An Arithmetic and Combinatorial Approach to three-dimensional Discrete Lines

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Université du Québec à Montréal

16th IAPR International Conference on
Discrete Geometry for Computer Imagery
6-8 April 2011
Nancy, France

Outline

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1. Goals

2. Preliminaries
   - 2D and 3D Discrete Lines defined by offset
   - Tribonacci Example from Rauzy (1982)

3. Arithmetic and Combinatorial Approach to 3D Discrete Lines

4. Experimental results

5. Conclusion

Goals of this talk

The Goal of this talk is to construct 3D 6-connected discrete lines that behave like 2D ones, that is :

- minimal for 6-connectedness,
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- minimal for 6-connectedness,
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In our approach, we also want the 3D lines to

- be defined by a dynamical system,
- be generated by substitutions.

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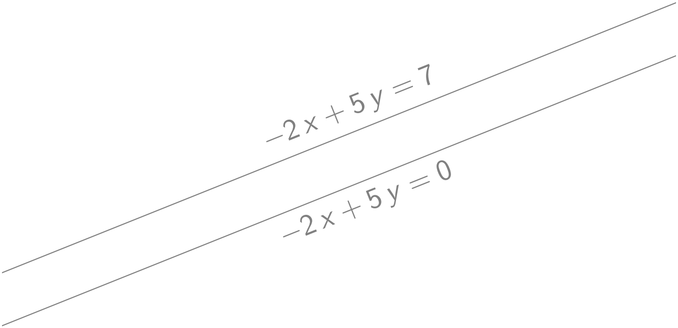

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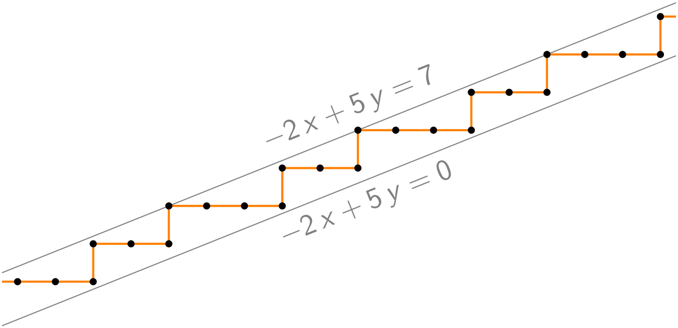

The line is 4-connected minimal and tiles the plane $\mathbb{Z}^2$ by translation.

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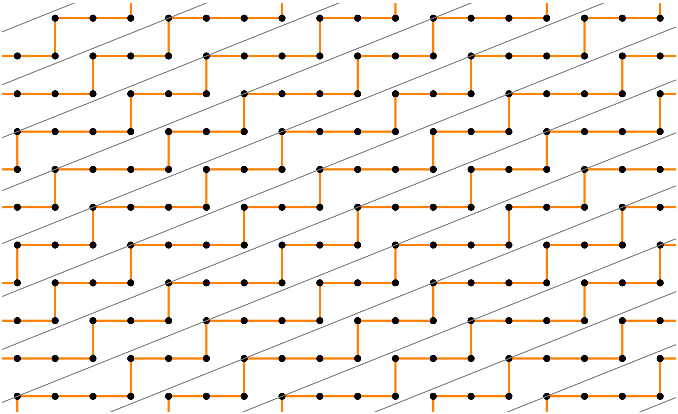

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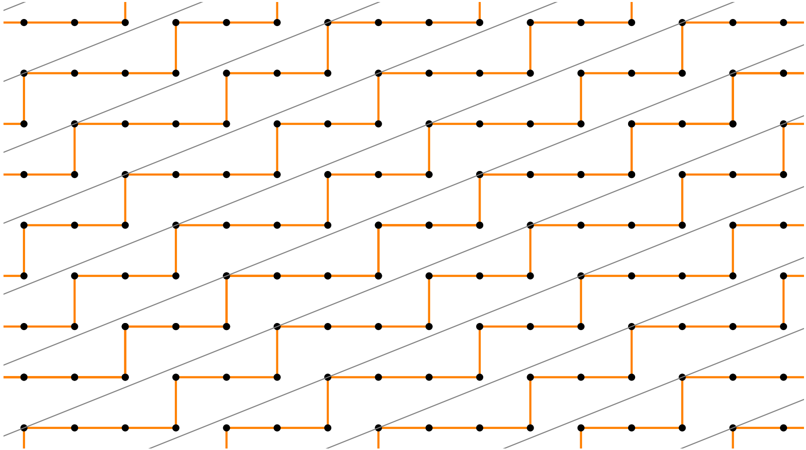
One good thing about 2d lines



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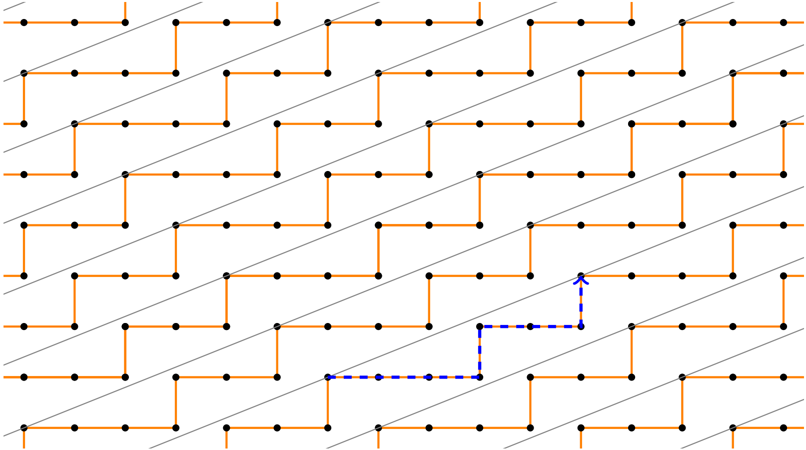

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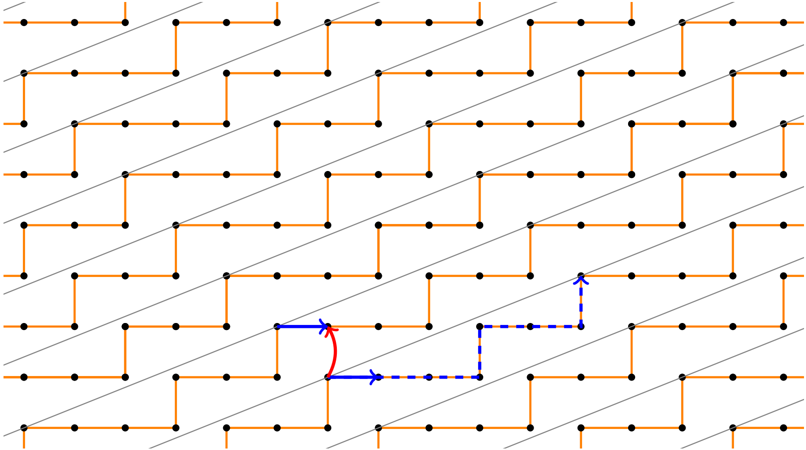

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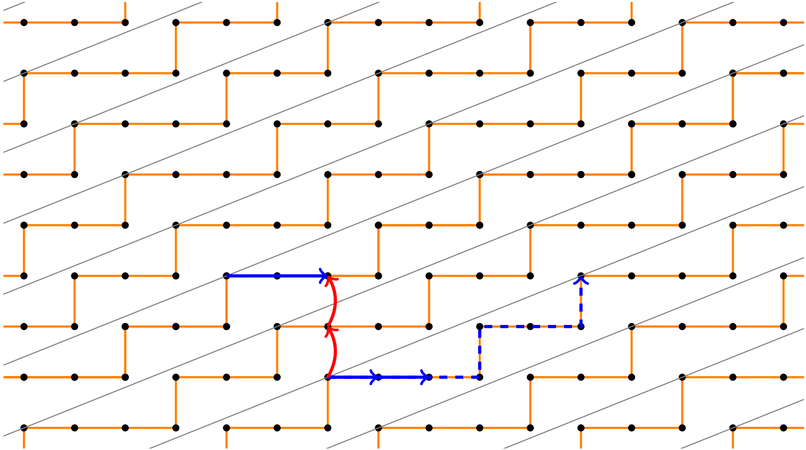

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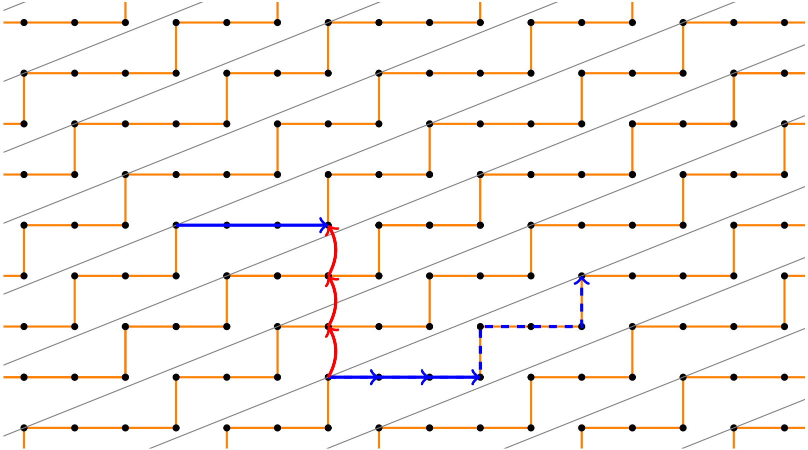

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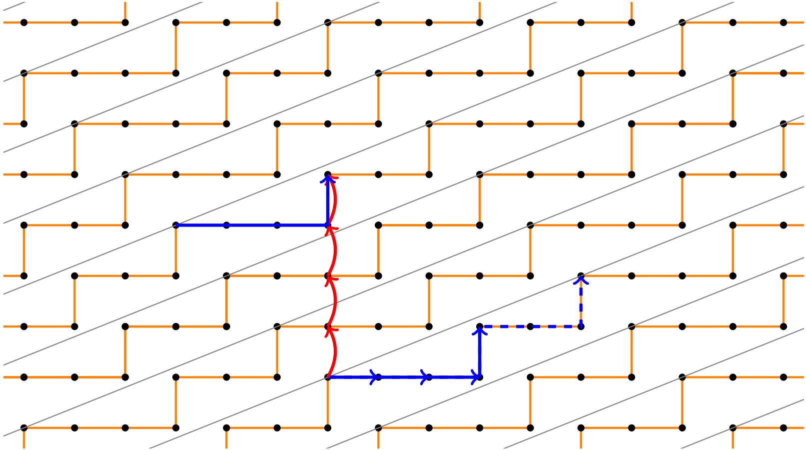

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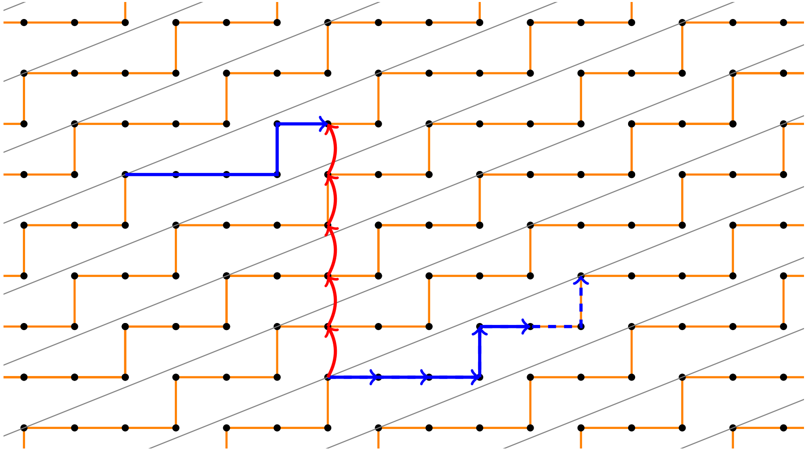

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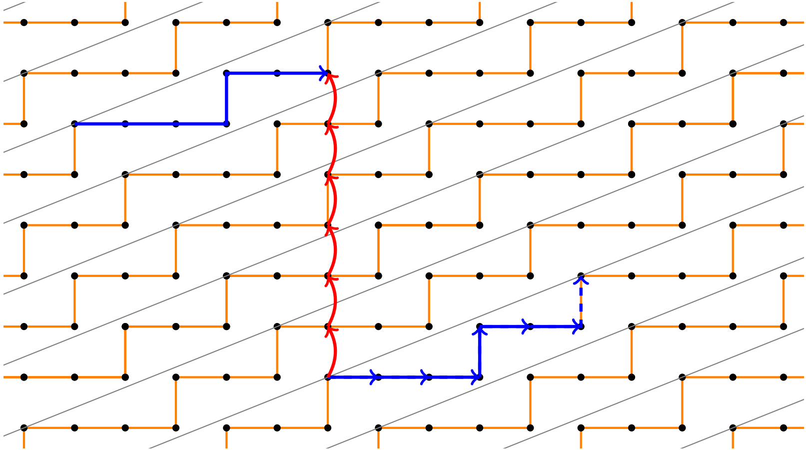

