.FibonacciWord([2,0])
word: 2022020220220202202022020220220202202202...
```

Fibonacci Tiles

Theorem (Blondin Massé, Brlek, L., Mendès France, 2011)

The limit ratio between the area of the n-th Fibonacci tile $A(n)$ and the area of its convex hull $H(n)$ is

$$\lim_{n\to\infty} \frac{A(n)}{H(n)} = 2 - \sqrt{2} = 0.58578643\cdots$$

Theorem (Blondin Massé, Brlek, L., Mendès France, 2012)

The fractal dimension of the n-th Fibonacci tile is

$$d = \frac{\log(2+\sqrt{5})}{\log(1+\sqrt{2})} = 1.637938210\cdots$$

Christoffel Tiles are double squares

Let λ defined by

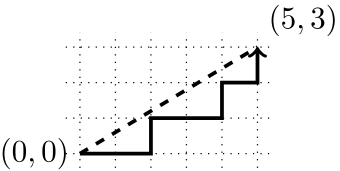
$$\mathbf{0} \mapsto \mathbf{0301}, \mathbf{1} \mapsto \mathbf{01}, \mathbf{2} \mapsto \mathbf{2123}, \mathbf{3} \mapsto \mathbf{23}.$$

Let $\overline{\alpha} = \alpha + 2 \in \mathbb{Z}_4$ for all $\alpha \in \mathbb{Z}_4$.

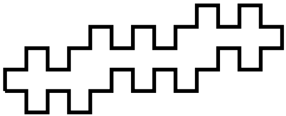
Theorem (Blondin Massé, Brlek, Garon, L., 2011)

Let $w = \mathbf{0}v\mathbf{1} \in \{\mathbf{0}, \mathbf{1}\}^*$.

- *w is a Christoffel word if and only if $\lambda(w\overline{w})$ is a double square.*



$w = \mathbf{00100101}$

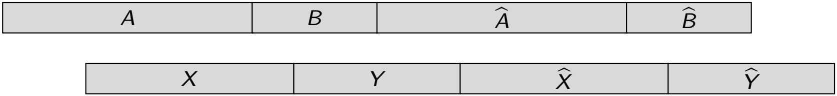


$\lambda(w\overline{w})$

Plan

Periods in the boundary of double square tiles

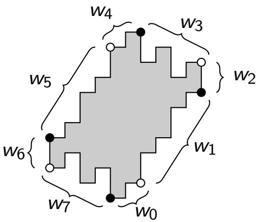
Let $AB\widehat{A}\widehat{B} \equiv XY\widehat{X}\widehat{Y}$ be the factorizations of a double square tile.



Periods in the boundary of double square tiles

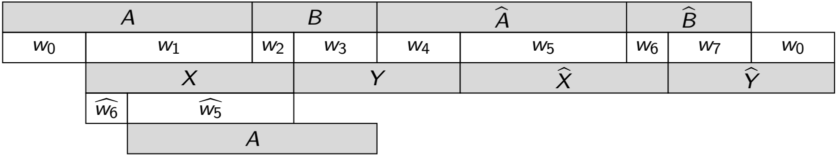
Let $AB\widehat{A}\widehat{B} \equiv XY\widehat{X}\widehat{Y}$ be the factorizations of a double square tile.

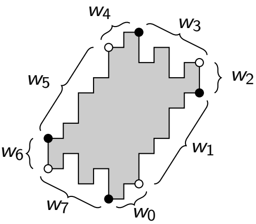
A		B		$\widehat{A}$		$\widehat{B}$		
w_0	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_0
	X		Y		$\widehat{X}$		$\widehat{Y}$	



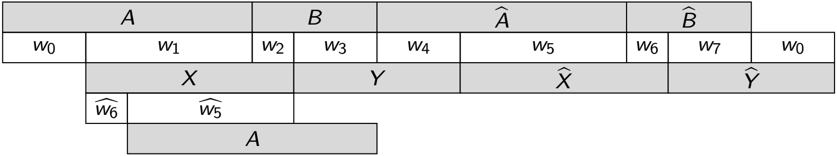
Periods in the boundary of double square tiles

Let $AB\widehat{A}\widehat{B} \equiv XY\widehat{X}\widehat{Y}$ be the factorizations of a double square tile.





