Avancées récentes sur des questions issues des pavages du plan par translation d'une tuile

Sébastien Labbé

Laboratoire d'Informatique Algorithmique : Fondements et Applications
Université Paris Diderot
&
Laboratoire de Combinatoire et d'Informatique Mathématique
Université du Québec à Montréal

Séminaire Structures Algébriques et Géométriques
Université de Sherbrooke
2 mai 2011

Travail en collaboration avec Alexandre Blondin Massé, Srečko Brlek et Ariane Garon

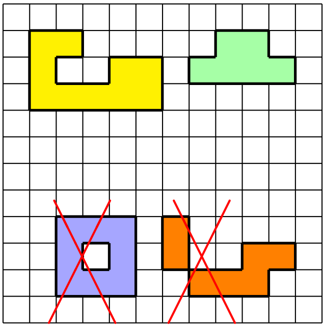
Outline

1. Introduction
   - Discrete Figures
   - Tilings
   - Beauquier and Nivat
   - Hexagonal and Square Tiles
   - A conjecture of Brlek, Dulucq, Fédou and Provençal, 2007
   - A conjecture of Provençal and Vuillon, 2008
2. (Idea of the) Proof of the first conjecture
3. (Idea of the) Proof of the second conjecture
4. Open problems

Outline

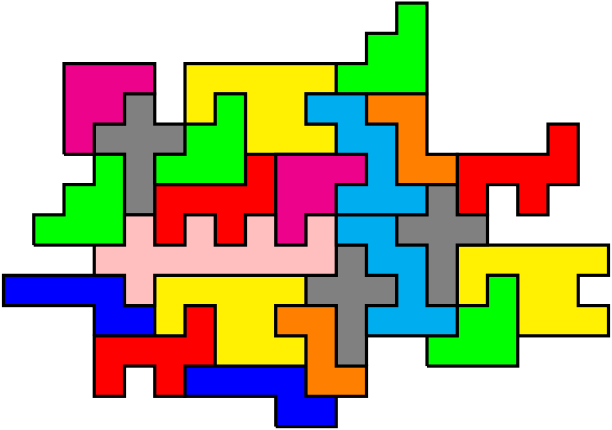
Discrete Figures and Polyominoes

- Discrete plane : $\mathbb{Z}^2$
- **Definition** : A polyomino is a finite, 4-connected subset of the plane, without holes.



Tilings

A set $S = \{P_1, P_2, \ldots, P_k\}$ of polyominoes tiles the plane if there exists a partition of $\mathbb{Z}^2$ into translated copies of P_i.



Results on tilings

Theorem (Berger, 1966)

Il existe un ensemble de polyominos qui pavent le plan uniquement de manière apériodique.

Corollary (Berger, 1966)

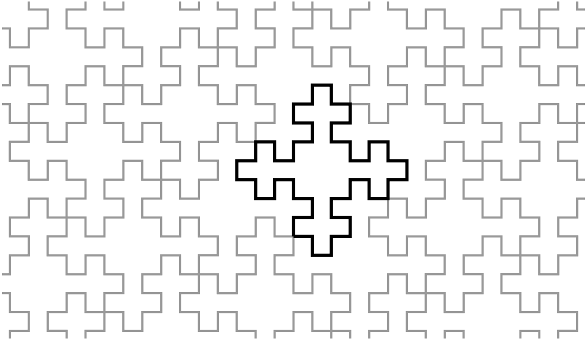
Le problème de pavage du plan par un ensemble fini de polyominos est indécidable.

