An Arithmetic and Combinatorial Approach to 3D Discrete Lines

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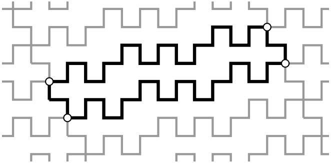
LIAFA
Université Paris-Diderot
Journée de rentrée de l'équipe automates
5 octobre 2012

Joint work with Valérie Berthé

No Double Square tiles talk today !



Talk on double square tiles is postponed for next Wednesday, October 10th (Journée de la rentrée du LIAFA).

Plan

Plan

2D Discrete Lines

$$0 < -2\,x + 5\,y \leq 7$$

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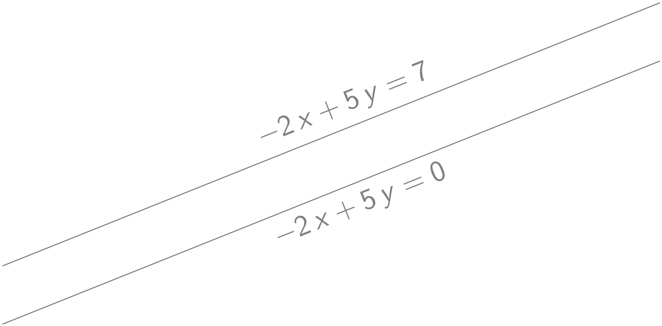
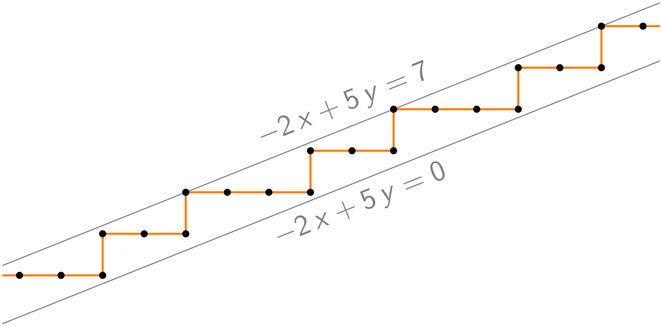


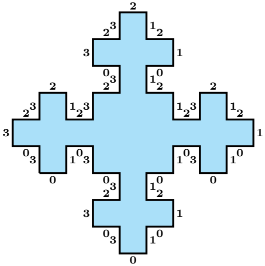
2D Discrete Lines

$$0 < -2\,x + 5\,y \leq 7$$



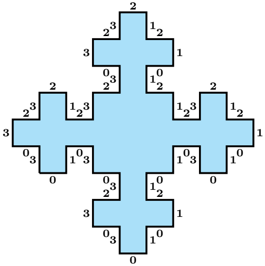
The line is 4-connected minimal and tiles the plane $\mathbb{Z}^2$ by translation.
For irrational slope, the factor complexity is $p(n) = n + 1$.

Factor Complexity

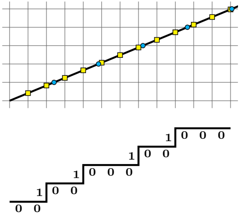


n	$p(n)$	$L(n)$
0	1	$\{\varepsilon\}$
1	4	$\{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}\}$
2	8	$\{\mathbf{01}, \mathbf{03}, \mathbf{10}, \mathbf{12}, \mathbf{21}, \mathbf{23}, \mathbf{30}, \mathbf{32}\}$
3	16	$\{\mathbf{030}, \mathbf{212}, \mathbf{301}, \mathbf{032}, \mathbf{103}, \cdots\}$
4	24	$\{\mathbf{1030}, \mathbf{2303}, \mathbf{0121}, \mathbf{1232}, \cdots\}$
5	32	$\{\mathbf{21010}, \mathbf{10303}, \mathbf{32123}, \cdots\}$

Factor Complexity



n	$p(n)$	$L(n)$
0	1	$\{\varepsilon\}$
1	4	$\{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}\}$
2	8	$\{\mathbf{01}, \mathbf{03}, \mathbf{10}, \mathbf{12}, \mathbf{21}, \mathbf{23}, \mathbf{30}, \mathbf{32}\}$
3	16	$\{\mathbf{030}, \mathbf{212}, \mathbf{301}, \mathbf{032}, \mathbf{103}, \cdots\}$
4	24	$\{\mathbf{1030}, \mathbf{2303}, \mathbf{0121}, \mathbf{1232}, \cdots\}$
5	32	$\{\mathbf{21010}, \mathbf{10303}, \mathbf{32123}, \cdots\}$