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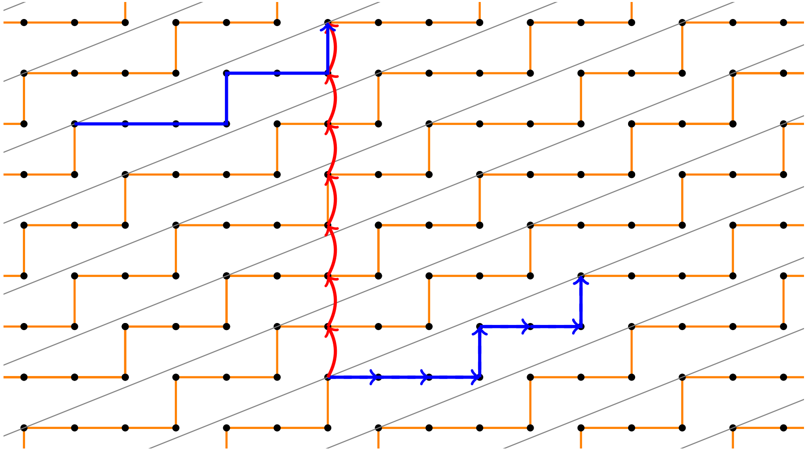

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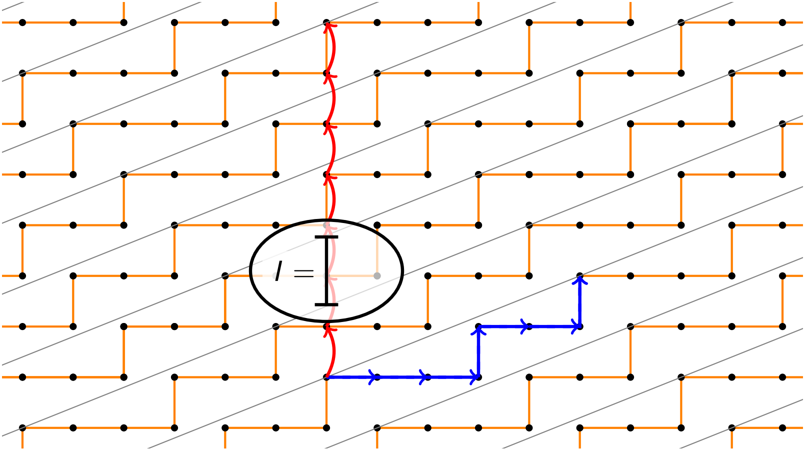

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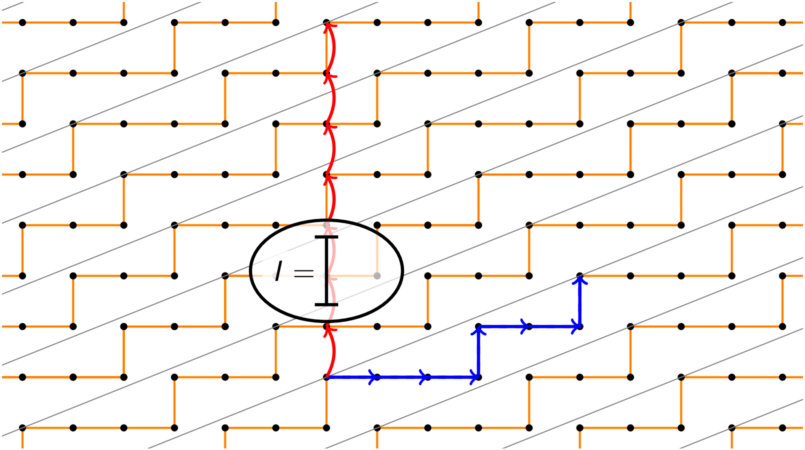

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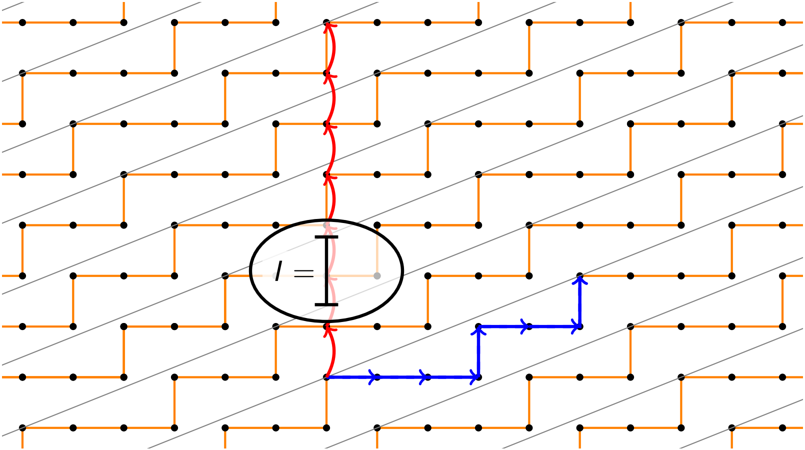

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The interval I is the fundamental domain of a dynamical system.
Can we get such a dynamical system for 3d discrete lines ?

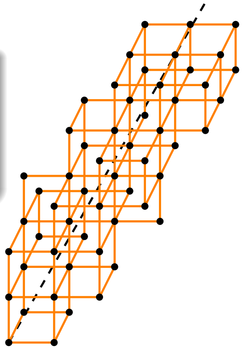
Cylinder offset approach for 3D Discrete Lines

Proposition (Brimkov et al. 2008)

Let D be a digital line defined by cylindrical offset of radius ω. If $\omega \geq \sqrt{3}$, then D is at least 18-connected.

Brimkov et al. 2008 :

> *"Moreover, the experiments showed that if ω is chosen to be equal to $\sqrt{2}$ or to 1 rather than to $\sqrt{3}$, then D is still always 6-connected (and thus also 18- and 26-connected)."*

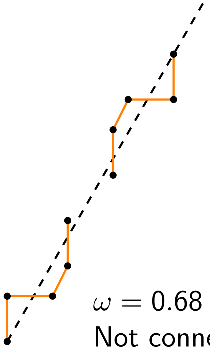


$\omega = \sqrt{3} = 1.73$
Proved to be 18-connected

Cylinder offset approach for 3D Discrete Lines

With a directive vector $(2, 3, 5)$ passing trough $(0, 0, 0)$, one gets the following formula for the cylinder :

$$\frac{17}{19}x^2 - \frac{6}{19}xy - \frac{10}{19}xz + \frac{29}{38}y^2 - \frac{15}{19}yz + \frac{13}{38}z^2 < \omega^2.$$



$\omega = 0.68$

Not connected

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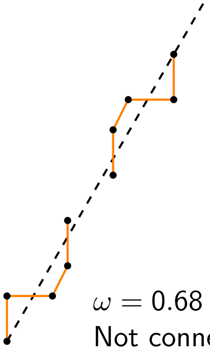

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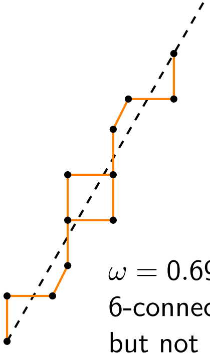

$\omega = 0.69$
6-connected
but not minimal

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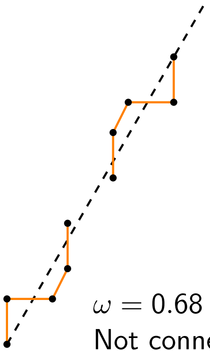

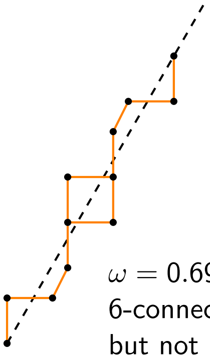

Which shape (instead of cylinder) would give a minimal connected 3d line ?

Tribonacci Example from Rauzy (1982)

Let σ be the substitution $1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 1$.
Iterating σ on the letter 1 yields increasing prefixes :

$$\begin{aligned}
\sigma^1(1) &= 12\\
\sigma^2(1) &= 1213\\
\sigma^3(1) &= 1213121\\
\sigma^4(1) &= 1213121121312\\
\sigma^5(1) &= 121312112131212131211213\\
\vdots \quad & \quad \vdots\\
\sigma^\infty(1) &= 12131211213121213121121312131211213121 \ldots
\end{aligned}$$

$\sigma^\infty(1)$ is the fixed point of σ and $M_\sigma = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ is its incidence matrix.

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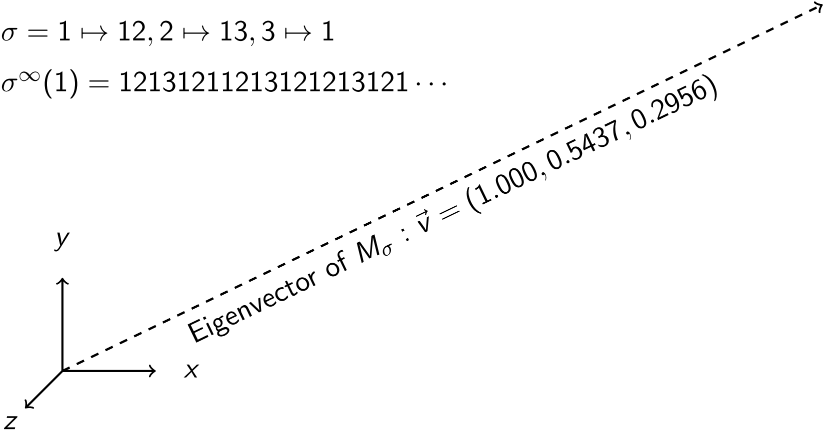

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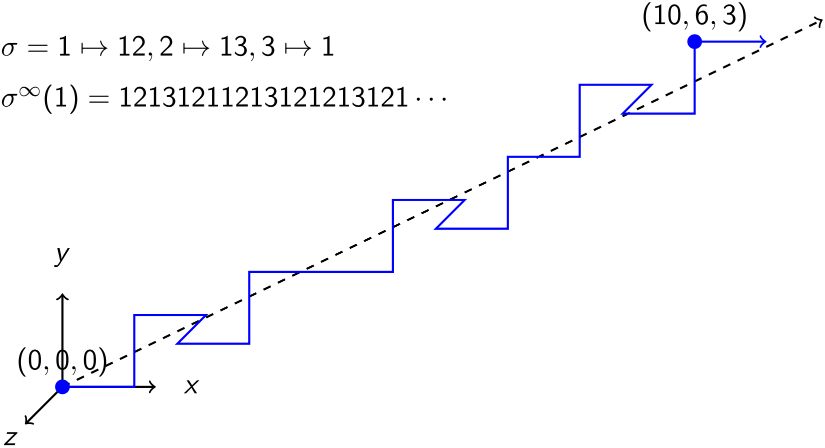

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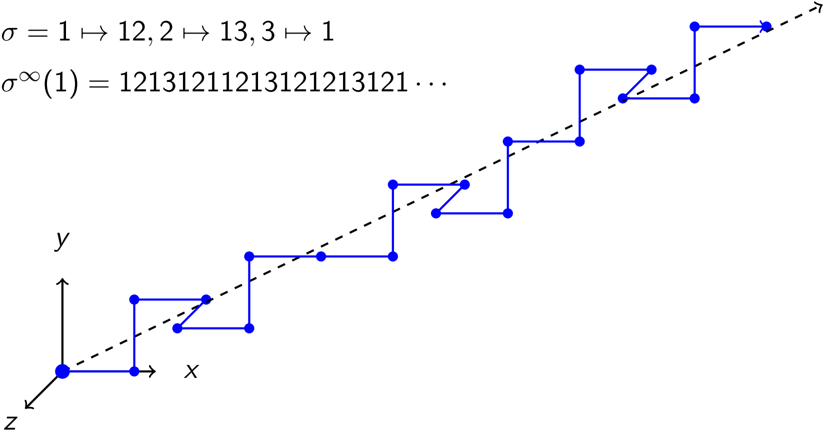

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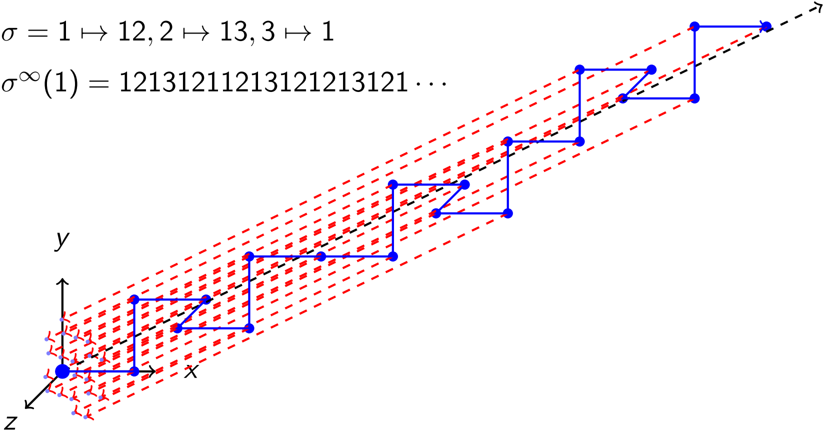