Let $AB\widehat{A}\widehat{B} \equiv XY\widehat{X}\widehat{Y}$ be the factorizations of a double square tile.



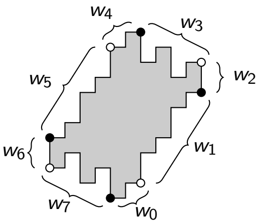
In general

- $d_i = |w_{i-1}| + |w_{i+1}|$ is a period of $w_{i-1}w_iw_{i+1}$.

Hence we write

- $w_i = (u_iv_i)^{n_i}u_i$ where $|u_iv_i| = d_i$.

Remark : u_i and v_i always exist even if $|w_i| < d_i$.

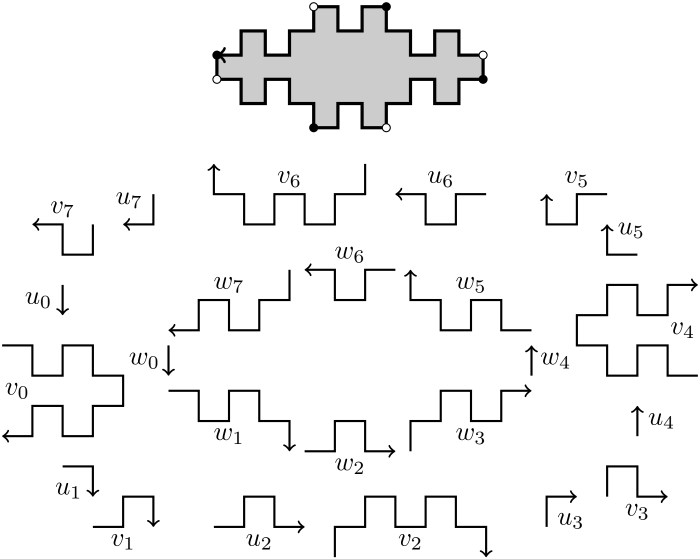


Double Square in Sage free software

Double Square tile from the words (w_0, w_1, w_2, w_3) :

```
sage: from sage.combinat.double_square_tile import DoubleSquare
sage: DoubleSquare(([3,2], [3], [0,3], [0,1,0,3,0]))
Double Square Tile
  w0 = 32                 w4 = 10
  w1 = 3                  w5 = 1
  w2 = 03                 w6 = 21
  w3 = 01030              w7 = 23212
(|w0|, |w1|, |w2|, |w3|) = (2, 1, 2, 5)
(d0, d1, d2, d3)         = (6, 4, 6, 4)
(n0, n1, n2, n3)         = (0, 0, 0, 1)
```

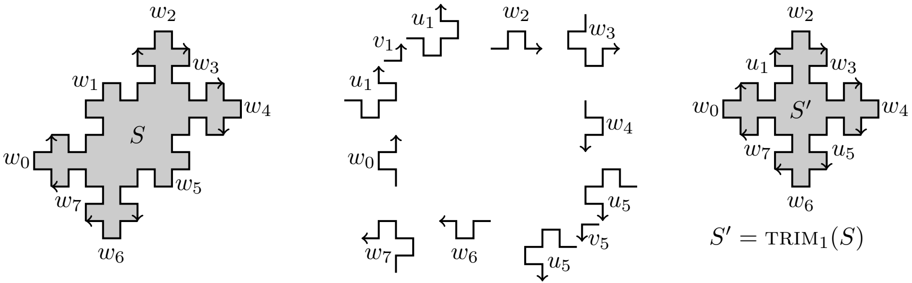
The factor u_i and v_i



TRIM_i : removes a period in w_i and w_{i+4}

Let $S = (w_0, w_1, w_2, w_3, w_4, w_5, w_6, w_7)$.