$c_{1/\sqrt{2}} = \mathbf{0010010001001000100}\cdots$

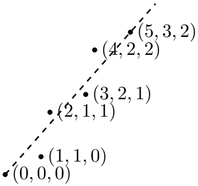
n	$p(n)$	$L(n)$
0	1	$\{\varepsilon\}$
1	2	$\{\mathbf{0}, \mathbf{1}\}$
2	3	$\{\mathbf{01}, \mathbf{10}, \mathbf{00}\}$
3	4	$\{\mathbf{001}, \mathbf{000}, \mathbf{010}, \mathbf{100}\}$
4	5	$\{\mathbf{1001}, \mathbf{1000}, \mathbf{0100}, \mathbf{0001}, \mathbf{0010}\}$
5	6	$\{\mathbf{00010}, \mathbf{01001}, \mathbf{10010}, \mathbf{00100}, \cdots\}$

Plan

Reveillès Discrete Line and standard model of E. Andres

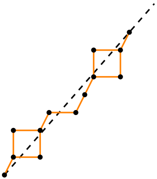
With a directive vector $(5, 3, 2)$ passing trough $(0, 0, 0)$, one gets :

Reveillès (1995) :
26-connexe



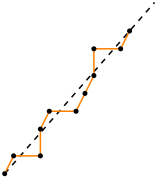
$$-5/2 \leq 2x - 5z < 5/2$$
$$-5/2 \leq 3x - 5y < 5/2$$

Reveillès (1995) :
6-connexe



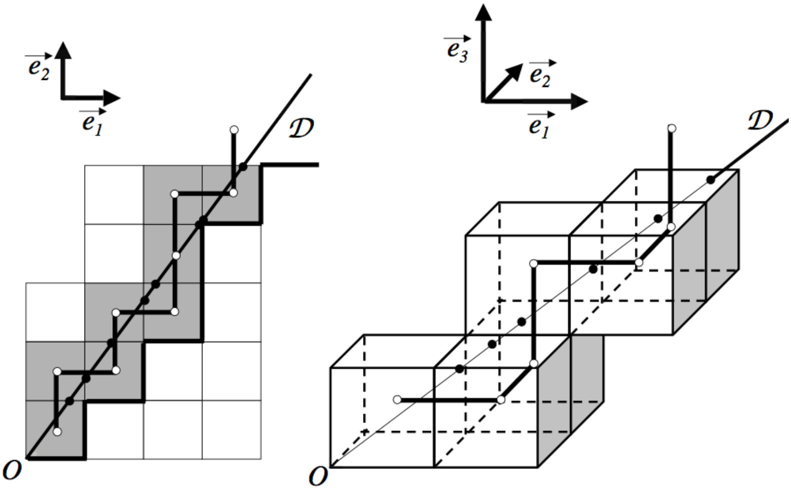
$$-7/2 \leq 2x - 5z < 7/2$$
$$-8/2 \leq 3x - 5y < 8/2$$

Andres (2003) :
6-connexe et minimale



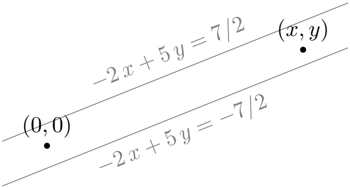
$$-7/2 \leq 2x - 5z < 7/2$$
$$-8/2 \leq 3x - 5y < 8/2$$
$$-5/2 \leq 2y - 3z < 5/2$$

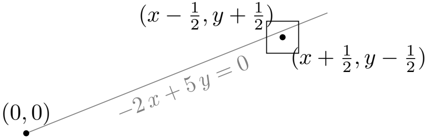
Billard in a cube



Source : J.-P. Borel, Complexity of Degenerated Three Dimensional Billiard Words, Developments in Language Theory 4036 (2006) 386-396.

Andres 3D Discrete Lines correspond to a Billiard word





$$\Updownarrow \qquad\qquad \Updownarrow$$

$$-7/2 < -2x+5y \leq 7/2 \iff -2(x+\tfrac{1}{2})+5(y-\tfrac{1}{2}) \leq 0 < -2(x-\tfrac{1}{2})+5(y+\tfrac{1}{2})$$

$$-2x+5y-\tfrac{7}{2} \leq 0 < -2x+5y+\tfrac{7}{2}$$

The word associated to Andres Discrete Line codes the trajectory of a billiard in a cube.