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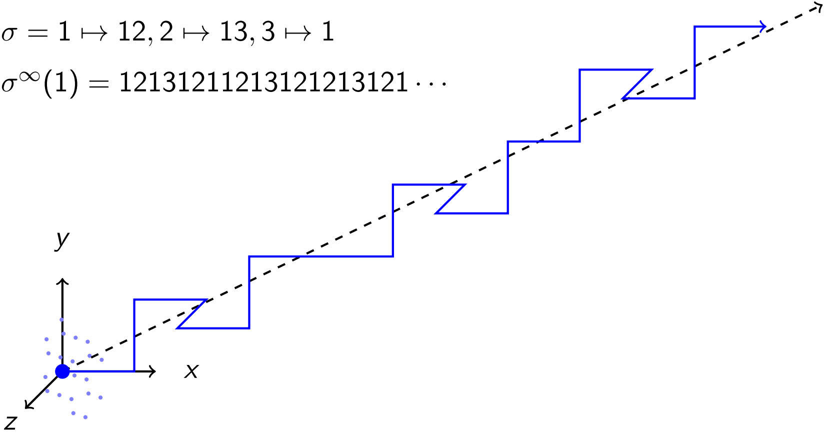

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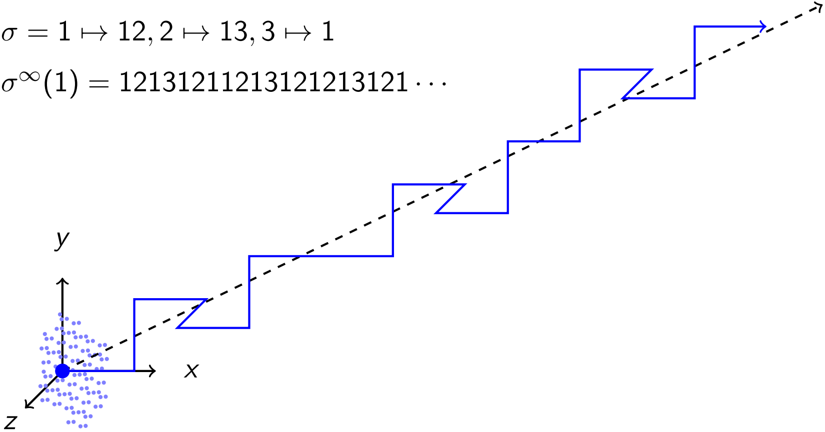

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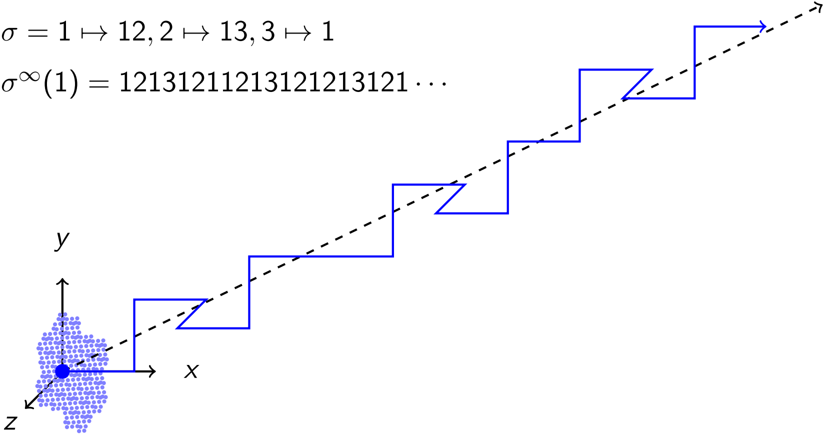

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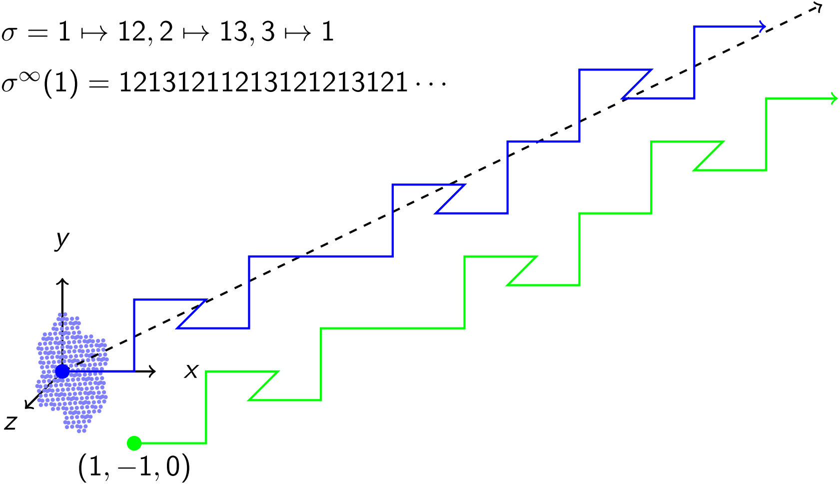

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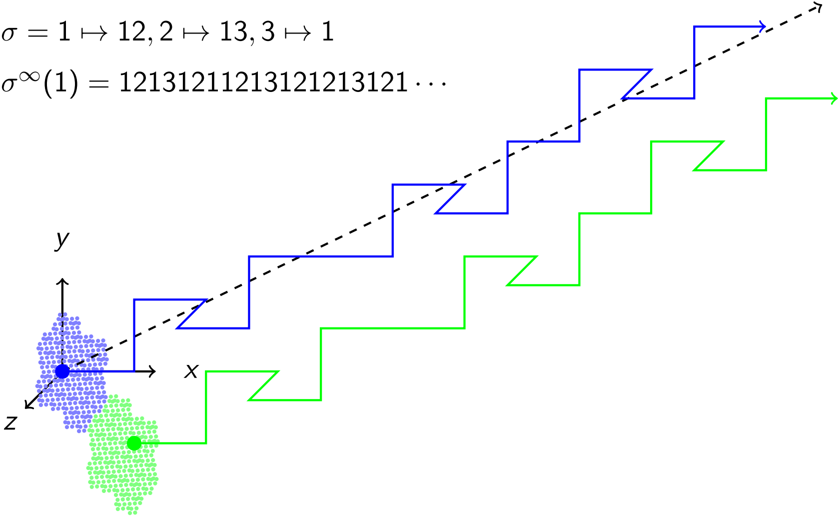

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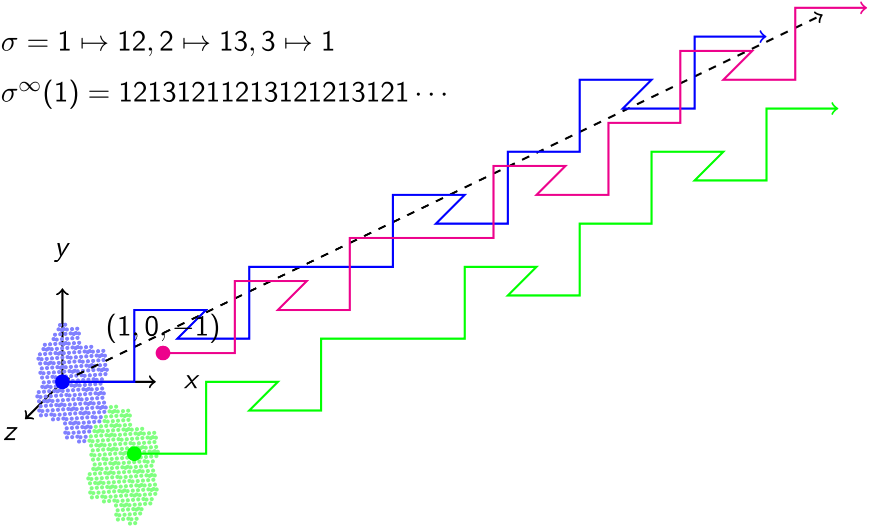

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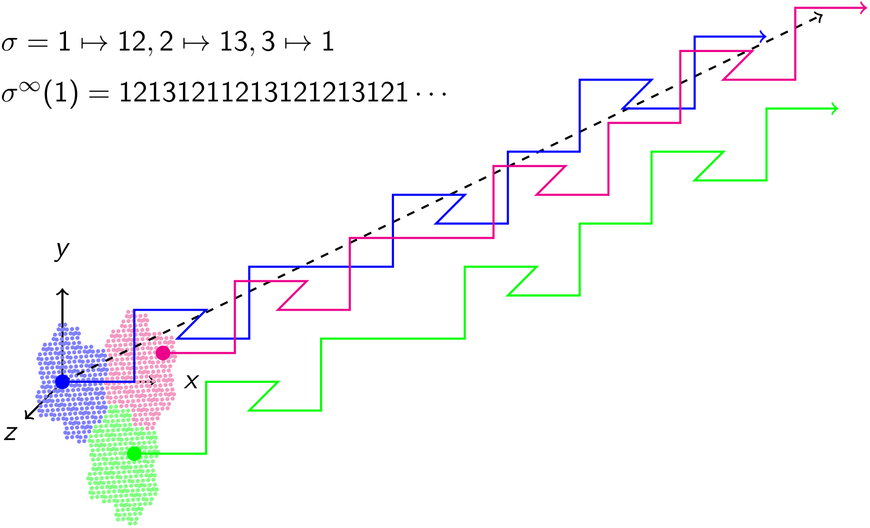

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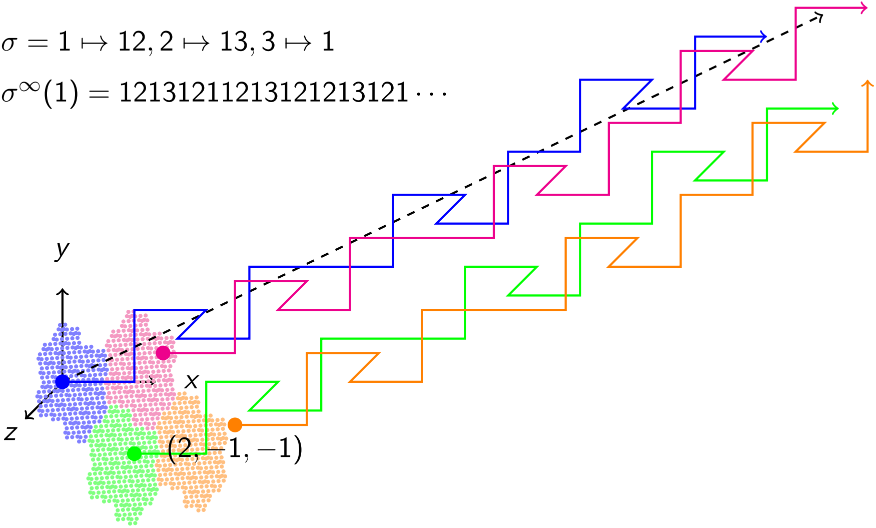

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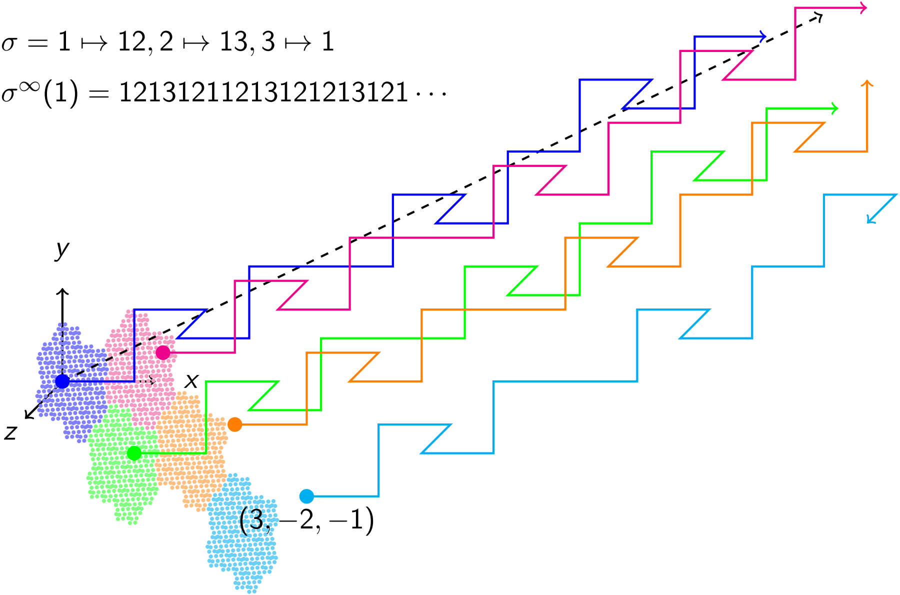