Theorem (Wijshoff et van Leuveen, 1984)

- *Si un polyomino pave le plan par translation, alors il peut également le faire de manière régulière.*
- *Donc, le problème du pavage du plan par un polyomino est décidable.*

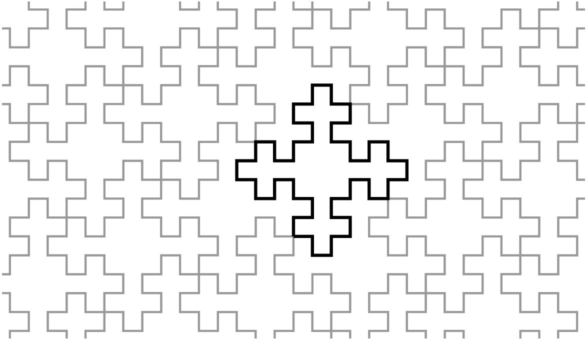
The Tiling by Translation Problem

A polyomino P is called a tile if it tiles the plane.



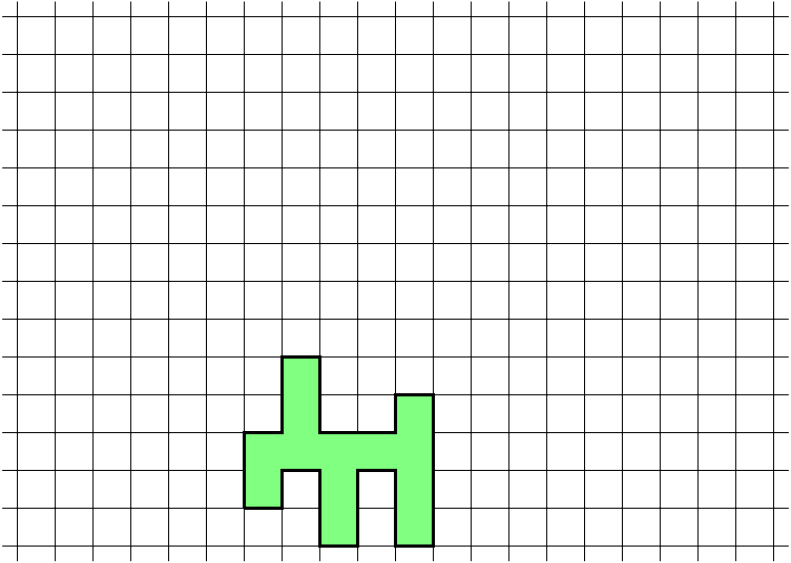
The Tiling by Translation Problem

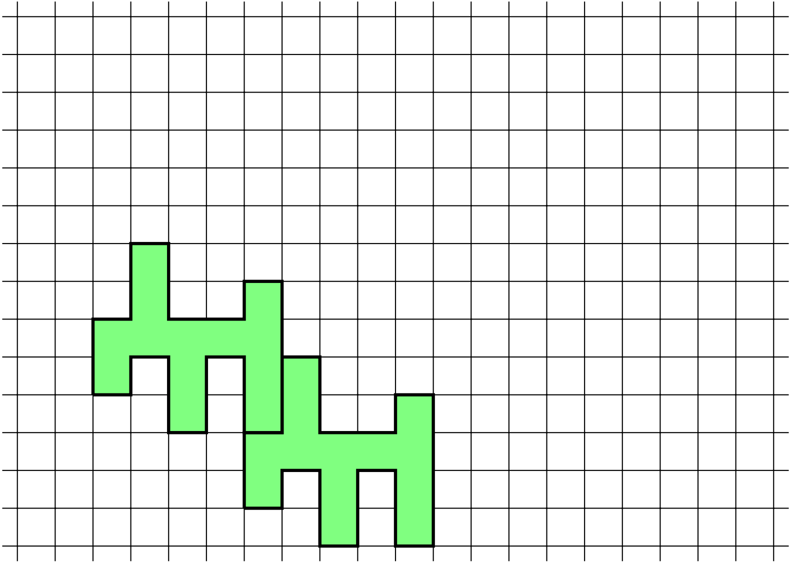
A polyomino P is called a tile if it tiles the plane.