Factor Complexity of the Billiard word

Theorem (Baryshnikov, 1995 ; Bédaride, 2003)

If both the direction $(\alpha_1, \alpha_2, \alpha_3)$ *and* $(\alpha_1^{-1}, \alpha_2^{-1}, \alpha_3^{-1})$ *are* $\mathbb{Q}$ *independent, the number of factors appearing in the Billiard word in a cube is exactly* $p(n) = n^2 + n + 1$.

Thus, the factor complexity of the word associated to the Andres Discrete Line is quadratic as well.

Question

Given $(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3$ *such that* $\alpha_1 + \alpha_2 + \alpha_3 = 1$*, can we construct an infinite word w over a 3-letter alphabet such that its letter frequencies is* $(\alpha_1, \alpha_2, \alpha_3)$ *and having a linear factor complexity ?*

Plan

Tribonacci Example from Rauzy (1982)

Let σ be the substitution $1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 1$.
Iterating σ on the letter 1 yields increasing prefixes :

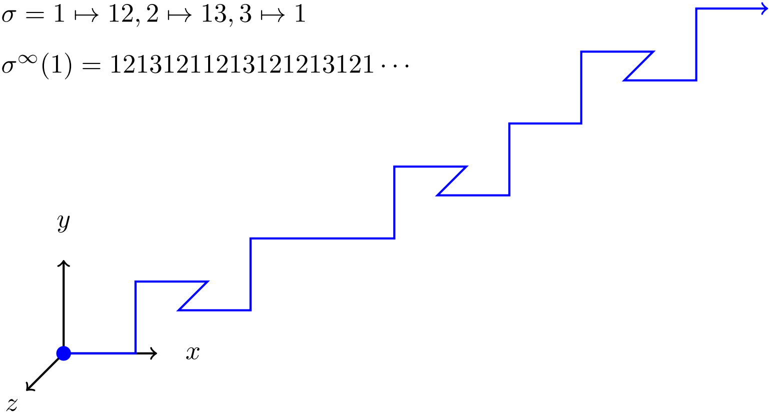
$$\begin{aligned}
\sigma^1(1) &= 12 \\
\sigma^2(1) &= 1213 \\
\sigma^3(1) &= 1213121 \\
\sigma^4(1) &= 1213121121312 \\
\sigma^5(1) &= 121312112131212131211213 \\
\vdots &\quad \vdots \\
\sigma^\infty(1) &= 12131211213121213121121312131211213121...
\end{aligned}$$

$\sigma^\infty(1)$ is the fixed point of σ and $M_\sigma = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ is its incidence matrix.

Tribonacci Example from Rauzy (1982)

$$\sigma = 1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 1$$

$$\sigma^{\infty}(1) = 1213121121312121 3121 \cdots$$



But, this line contains exactly $2n + 1$ factors of length n.

2D : Euclid algorithm on $(11, 4)$

$$\begin{array}{|ccc|} 11 & = & 2 \cdot 4 + 3 \\ 4 & = & 1 \cdot 3 + 1 \\ 3 & = & 3 \cdot 1 + 0 \end{array}$$

$$\boxed{\frac{4}{11} = 0 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{3}}}}$$

$$(11,4) \xleftarrow{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^2} (3,4) \xleftarrow{\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}} (3,1) \xleftarrow{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^3} (0,1)$$

$$\mathbf{w} = \mathbf{w}_0 \xleftarrow{\begin{array}{ccc} a & \mapsto & a \\ b & \mapsto & aab \end{array}} \mathbf{w}_1 \xleftarrow{\begin{array}{ccc} a & \mapsto & ab \\ b & \mapsto & b \end{array}} \mathbf{w}_2 \xleftarrow{\begin{array}{ccc} a & \mapsto & a \\ b & \mapsto & aaab \end{array}} \mathbf{w}_3 = b$$