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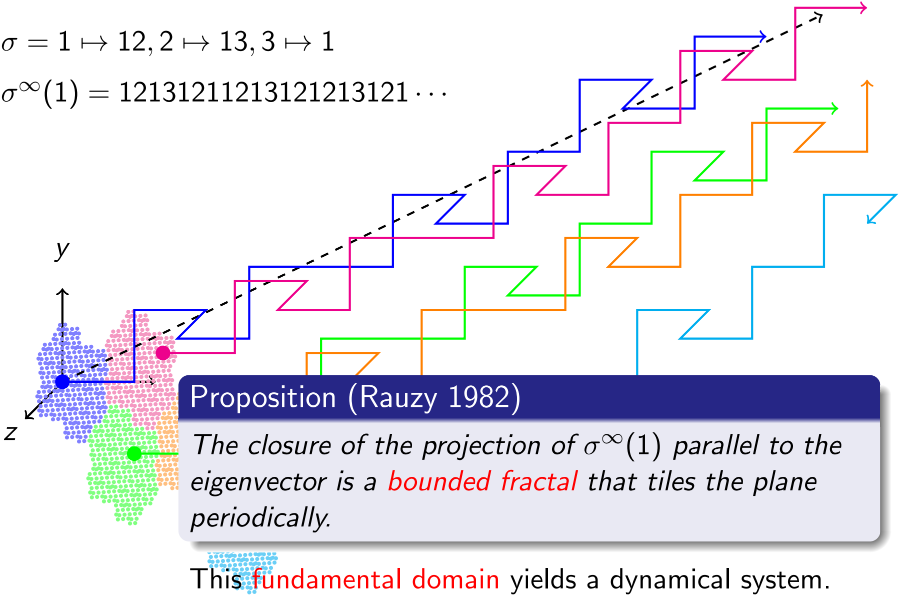

Proposition (Rauzy 1982)

The closure of the projection of $\sigma^\infty(1)$ parallel to the eigenvector is a bounded fractal that tiles the plane periodically.

This fundamental domain yields a dynamical system.

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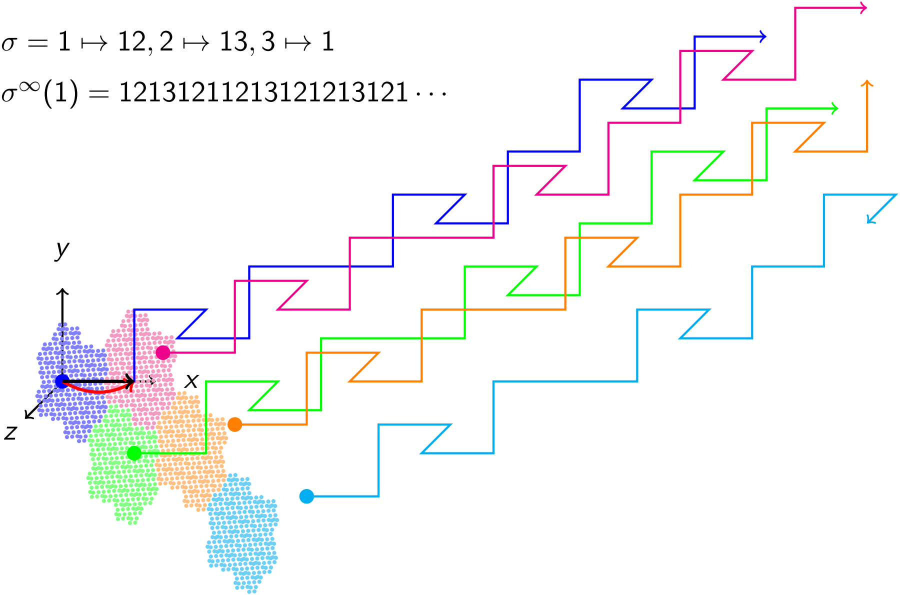

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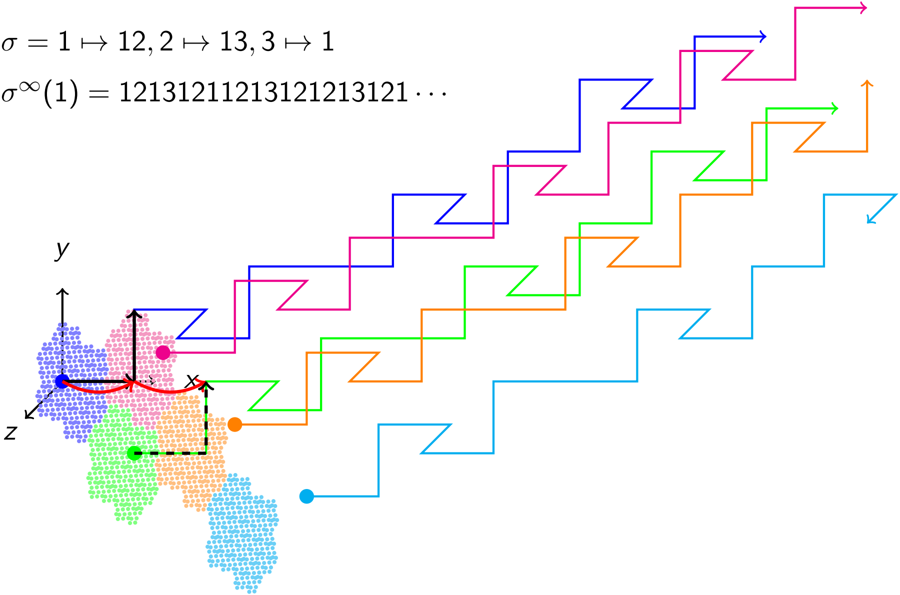

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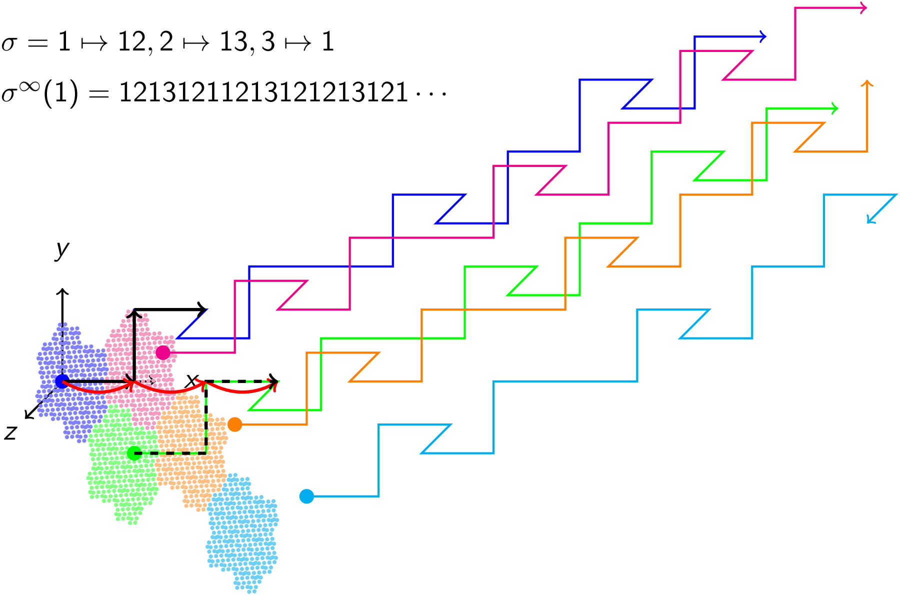

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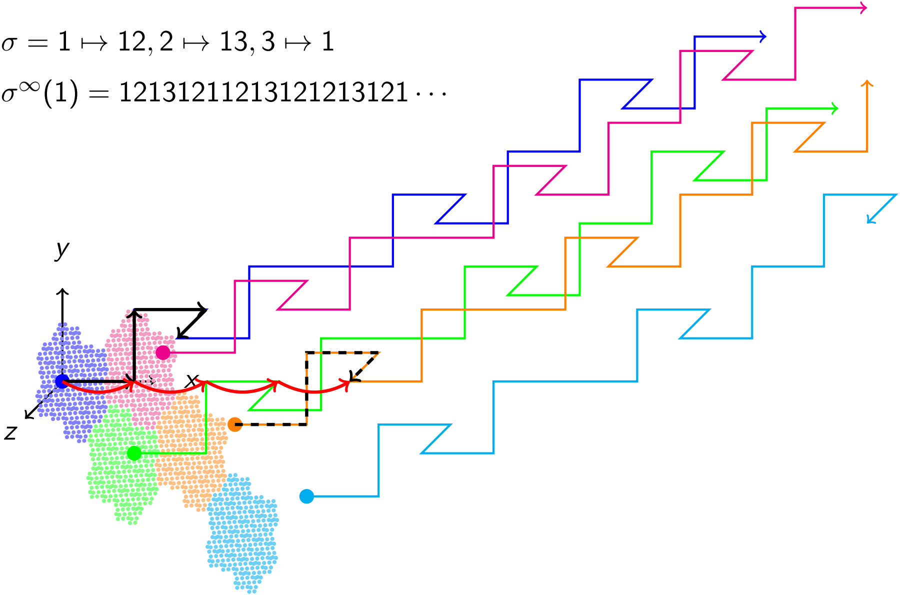

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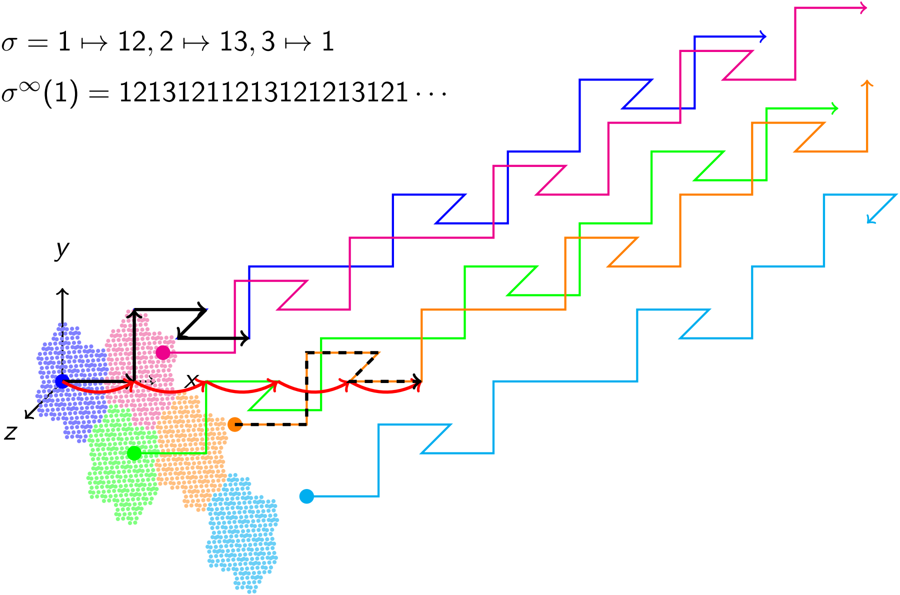

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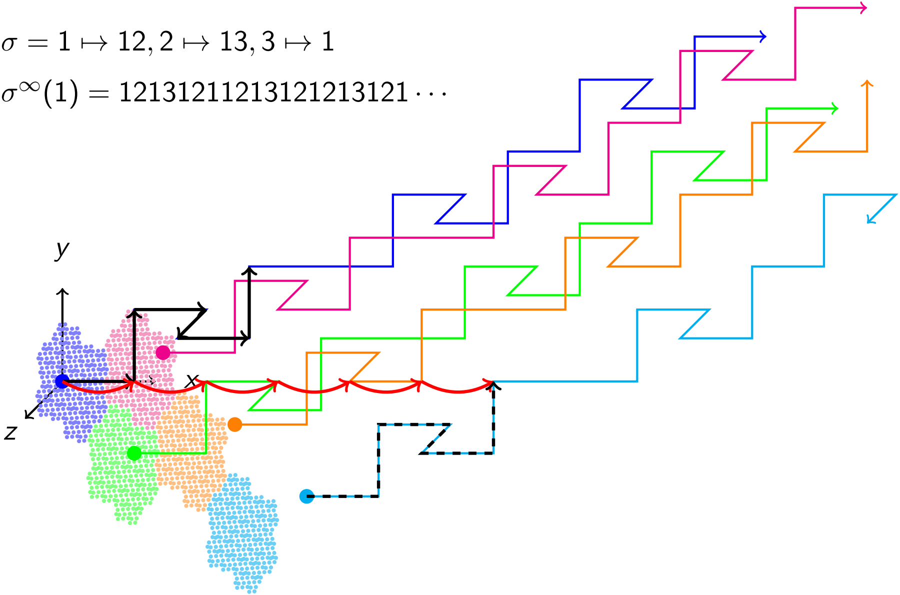