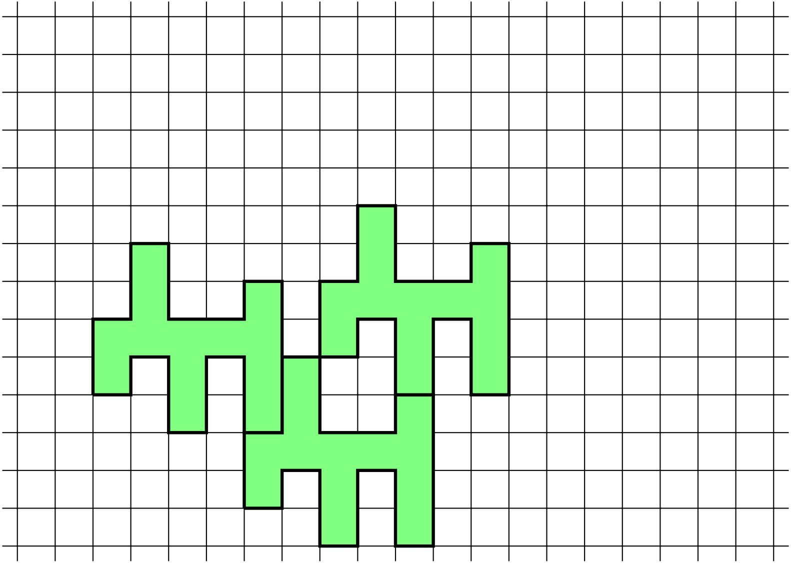


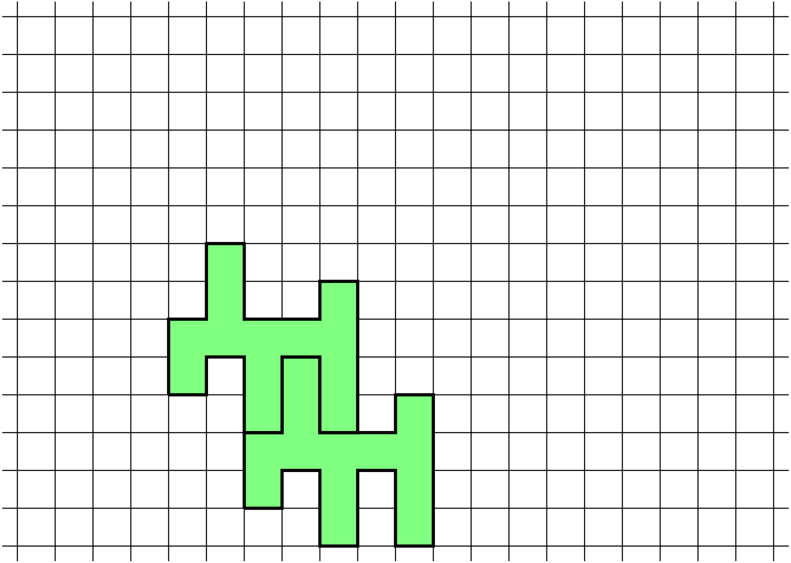
Problem

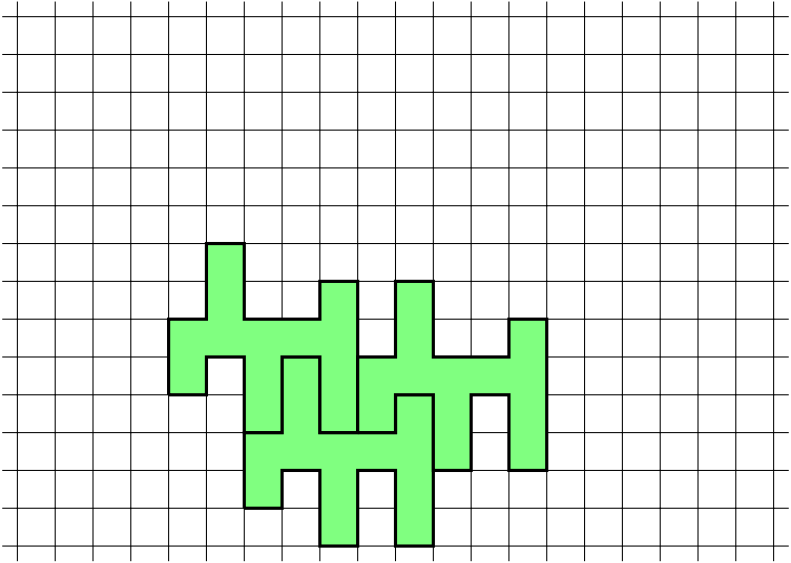
Does a given polyomino P tile the plane ?