2D : Euclid algorithm on $(11, 4)$

$$
\begin{array}{ccc}
11 & = & 2 \cdot 4 + 3 \\
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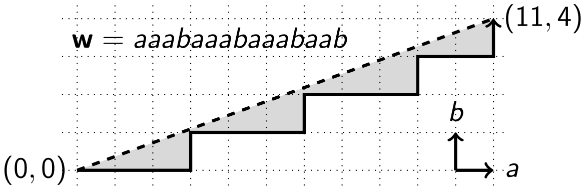
3 & = & 3 \cdot 1 + 0
\end{array}
$$

$$
\frac{4}{11} = 0 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{3}}}
$$



$$
(11,4) \xleftarrow{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^2} (3,4) \xleftarrow{\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}} (3,1) \xleftarrow{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^3} (0,1)
$$

$$
\mathbf{w} = \mathbf{w}_0 \xleftarrow[\;b \mapsto aab\;]{\;a \mapsto a\;} \mathbf{w}_1 \xleftarrow[\;b \mapsto b\;]{\;a \mapsto ab\;} \mathbf{w}_2 \xleftarrow[\;b \mapsto aaab\;]{\;a \mapsto a\;} \mathbf{w}_3 = b
$$

$$
(7,4,6) \xleftarrow{\begin{pmatrix}1&0&1\\0&1&0\\0&0&1\end{pmatrix}} (1,4,6) \xleftarrow{\begin{pmatrix}1&0&0\\0&1&0\\0&1&1\end{pmatrix}} (1,4,2) \xleftarrow{\begin{pmatrix}1&0&0\\0&1&2\\0&0&1\end{pmatrix}} (1,0,2) \xleftarrow{\begin{pmatrix}1&0&0\\0&1&0\\2&0&1\end{pmatrix}} (1,0,0)
$$

$$
\mathbf{w}_0 \xleftarrow{\begin{array}{ccc}1&\mapsto&1\\2&\mapsto&2\\3&\mapsto&13\end{array}} \mathbf{w}_1 \xleftarrow{\begin{array}{ccc}1&\mapsto&1\\2&\mapsto&23\\3&\mapsto&3\end{array}} \mathbf{w}_2 \xleftarrow{\begin{array}{ccc}1&\mapsto&1\\2&\mapsto&2\\3&\mapsto&223\end{array}} \mathbf{w}_3 \xleftarrow{\begin{array}{ccc}1&\mapsto&133\\2&\mapsto&2\\3&\mapsto&3\end{array}} \mathbf{w}_4
$$

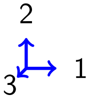
Its (Hausdorff) distance to the euclidean line is 1.3680.

3D : Imitation of Euclid algorithm on $(7, 4, 6)$

$$(7,4,6) \xleftarrow{\begin{pmatrix}1&0&1\\0&1&0\\0&0&1\end{pmatrix}} (1,4,6) \xleftarrow{\begin{pmatrix}1&0&0\\0&1&0\\0&1&1\end{pmatrix}} (1,4,2) \xleftarrow{\begin{pmatrix}1&0&0\\0&1&2\\0&0&1\end{pmatrix}} (1,0,2) \xleftarrow{\begin{pmatrix}1&0&0\\0&1&0\\2&0&1\end{pmatrix}} (1,0,0)$$

$$\mathbf{w}_0 \xleftarrow{\substack{1\mapsto 1\\ 2\mapsto 2\\ 3\mapsto 13}} \mathbf{w}_1 \xleftarrow{\substack{1\mapsto 1\\ 2\mapsto 23\\ 3\mapsto 3}} \mathbf{w}_2 \xleftarrow{\substack{1\mapsto 1\\ 2\mapsto 2\\ 3\mapsto 223}} \mathbf{w}_3 \xleftarrow{\substack{1\mapsto 133\\ 2\mapsto 2\\ 3\mapsto 3}} \mathbf{w}_4$$

$$\mathbf{w} = \mathbf{w}_0 = 1213213132132313$$

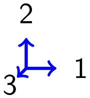


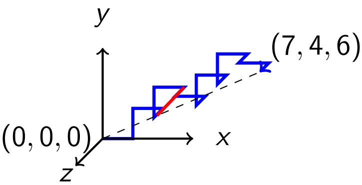
Its (Hausdorff) distance to the euclidean line is 1.3680.