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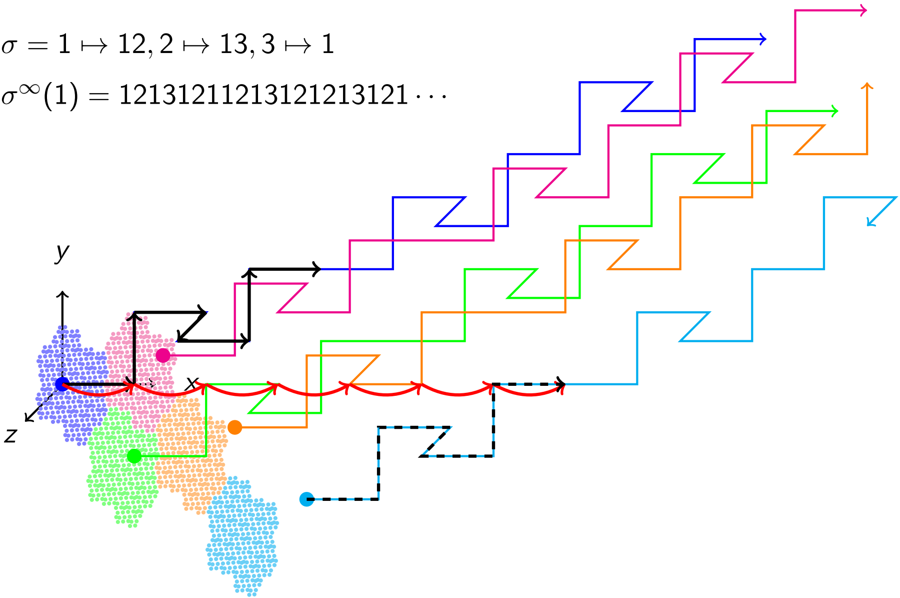

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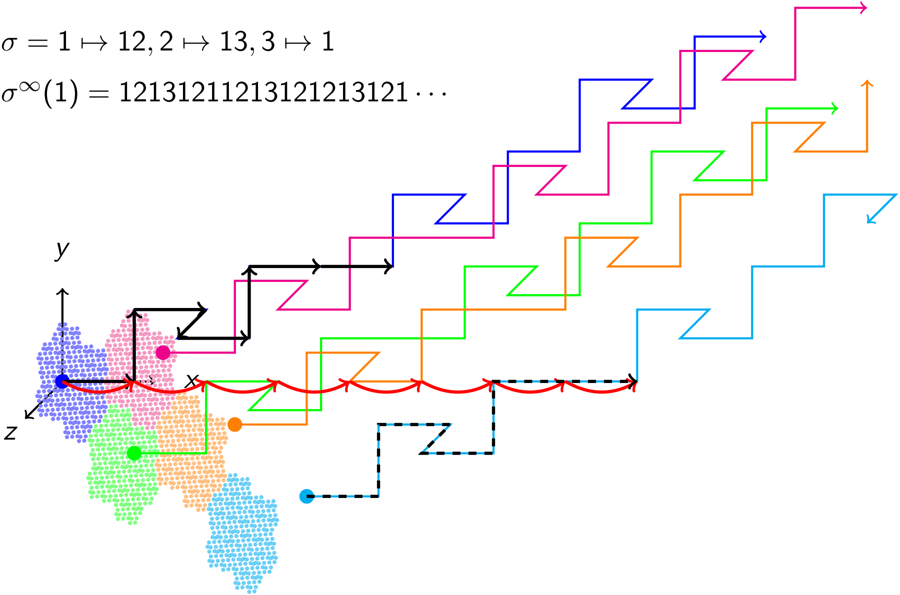

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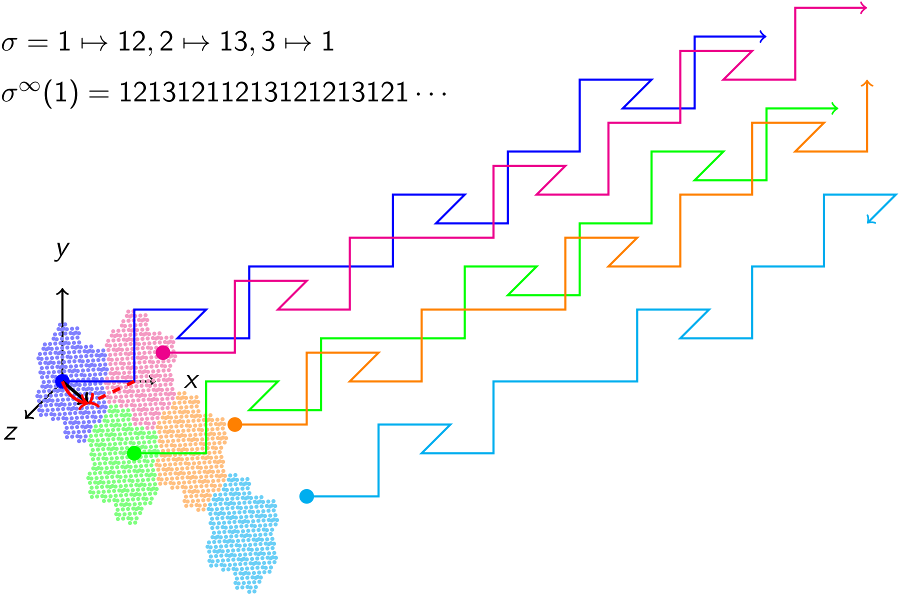

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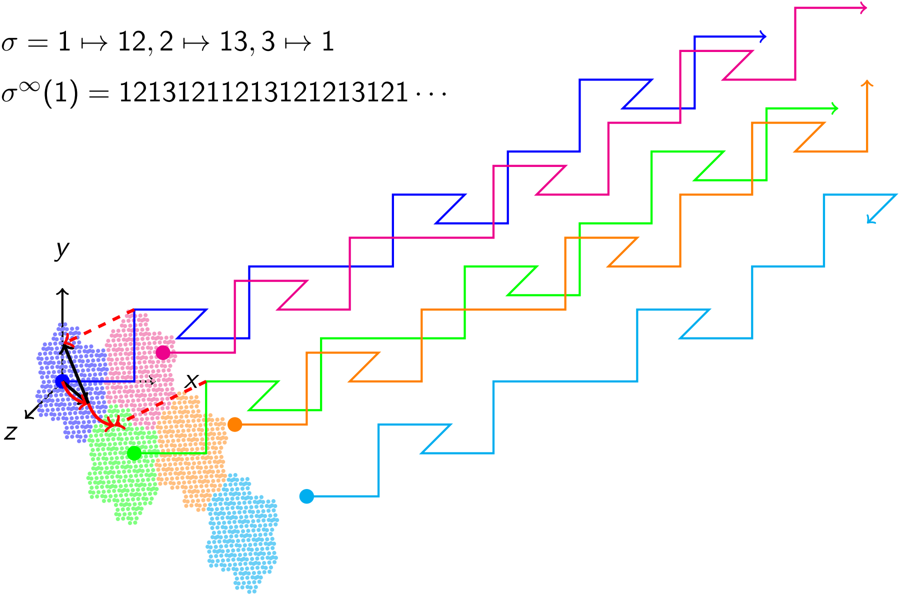

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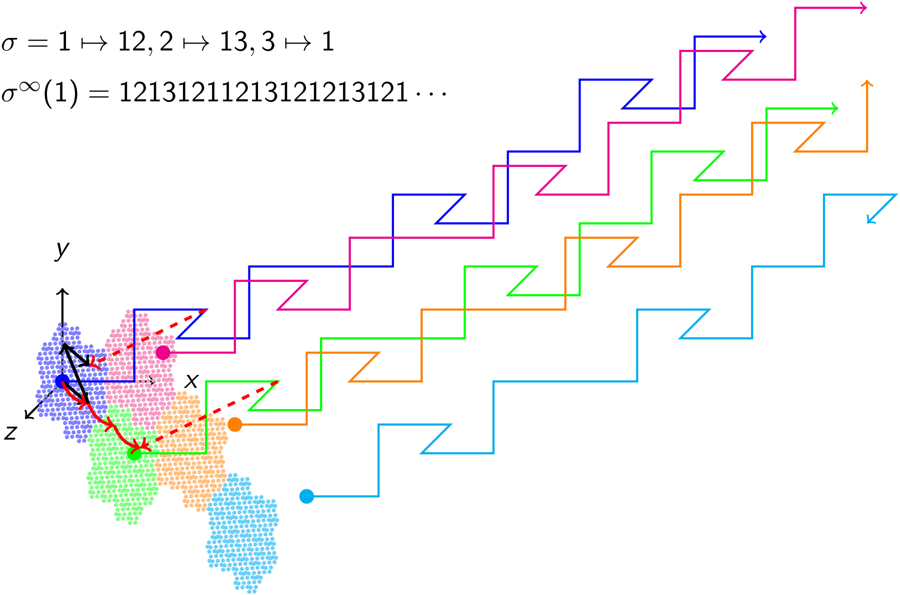

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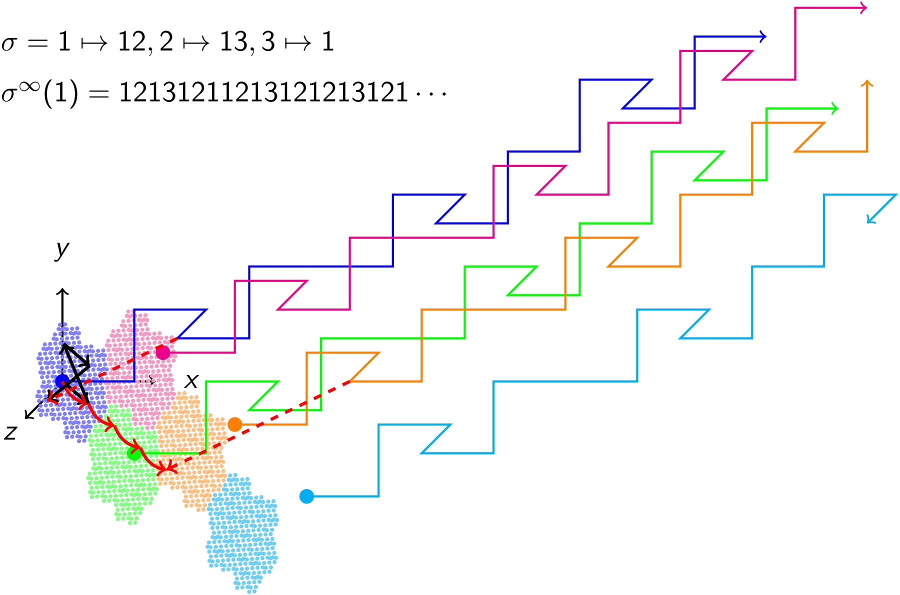

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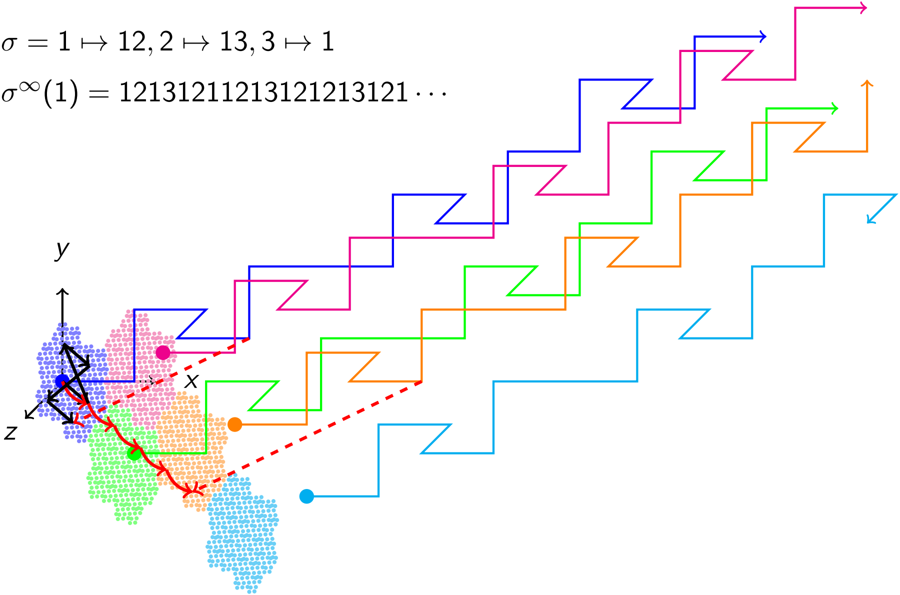
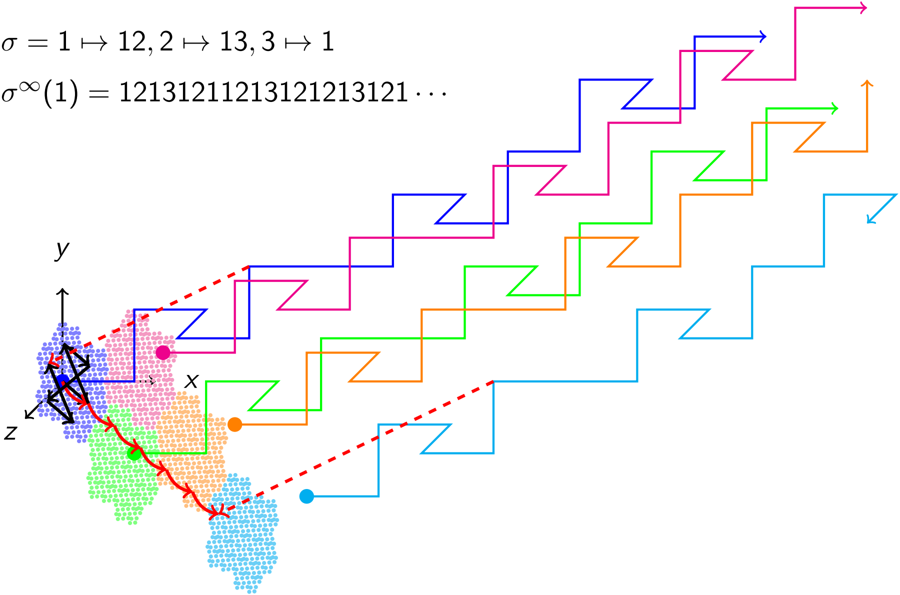