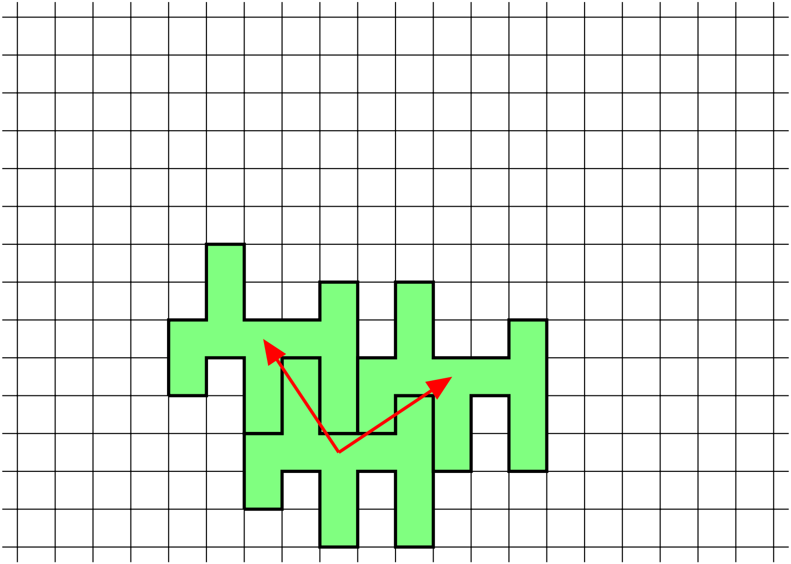


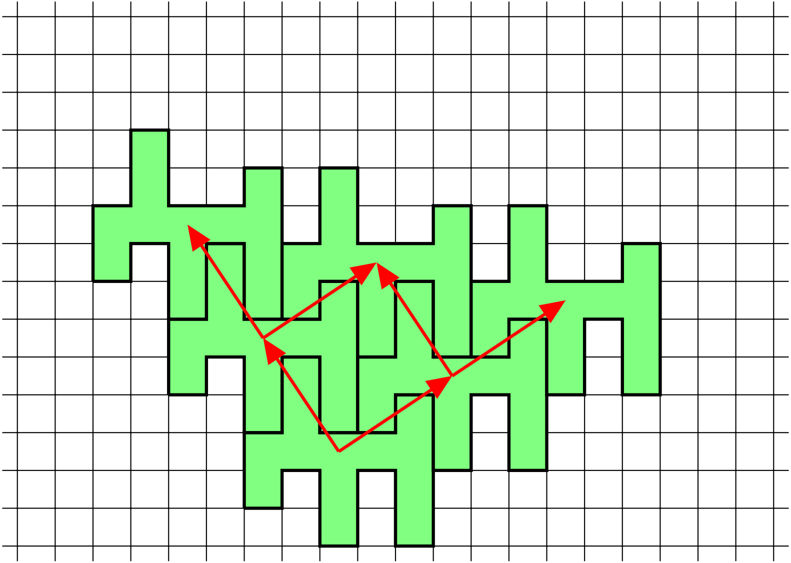


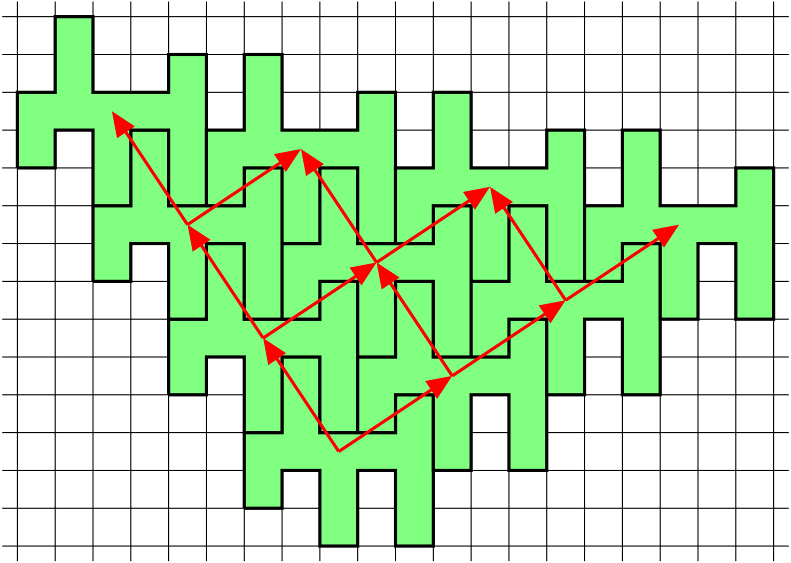


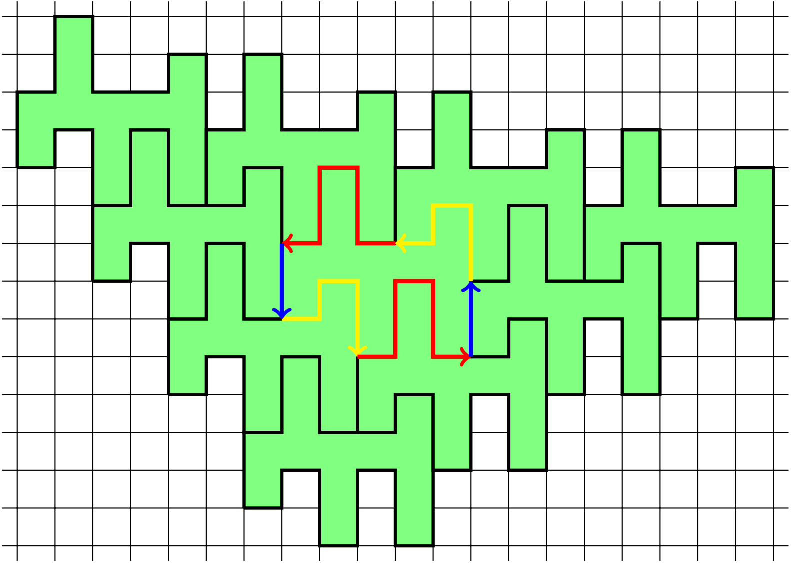






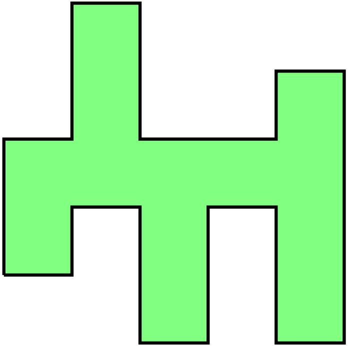


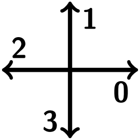




Freeman Chain Code

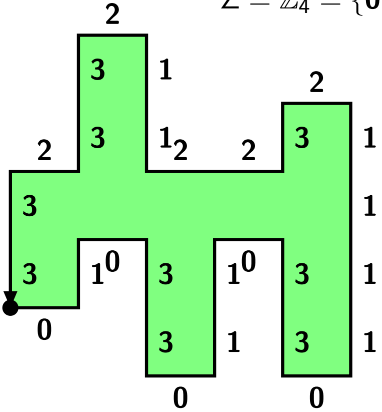
$$\Sigma = \mathbb{Z}_4 = \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}\}$$