$$(7,4,6) \xleftarrow{\begin{pmatrix}1&0&1\\0&1&0\\0&0&1\end{pmatrix}} (1,4,6) \xleftarrow{\begin{pmatrix}1&0&0\\0&1&0\\0&1&1\end{pmatrix}} (1,4,2) \xleftarrow{\begin{pmatrix}1&0&0\\0&1&2\\0&0&1\end{pmatrix}} (1,0,2) \xleftarrow{\begin{pmatrix}1&0&0\\0&1&0\\2&0&1\end{pmatrix}} (1,0,0)$$

$$\mathbf{w}_0 \xleftarrow{\begin{array}{ccc}1&\mapsto&1\\2&\mapsto&2\\3&\mapsto&13\end{array}} \mathbf{w}_1 \xleftarrow{\begin{array}{ccc}1&\mapsto&1\\2&\mapsto&23\\3&\mapsto&3\end{array}} \mathbf{w}_2 \xleftarrow{\begin{array}{ccc}1&\mapsto&1\\2&\mapsto&2\\3&\mapsto&223\end{array}} \mathbf{w}_3 \xleftarrow{\begin{array}{ccc}1&\mapsto&133\\2&\mapsto&2\\3&\mapsto&3\end{array}} \mathbf{w}_4$$

$$\mathbf{w} = \mathbf{w}_0 = 12132131321321313$$





Its (Hausdorff) distance to the euclidean line is 1.3680.

3D Continued fraction algorithms

Brun's Algorithm : Subtract the second largest to the largest.

$(7, 4, 6) \to (1, 4, 6) \to (1, 4, 2) \to (1, 2, 2) \to (1, 0, 2) \to (1, 0, 1) \to (0, 0, 1).$

Selmer's Algorithm : Subtract the smallest to the largest.

$(7, 4, 6) \to (3, 4, 6) \to (3, 4, 3) \to (3, 1, 3) \to (2, 1, 3) \to (2, 1, 2) \to (1, 1, 2)$
$\to (1, 1, 1) \to (0, 1, 1) \to (0, 0, 1)$

Poincaré's Algorithm : Subtract the smallest to the mid and the mid to the largest.

$$(7, 4, 6) \to (1, 4, 2) \to (1, 2, 1) \to (1, 1, 0) \to (1, 0, 0)$$

Arnoux-Rauzy's Algorithm : Subtract the sum of the two smallest to the largest (not always possible).

$$(7, 4, 6) \to \text{Impossible}$$

Fully subtractive's Algorithm : Subtract the smallest to the other two.

$$(7, 4, 6) \to (3, 4, 2) \to (1, 2, 2) \to (1, 1, 1) \to (1, 0, 0)$$

Plan

Discrépance

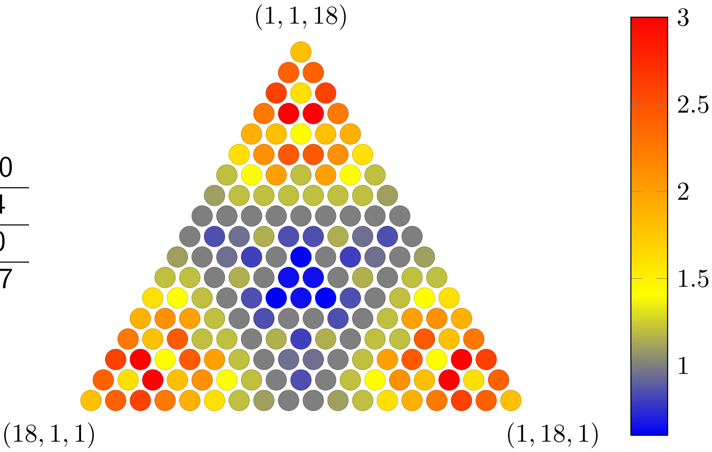
Définition

Soit C un chemin discret définissant une suite $(P_k)_{0\leq k<n}$ de points $P_k = (x_k, y_k, z_k)$ de $\mathbb{Z}^3$ telle que $P_0 = (0,0,0)$ et $P_{n-1} = (a,b,c)$. On définit la discrépance du chemin comme la valeur

$$\max_{0\leq k<n}\left\{\left|\frac{a}{a+b+c}k - x_k\right|, \left|\frac{b}{a+b+c}k - y_k\right|, \left|\frac{c}{a+b+c}k - z_k\right|\right\}.$$

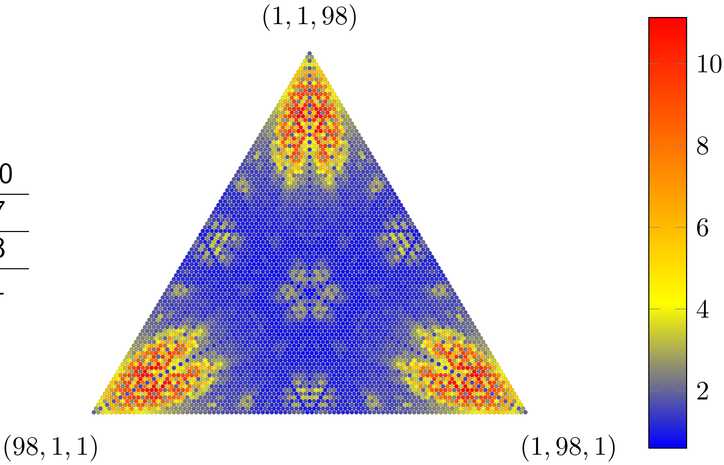
Discrépance pour les triplets d'entiers strictement positifs (a_1, a_2, a_3) tels que $a_1 + a_2 + a_3 = N$ et $N = 20$ pour l'algorithme **Poincaré**.