Tribonacci Example from Rauzy (1982)

$\sigma = 1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 1$

$\sigma^\infty(1) = 12131211213121213121\cdots$



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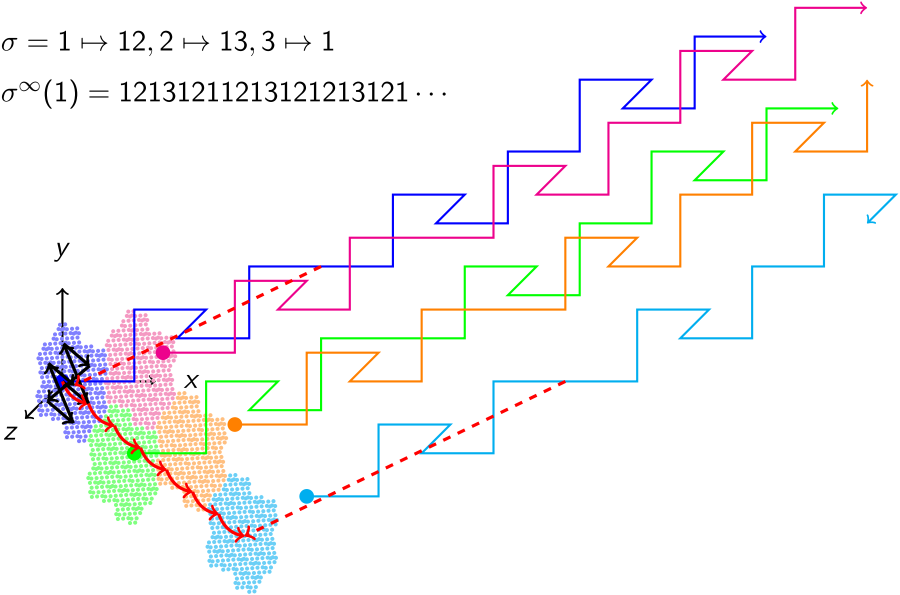

Tribonacci Example from Rauzy (1982)

$\sigma = 1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 1$

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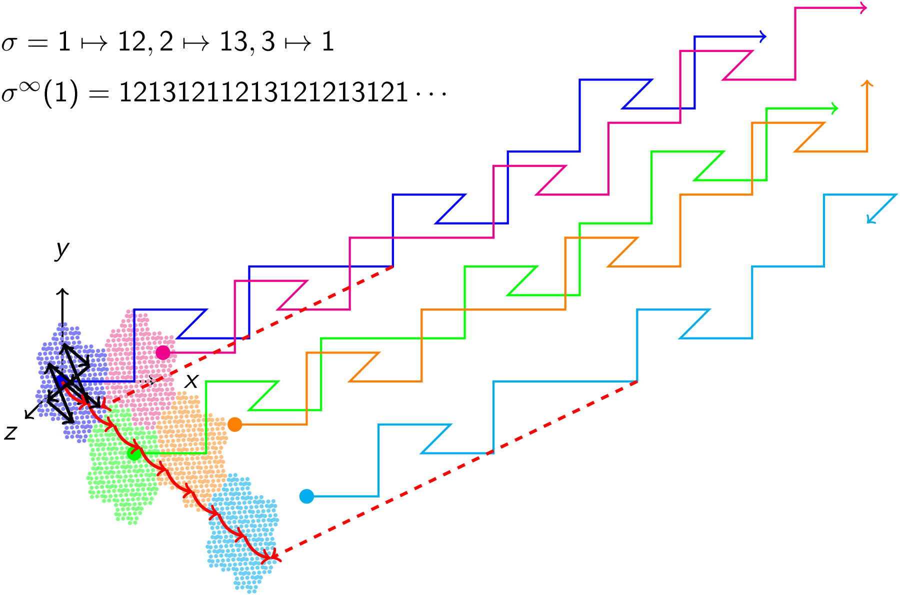

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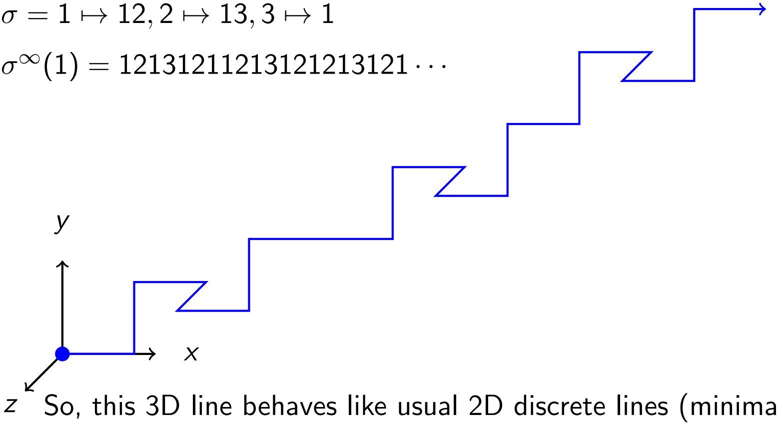

So, this 3D line behaves like usual 2D discrete lines (minimal 6-connected, dynamical system, substitutive).

Can we get the same in any 3D direction and how ?

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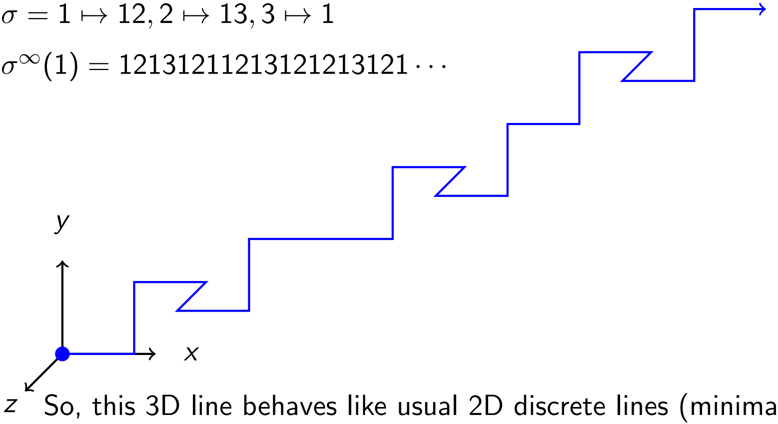

So, this 3D line behaves like usual 2D discrete lines (minimal 6-connected, dynamical system, substitutive).

Can we get the same in any 3D direction and how ?

This question is related to the Pisot Conjecture.

Outline

2D : Euclid algorithm on $(11, 4)$

$$
\begin{array}{|rcl|}
\hline
11 & = & 2 \cdot 4 + 3 \\
4 & = & 1 \cdot 3 + 1 \\
3 & = & 3 \cdot 1 + 0 \\
\hline
\end{array}
$$

$$
\boxed{\frac{4}{11} = 0 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{3}}}}
$$

$$
(11,4) \xleftarrow{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^2} (3,4) \xleftarrow{\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}} (3,1) \xleftarrow{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^3} (0,1)
$$

$$
\mathbf{w} = \mathbf{w}_0 \xleftarrow{\begin{array}{rcl} a & \mapsto & a \\ b & \mapsto & aab \end{array}} \mathbf{w}_1 \xleftarrow{\begin{array}{rcl} a & \mapsto & ab \\ b & \mapsto & b \end{array}} \mathbf{w}_2 \xleftarrow{\begin{array}{rcl} a & \mapsto & a \\ b & \mapsto & aaab \end{array}} \mathbf{w}_3 = b
$$

2D : Euclid algorithm on $(11, 4)$

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\begin{array}{|ccc|}
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11 & = & 2 \cdot 4 + 3 \\
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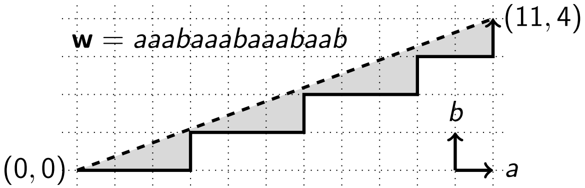
\hline
\end{array}
$$

$$\frac{4}{11} = 0 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{3}}}$$



$$(11,4) \xleftarrow{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^2} (3,4) \xleftarrow{\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}} (3,1) \xleftarrow{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^3} (0,1)$$

$$\mathbf{w} = \mathbf{w}_0 \xleftarrow{\substack{a \mapsto a \\ b \mapsto aab}} \mathbf{w}_1 \xleftarrow{\substack{a \mapsto ab \\ b \mapsto b}} \mathbf{w}_2 \xleftarrow{\substack{a \mapsto a \\ b \mapsto aaab}} \mathbf{w}_3 = b$$

3D : Imitation of Euclid algorithm on $(7,4,6)$

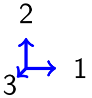
$$
\begin{array}{ccccccccc}
(7,4,6) & \xleftarrow{\begin{pmatrix}1&0&1\\0&1&0\\0&0&1\end{pmatrix}} & (1,4,6) & \xleftarrow{\begin{pmatrix}1&0&0\\0&1&0\\0&1&1\end{pmatrix}} & (1,4,2) & \xleftarrow{\begin{pmatrix}1&0&0\\0&1&2\\0&0&1\end{pmatrix}} & (1,0,2) & \xleftarrow{\begin{pmatrix}1&0&0\\0&1&0\\2&0&1\end{pmatrix}} & (1,0,0) \\
\mathbf{w}_0 & \xleftarrow[\begin{array}{ccc}1&\mapsto&1\\2&\mapsto&2\\3&\mapsto&13\end{array}]{} & \mathbf{w}_1 & \xleftarrow[\begin{array}{ccc}1&\mapsto&1\\2&\mapsto&23\\3&\mapsto&3\end{array}]{} & \mathbf{w}_2 & \xleftarrow[\begin{array}{ccc}1&\mapsto&1\\2&\mapsto&2\\3&\mapsto&223\end{array}]{} & \mathbf{w}_3 & \xleftarrow[\begin{array}{ccc}1&\mapsto&133\\2&\mapsto&2\\3&\mapsto&3\end{array}]{} & \mathbf{w}_4
\end{array}
$$

Its (Hausdorff) distance to the euclidean line is 1.3680.