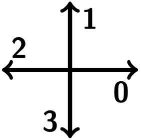




Freeman Chain Code

$$\Sigma = \mathbb{Z}_4 = \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}\}$$

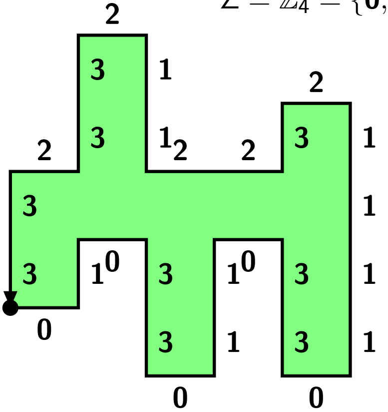


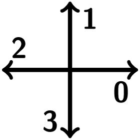


$$w = \mathbf{0103301103301111232211233233}$$

Freeman Chain Code

$\Sigma = \mathbb{Z}_4 = \{\mathbf{0},\mathbf{1},\mathbf{2},\mathbf{3}\}$



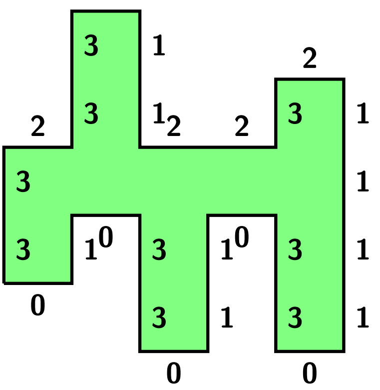


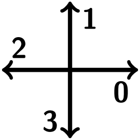
Any conjugate w' of w codes the same polyomino.

w and w' are conjugate if there exist $u, v \in \Sigma^*$ such that $w = uv$ and $w' = vu$. We write $w \equiv_{|u|} w'$.

$w = \mathbf{0103301103301111232211233233}$

$$\Sigma = \mathbb{Z}_4 = \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}\}$$





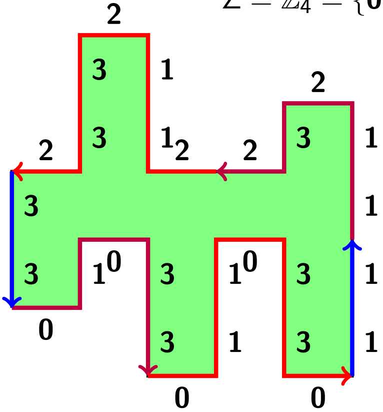
Any conjugate w' of w codes the same polyomino.

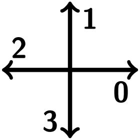
w and w' are conjugate if there exist $u, v \in \Sigma^*$ such that $w = uv$ and $w' = vu$. We write $w \equiv_{|u|} w'$.

$[w] \equiv \mathbf{0103301103301111232211233233}$

Freeman Chain Code

$\Sigma = \mathbb{Z}_4 = \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}\}$





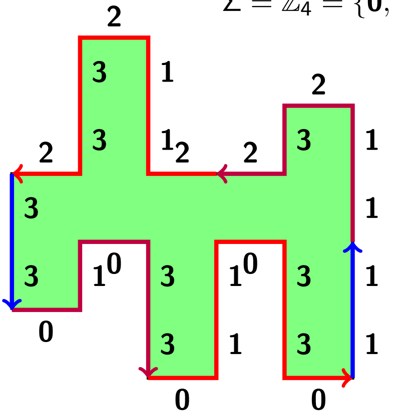
Any conjugate w' of w codes the same polyomino.

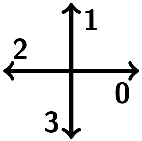
w and w' are conjugate if there exist $u, v \in \Sigma^*$ such that $w = uv$ and $w' = vu$. We write $w \equiv_{|u|} w'$.