min	0.6000
moy.	1.484
max	3.000
E.T.	0.6137



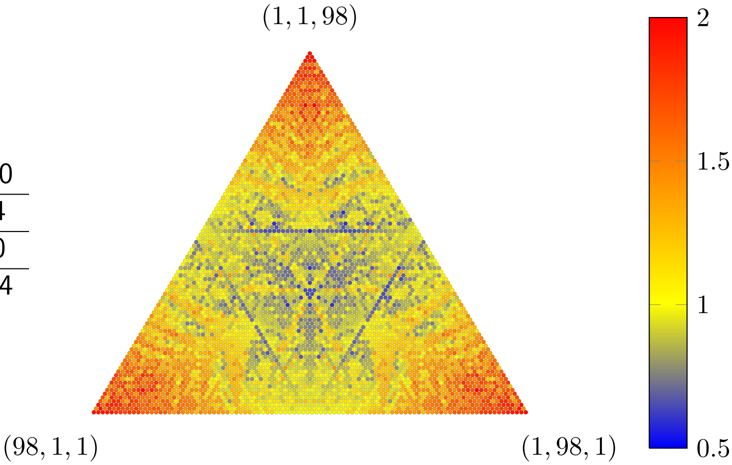
Discrépance pour les triplets d'entiers strictement positifs (a_1, a_2, a_3) tels que $a_1 + a_2 + a_3 = N$ et $N = 100$ pour l'algorithme **Poincaré**.

min	0.6000
moy.	2.527
max	11.13
E.T.	2.261



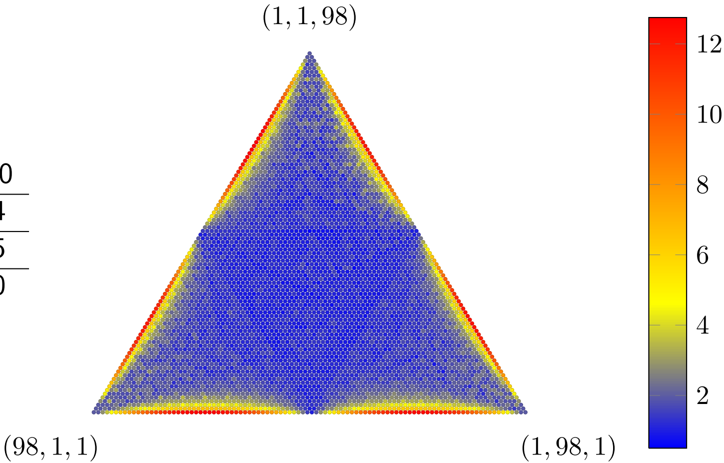
Discrépance pour les triplets d'entiers strictement positifs (a_1, a_2, a_3) tels que $a_1 + a_2 + a_3 = N$ et $N = 100$ pour l'algorithme **Brun**.

min	0.5000
moy.	1.114
max	2.000
E.T.	0.2664



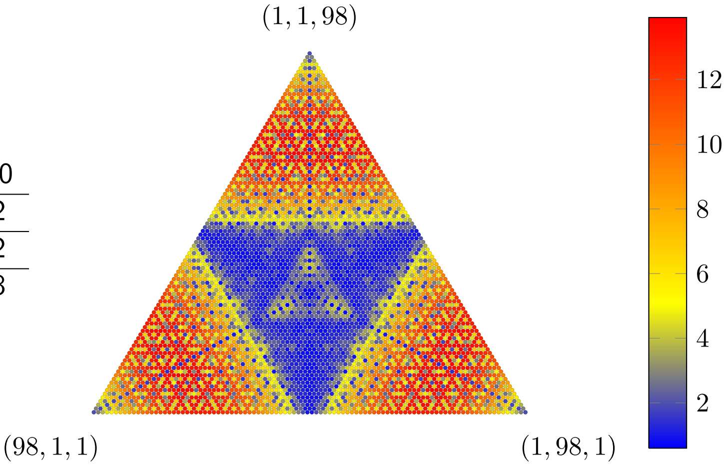
Discrépance pour les triplets d'entiers strictement positifs (a_1, a_2, a_3) tels que $a_1 + a_2 + a_3 = N$ et $N = 100$ pour l'algorithme **Selmer**.