3D : Imitation of Euclid algorithm on $(7, 4, 6)$

$$
\begin{array}{ccccccccc}
(7,4,6) & \xleftarrow{\begin{pmatrix}1&0&1\\0&1&0\\0&0&1\end{pmatrix}} & (1,4,6) & \xleftarrow{\begin{pmatrix}1&0&0\\0&1&0\\0&1&1\end{pmatrix}} & (1,4,2) & \xleftarrow{\begin{pmatrix}1&0&0\\0&1&2\\0&0&1\end{pmatrix}} & (1,0,2) & \xleftarrow{\begin{pmatrix}1&0&0\\0&1&0\\2&0&1\end{pmatrix}} & (1,0,0) \\
& \begin{array}{ccc}1&\mapsto&1\\2&\mapsto&2\\3&\mapsto&13\end{array} & & \begin{array}{ccc}1&\mapsto&1\\2&\mapsto&23\\3&\mapsto&3\end{array} & & \begin{array}{ccc}1&\mapsto&1\\2&\mapsto&2\\3&\mapsto&223\end{array} & & \begin{array}{ccc}1&\mapsto&133\\2&\mapsto&2\\3&\mapsto&3\end{array} & \\
\mathbf{w}_0 & \longleftarrow & \mathbf{w}_1 & \longleftarrow & \mathbf{w}_2 & \longleftarrow & \mathbf{w}_3 & \longleftarrow & \mathbf{w}_4
\end{array}
$$

$$\mathbf{w} = \mathbf{w}_0 = 1213213132132131\,3$$



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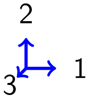
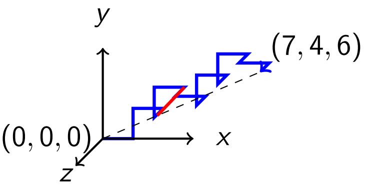
3D : Imitation of Euclid algorithm on $(7, 4, 6)$

$$(7,4,6) \xleftarrow{\begin{pmatrix}1&0&1\\0&1&0\\0&0&1\end{pmatrix}} (1,4,6) \xleftarrow{\begin{pmatrix}1&0&0\\0&1&0\\0&1&1\end{pmatrix}} (1,4,2) \xleftarrow{\begin{pmatrix}1&0&0\\0&1&2\\0&0&1\end{pmatrix}} (1,0,2) \xleftarrow{\begin{pmatrix}1&0&0\\0&1&0\\2&0&1\end{pmatrix}} (1,0,0)$$

$$\mathbf{w}_0 \xleftarrow[\begin{array}{l}1\mapsto 1\\2\mapsto 2\\3\mapsto 13\end{array}]{} \mathbf{w}_1 \xleftarrow[\begin{array}{l}1\mapsto 1\\2\mapsto 23\\3\mapsto 3\end{array}]{} \mathbf{w}_2 \xleftarrow[\begin{array}{l}1\mapsto 1\\2\mapsto 2\\3\mapsto 223\end{array}]{} \mathbf{w}_3 \xleftarrow[\begin{array}{l}1\mapsto 133\\2\mapsto 2\\3\mapsto 3\end{array}]{} \mathbf{w}_4$$

$$\mathbf{w} = \mathbf{w}_0 = 1213213132132131 3$$





Its (Hausdorff) distance to the euclidean line is 1.3680.

3D Continued fraction algorithms

Brun Subtract the second largest to the largest.

Poincaré Subtract the smallest to the mid and the mid to the largest.

Selmer Subtract the smallest to the largest.

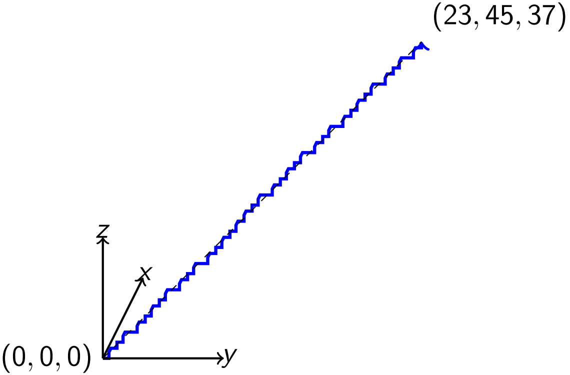
Fully subtractive Subtract the smallest to the other two.

Arnoux-Rauzy Subtract the sum of the two smallest to the largest (not always possible).

Outline

On $(23, 45, 37)$ using Brun algorithm

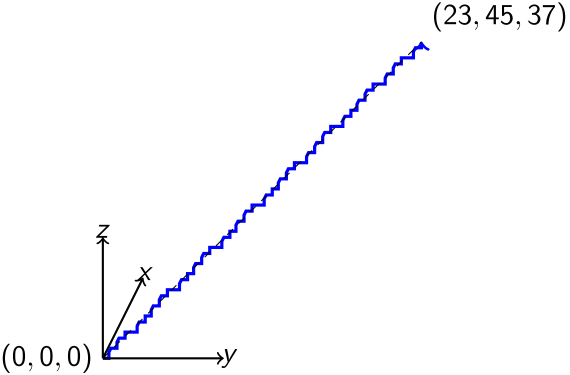
Let $\vec{u} = (23, 45, 37)$. Using Brun algorithm, one gets



and its distance to the Euclidean segment is 1.0753.

On $(23, 45, 37)$ using Poincaré algorithm

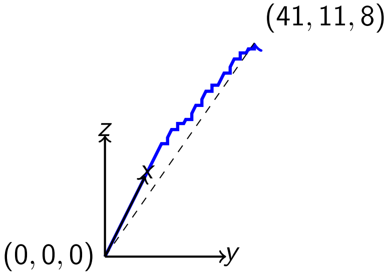
Let $\vec{u} = (23, 45, 37)$. Using Poincaré algorithm, one gets



and its distance to the Euclidean segment is 1.0340.

On $(41, 11, 8)$ using Fully subtractive algorithm

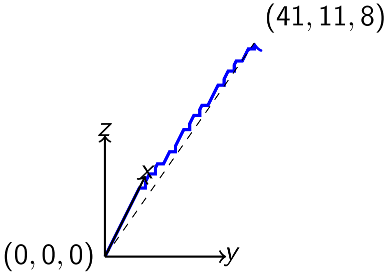
Let $\vec{u} = (41, 11, 8)$. Using Fully subtractive algorithm, one gets



and its distance to the Euclidean segment is 8.8163.

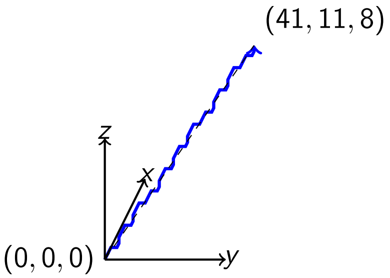
On $(41, 11, 8)$ using Poincaré algorithm

Let $\vec{u} = (41, 11, 8)$. Using Poincaré algorithm, one gets



and its distance to the Euclidean segment is 5.3528.

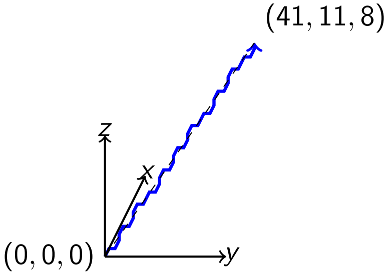
Let $\vec{u} = (41, 11, 8)$. Using Brun algorithm, one gets



and its distance to the Euclidean segment is 1.0348.

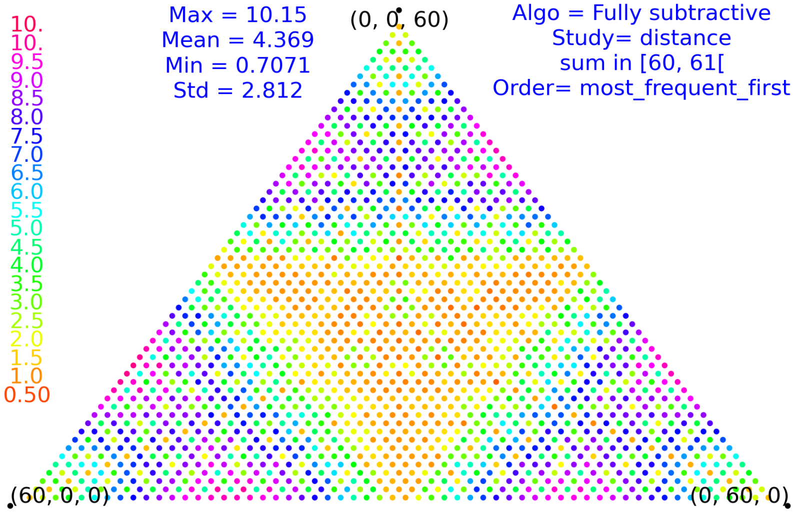
On $(41, 11, 8)$ using Arnoux-Rauzy algorithm

Let $\vec{u} = (41, 11, 8)$. Using Arnoux-Rauzy algorithm, one gets

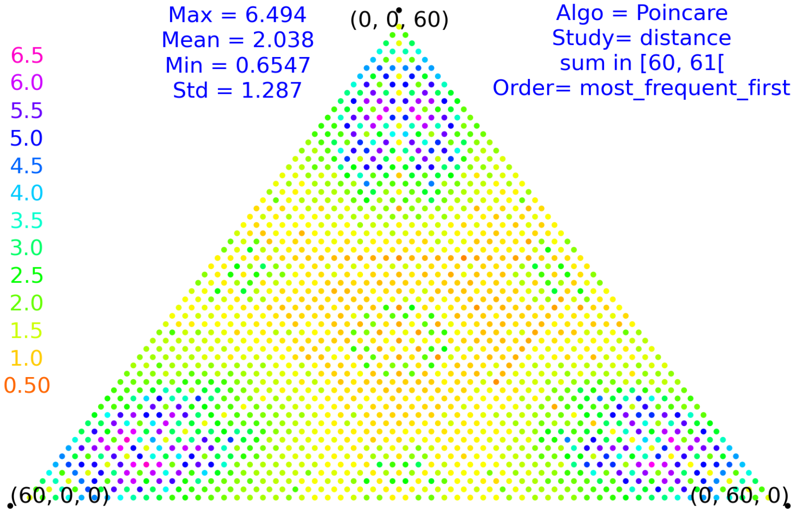


and its distance to the Euclidean segment is 0.98270.

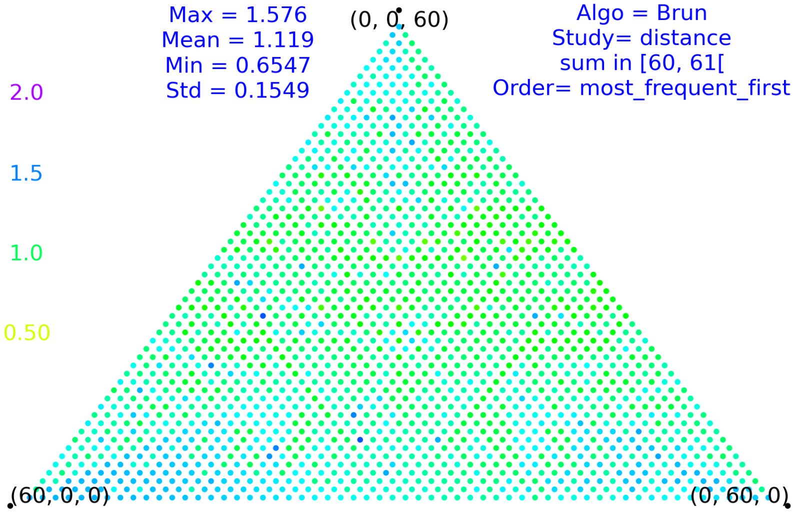
When $a + b + c = 60$ using Fully subtractive



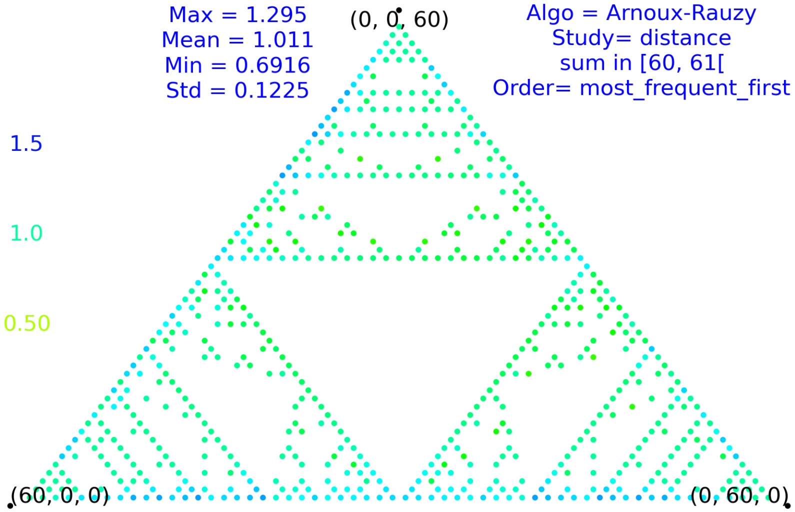
When $a + b + c = 60$ using Poincaré



When $a + b + c = 60$ using Brun



When $a + b + c = 60$ using Arnoux-Rauzy



3D Continued fraction algorithms : fusions

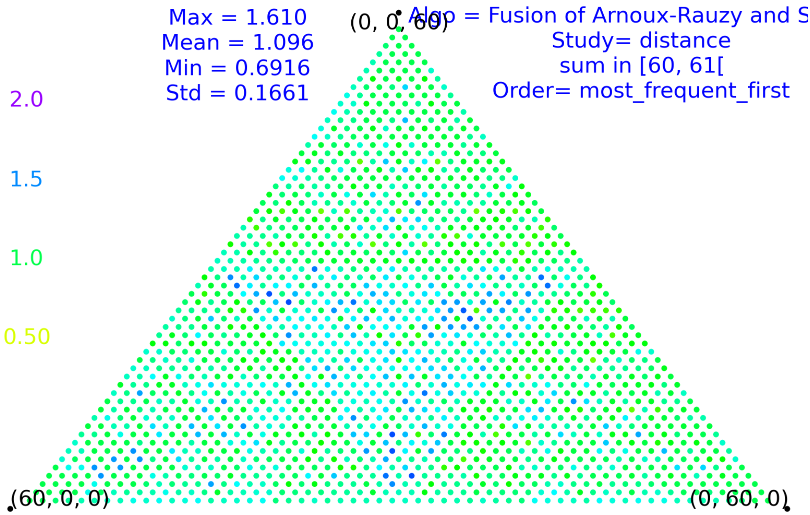
Arnoux-Rauzy and Selmer Do Arnoux-Rauzy if possible, otherwise Selmer.