$[w] \equiv \mathbf{0103301103301111232211233233}$

Freeman Chain Code

$$\Sigma = \mathbb{Z}_4 = \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}\}$$





Any conjugate w' of w codes the same polyomino.

w and w' are conjugate if there exist $u, v \in \Sigma^*$ such that $w = uv$ and $w' = vu$. We write $w \equiv_{|u|} w'$.

$[w] \equiv$

01033	0110330	11	11232	2112332	33
X	Y	Z	$\widehat{X}$	$\widehat{Y}$	$\widehat{Z}$

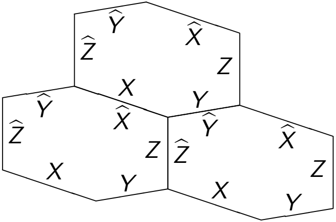
Beauquier and Nivat (1991)

Characterization : A polyomino P tiles the plane if and only if there exist $X, Y, Z \in \Sigma^*$ such that $[w] \equiv XYZ\widehat{X}\widehat{Y}\widehat{Z}$.

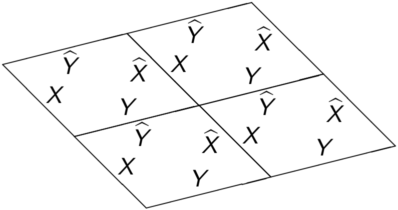
$X = \mathbf{0\,0\,1\,0\,3\,0\,1}$

$\widehat{X} = \mathbf{3\,2\,1\,2\,3\,2\,2}$

hexagon tiles



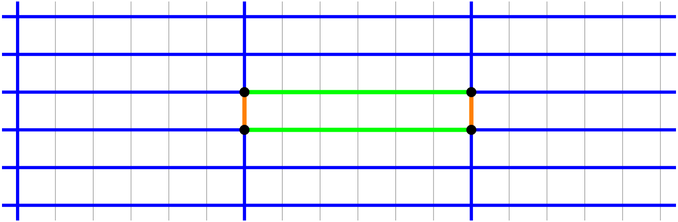
square tiles



Maurits Cornelis Escher (1898-1972). Hexagonal tiling. Square tiling.

Hexagonal Tilings

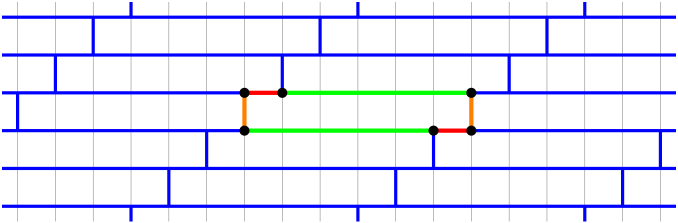
There are polyominoes admitting many hexagon tilings :



A $1 \times n$ rectangle tiles the plane as an hexagon in $n-1$ ways and as a square in only 1 way.

Hexagonal Tilings

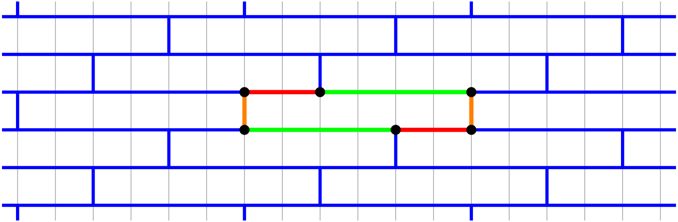
There are polyominoes admitting many hexagon tilings :



A $1 \times n$ rectangle tiles the plane as an hexagon in $n - 1$ ways and as a square in only 1 way.

Hexagonal Tilings

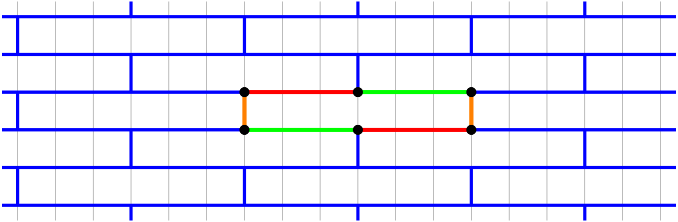
There are polyominoes admitting many hexagon tilings :



A $1 \times n$ rectangle tiles the plane as an hexagon in $n-1$ ways and as a square in only 1 way.