min	0.5000
moy.	2.184
max	12.75
E.T.	2.070



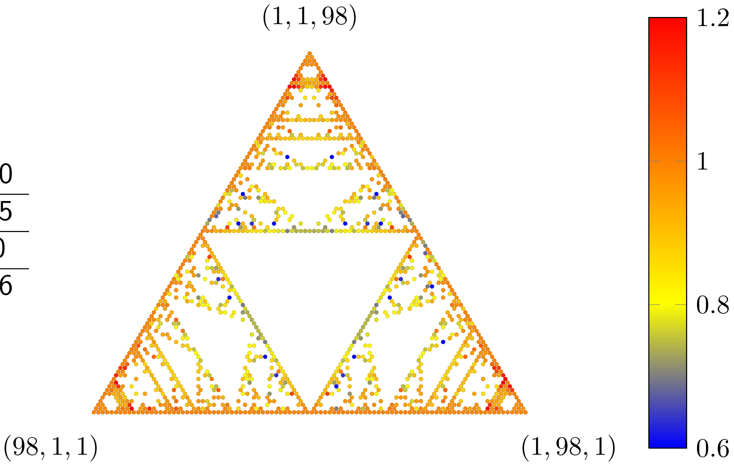
Discrépance pour les triplets d'entiers strictement positifs (a_1, a_2, a_3) tels que $a_1 + a_2 + a_3 = N$ et $N = 100$ pour l'algorithme **Fully subtractive**.

min	0.6000
moy.	5.982
max	13.92
E.T.	4.388



Discrépance pour les triplets d'entiers strictement positifs (a_1, a_2, a_3) tels que $a_1 + a_2 + a_3 = N$ et $N = 100$ pour l'algorithme **Arnoux-Rauzy**.

min	0.6000
moy.	0.9055
max	1.200
E.T.	0.1006



3D Continued fraction algorithms : fusions

Arnoux-Rauzy and Selmer Do Arnoux-Rauzy if possible, otherwise Selmer.

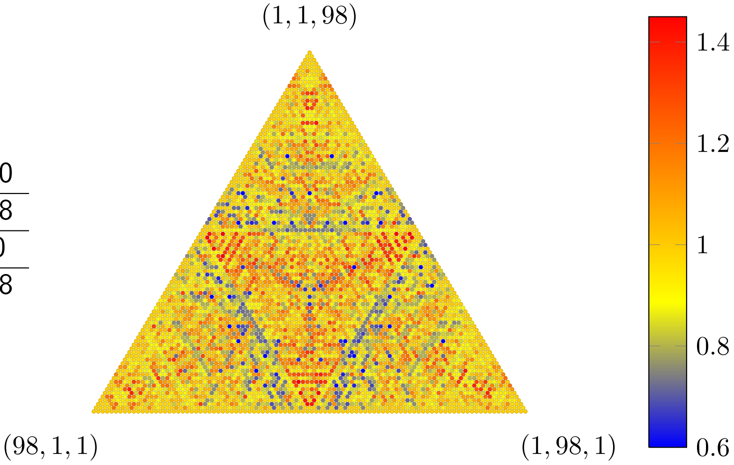
Arnoux-Rauzy and Fully Do Arnoux-Rauzy if possible, otherwise Fully subtractive.

Arnoux-Rauzy and Brun Do Arnoux-Rauzy if possible, otherwise Brun.

Arnoux-Rauzy and Poincaré Do Arnoux-Rauzy if possible, otherwise Poincaré.