$$\mathrm{TRIM}_0(S) = ((u_0 v_0)^{n_i - 1} u_0, w_1, w_2, w_3, (u_4 v_4)^{n_4 - 1} u_4, w_5, w_6, w_7)$$



$S' = \mathrm{TRIM}_1(S)$

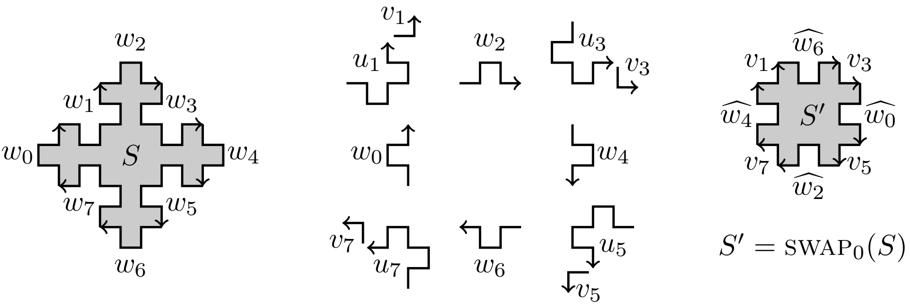
and its conjugates

- $\mathrm{SHIFT}(S) = (w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_0)$,
- $\mathrm{TRIM}_i(S) = \mathrm{SHIFT}^{-i} \circ \mathrm{TRIM}_0 \circ \mathrm{SHIFT}^i(S)$,

SWAP_i : the curious involution

Let $S = (w_0, w_1, w_2, w_3, w_4, w_5, w_6, w_7)$.

$\mathrm{SWAP}_0(S) = (\widehat{w_4}, (v_1 u_1)^{n_1} v_1, \widehat{w_6}, (v_3 u_3)^{n_3} v_3, \widehat{w_0}, (v_5 u_5)^{n_5} v_5, \widehat{w_2}, (v_7 u_7)^{n_7} v_7)$

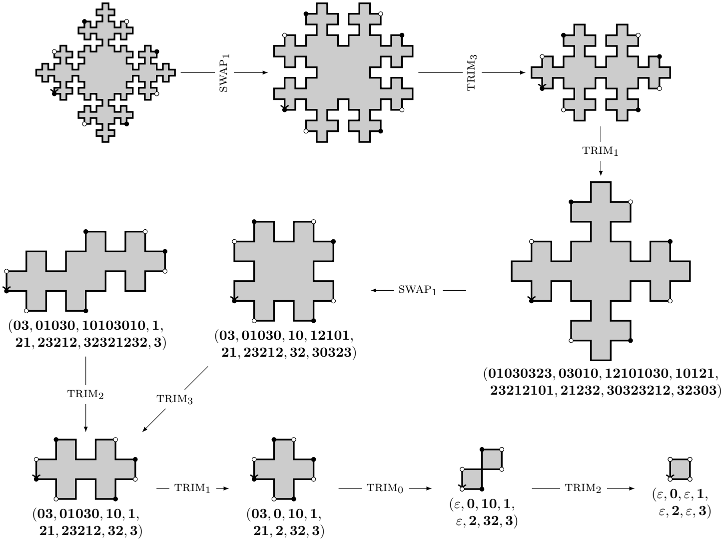


and its conjugates

- $\mathrm{SHIFT}(S) = (w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_0)$,
- $\mathrm{SWAP}_i(S) = \mathrm{SHIFT}^{-i} \circ \mathrm{SWAP}_0 \circ \mathrm{SHIFT}^i(S)$,

Theorem (Blondin Massé, Brlek, Garon, L.)

Every double square tile *reduces to a square tile* *with* TRIM *and* SWAP.