Arnoux-Rauzy and Fully Do Arnoux-Rauzy if possible, otherwise Fully subtractive.

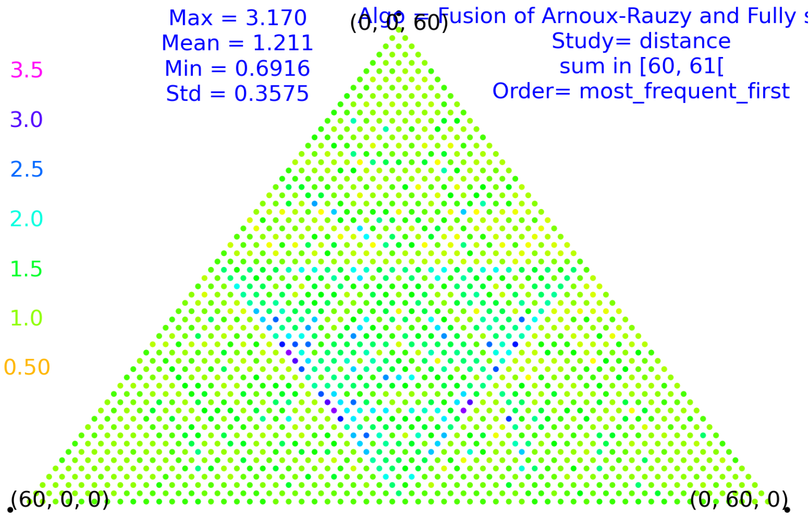
Arnoux-Rauzy and Brun Do Arnoux-Rauzy if possible, otherwise Brun.

Arnoux-Rauzy and Poincaré Do Arnoux-Rauzy if possible, otherwise Poincaré.

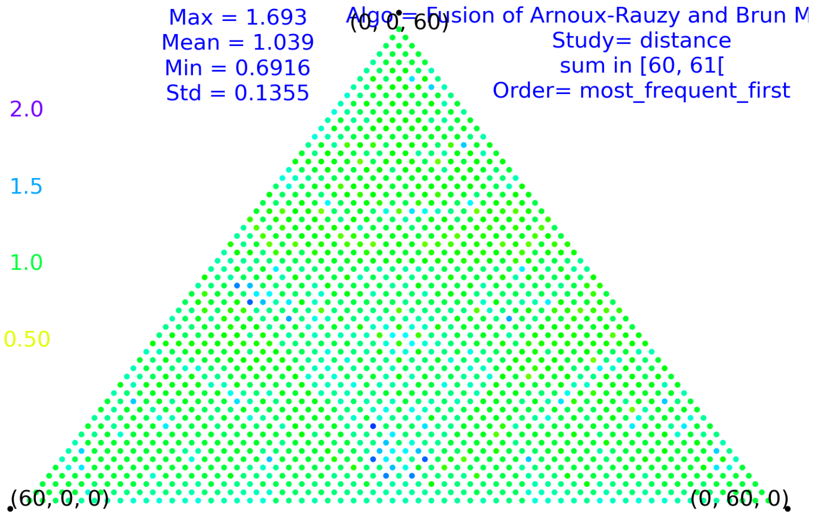
When $a + b + c = 60$ using Arnoux-Rauzy and Selmer



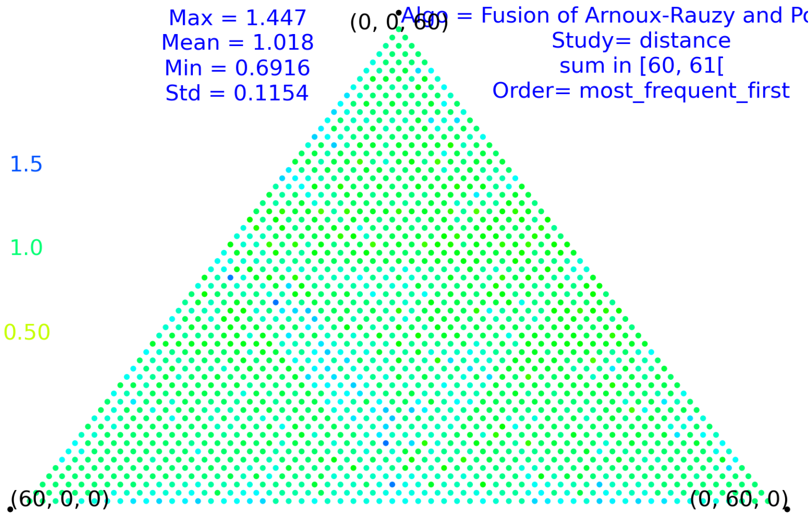
When $a + b + c = 60$ using Arnoux-Rauzy and Fully Subtractive



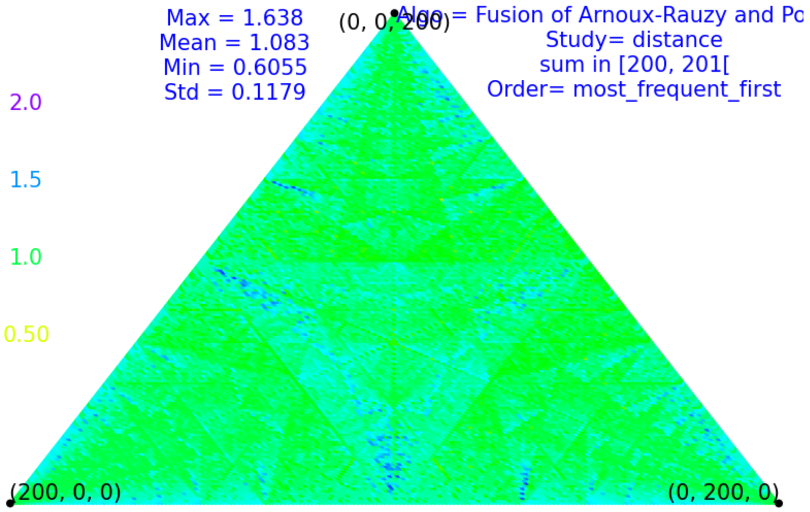
When $a + b + c = 60$ using Arnoux-Rauzy and Brun



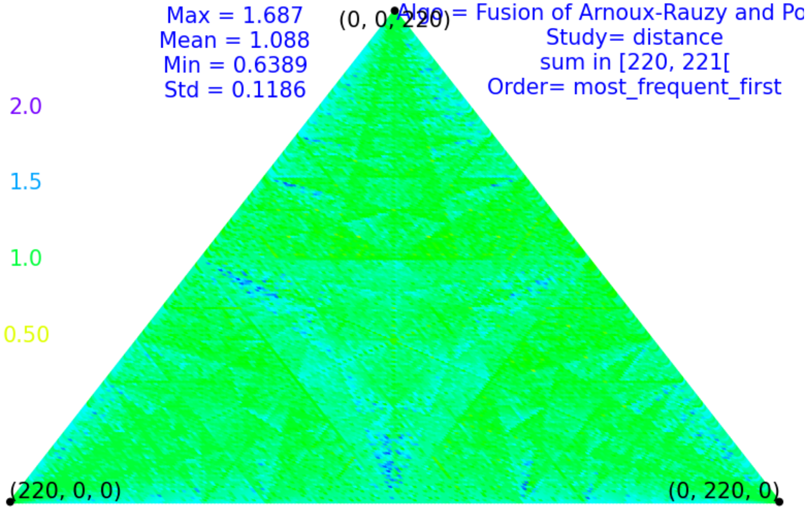
When $a + b + c = 60$ using Arnoux-Rauzy and Poincaré



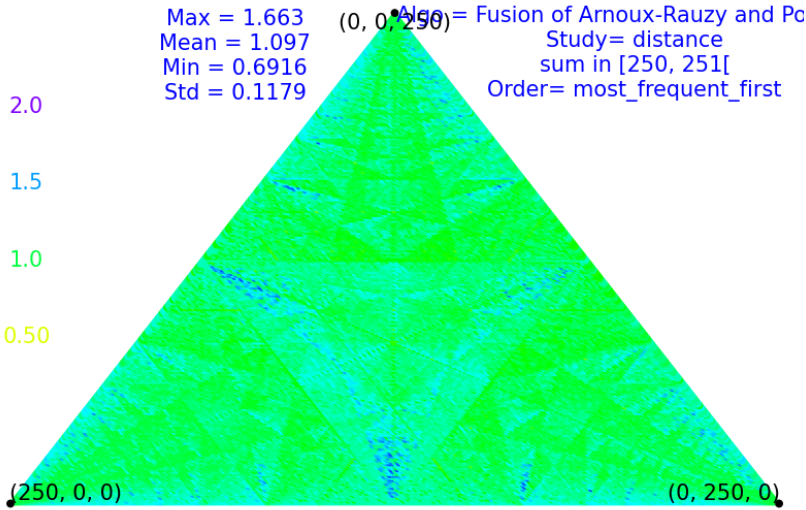
When $a + b + c = 200$ using Arnoux-Rauzy and Poincaré



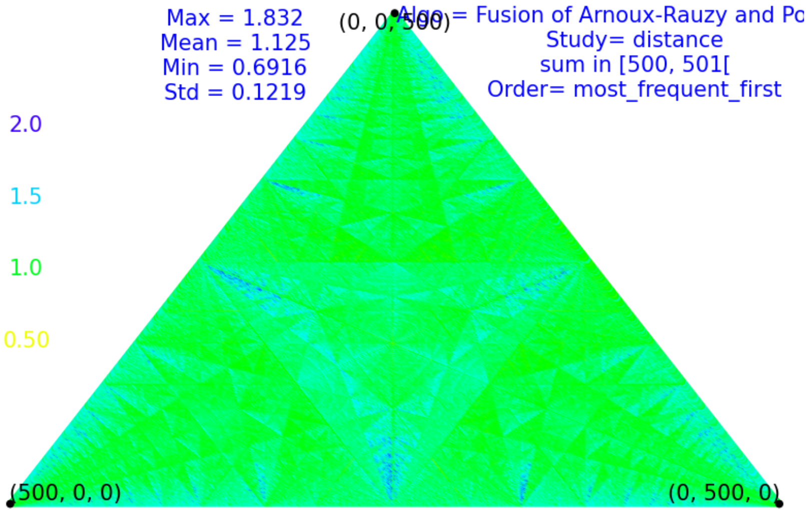
When $a + b + c = 220$ using Arnoux-Rauzy and Poincaré



When $a + b + c = 250$ using Arnoux-Rauzy and Poincaré



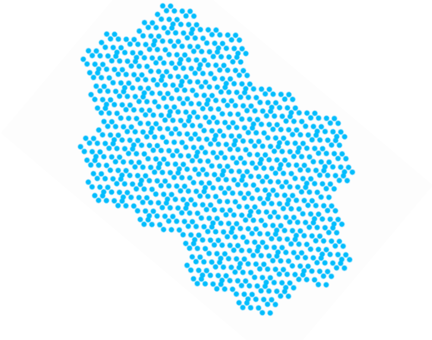
When $a + b + c = 500$ using Arnoux-Rauzy and Poincaré



Outline

Correspondence between discrete lines and planes

There is a bijection between the generated discrete line and the section of a discrete plane.



Projection of a finite prefix of $\sigma^\infty(1)$

$E_1^*(\sigma^{12})(C)$ where C is the upper unit cube

For more details about the dynamical system defined on the section of discrete plane and the bijection, consult the article (Theorem 2).

Conclusion

In brief, we proposed a new construction method for 3D discrete lines which

- is minimal and 6-connected,
- is close to the Euclidean line,
- is defined by an offset which is not a circle but a fractal,
- is generated by elementary substitutions,
- is the symbolic coding of a dynamical system.

Moreover we believe that it

- has linear word complexity,
- has low word balance value.

It has been experimentally verified that fusions of Arnoux-Rauzy and Brun or Poincaré algorithms behave very nicely (average distance is 1).

Further research

More work need to be done now :

- Prove that the distance is bounded for all integer directions.
- Do thorough comparisons with existing 3D discrete lines (O. Figueiredo, and J.-P. Reveilllès, J.-L. Toutant, ...)

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Credits :

- This research was driven by computer exploration using the open-source mathematical software `Sage`.
- Images of this document were produced using `pgf/tikz`.