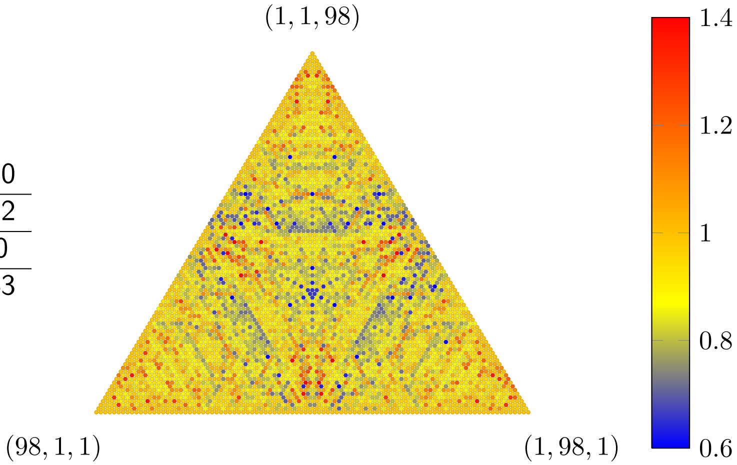
Discrépance pour les triplets d'entiers strictement positifs (a_1, a_2, a_3) tels que $a_1 + a_2 + a_3 = N$ et $N = 100$ pour l'algorithme **Arnoux-Rauzy-Selmer**.

min	0.6000
moy.	0.9678
max	1.450
E.T.	0.1438



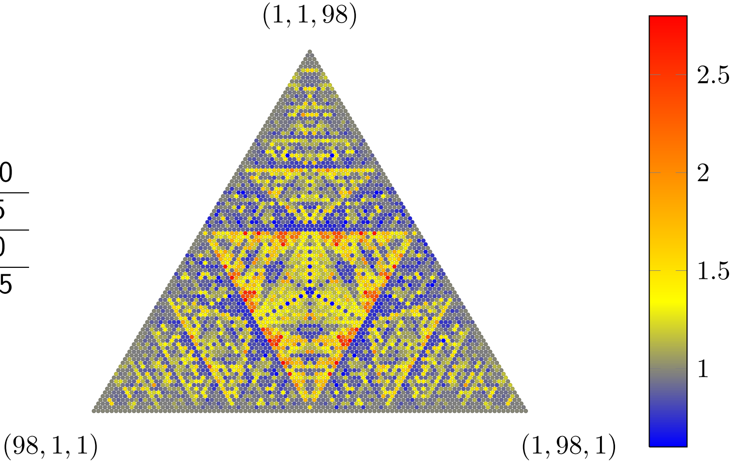
Discrépance pour les triplets d'entiers strictement positifs (a_1, a_2, a_3) tels que $a_1 + a_2 + a_3 = N$ et $N = 100$ pour l'algorithme **Arnoux-Rauzy-Brun**.

min	0.6000
moy.	0.9132
max	1.400
E.T.	0.1143



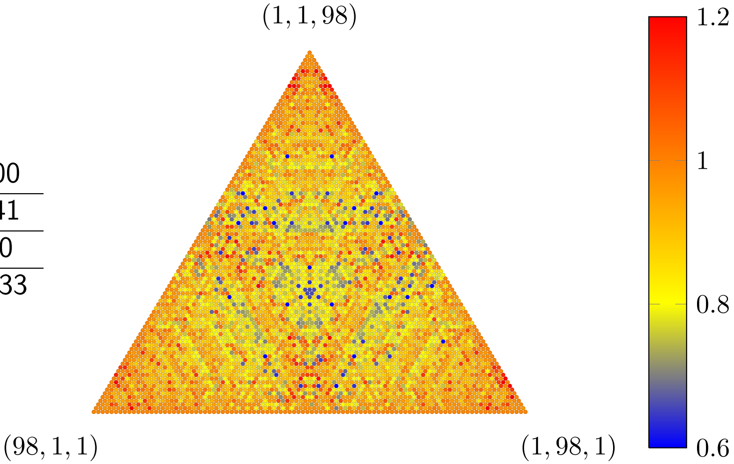
Discrépance pour les triplets d'entiers strictement positifs (a_1, a_2, a_3) tels que $a_1 + a_2 + a_3 = N$ et $N = 100$ pour l'algorithme **Arnoux-Rauzy-Fully subtractive**.

min	0.6000
moy.	1.095
max	2.800
E.T.	0.3105



Discrépance pour les triplets d'entiers strictement positifs (a_1, a_2, a_3) tels que $a_1 + a_2 + a_3 = N$ et $N = 100$ pour l'algorithme **Arnoux-Rauzy-Poincaré**.

min	0.6000
moy.	0.8941
max	1.200
E.T.	0.09733



Plan

Questions

- Prove or disprove that a multidimensional continued fraction algorithm may lead to 3D discrete lines having linear complexity.
- Study ergodic properties of those previous fusion algorithms.