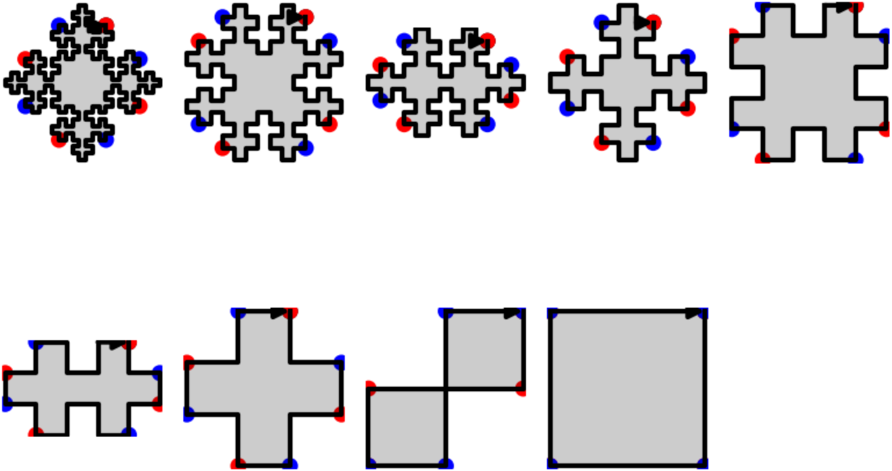
Double Square in Sage free software

```
sage: D = DoubleSquare(words.christoffel_tile(4,7))
sage: D.reduction()
['TRIM_1', 'TRIM_1', 'TRIM_2', 'TRIM_1', 'TRIM_0', 'TRIM_2']
sage: D.trim(1)
Double Square Tile
  w0 = 03                        w4 = 21
  w1 = 010301010301030           w5 = 232123232123212
  w2 = 10103010                  w6 = 32321232
  w3 = 1                         w7 = 3
(|w0|, |w1|, |w2|, |w3|) = (2, 15, 8, 1)
(d0, d1, d2, d3)         = (16, 10, 16, 10)
(n0, n1, n2, n3)         = (0, 1, 0, 0)
```

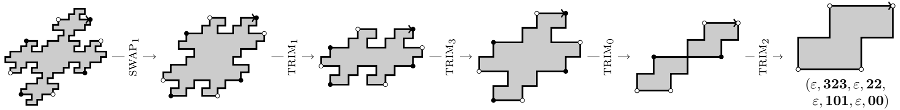
Double Square in Sage free software

Plot a double square tile and its reduction :

```
sage: D = DoubleSquare((34,21,34,21))
sage: D.plot_reduction(ncols=5)
```



Reduction of double square tiles



Moreover,

- The transformations TRIM_i and SWAP_i are invertible.
- The transformations TRIM_i^{-1} and SWAP_i^{-1} preserve palindromes.

Proposition (Blondin Massé, Brlek, Garon, L.)

Let $AB\widehat{A}\widehat{B} \equiv XY\widehat{X}\widehat{Y}$ be the boundary of a double square D. If D reduces to the unit square tile, then

- *A, B, X and Y are palindromes,*
- *D is invariant under a rotation of 180 degrees.*

Plan

Open problems

Some problems are left open :

- Find an algorithm that decides whether a polyomino is prime.
- If $\alpha\alpha$ appears in the boundary word of a double square tile D, where $\alpha \in \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}\}$, then D is not prime.
- Prove that if $S \circ P$ is a square tile, then so is P.
- Describe the distribution and the proportion of prime square tiles of half-perimeter n as n goes to infinity.
- Extend the results to 8-connected polyominoes.
- Extend the results to continuous paths and tiles.
- Understand the function $(|w_0|, |w_1|, |w_2|, |w_3|) \mapsto$ double square.
- Understand the tree of double squares under SWAP_i and EXTEND_i.

Double Square in Sage free software

Double Square tile from the lengths of the w_i :

```
sage: from sage.combinat.double_square_tile import DoubleSquare
sage: DoubleSquare((4,7,4,7))
Double Square Tile
  w0 = 3232           w4 = 1010
  w1 = 1212323        w5 = 3030101
  w2 = 2121           w6 = 0303
  w3 = 0101212        w7 = 2323030
(|w0|, |w1|, |w2|, |w3|) = (4, 7, 4, 7)
(d0, d1, d2, d3)         = (14, 8, 14, 8)
(n0, n1, n2, n3)         = (0, 0, 0, 0)
```

Credits

- This research was driven by computer exploration using the open-source mathematical software `Sage`.
- Les images de ce document ont été produites à l'aide de `pgf/tikz`.