Hexagonal Tilings

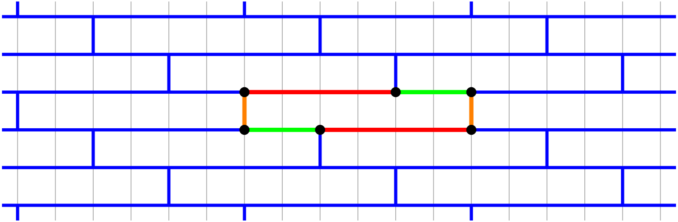
There are polyominoes admitting many hexagon tilings :



A $1 \times n$ rectangle tiles the plane as an hexagon in $n-1$ ways and as a square in only 1 way.

Hexagonal Tilings

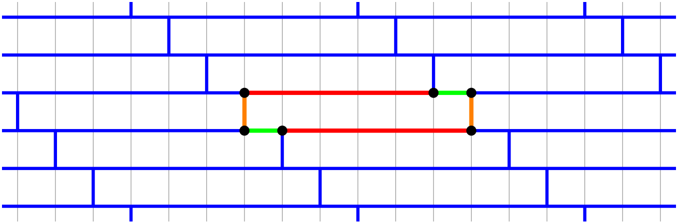
There are polyominoes admitting many hexagon tilings :



A $1 \times n$ rectangle tiles the plane as an hexagon in $n-1$ ways and as a square in only 1 way.

Hexagonal Tilings

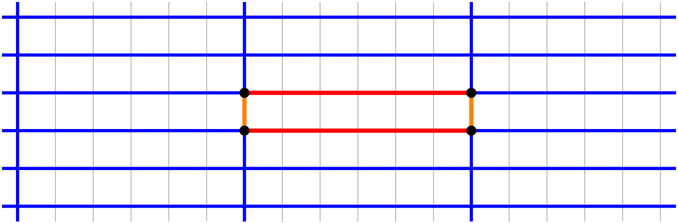
There are polyominoes admitting many hexagon tilings :



A $1 \times n$ rectangle tiles the plane as an hexagon in $n-1$ ways and as a square in only 1 way.

Hexagonal Tilings

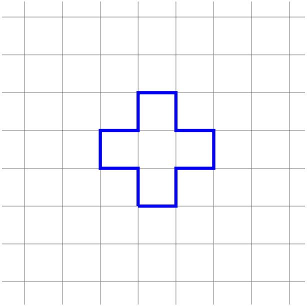
There are polyominoes admitting many hexagon tilings :

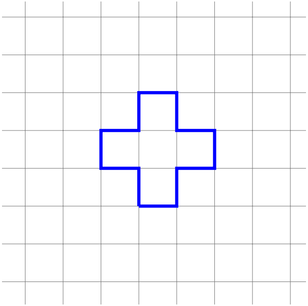


A $1 \times n$ rectangle tiles the plane as an hexagon in $n-1$ ways and as a square in only 1 way.

Square Tilings

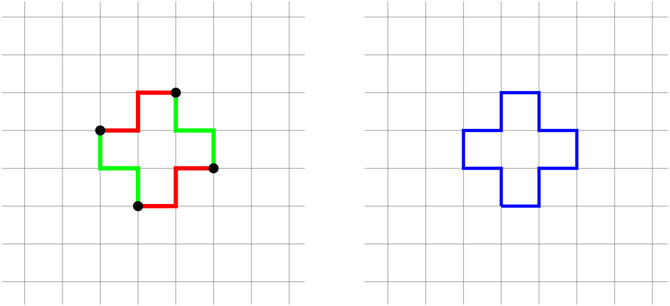
The pentomino has two distinct square factorizations :





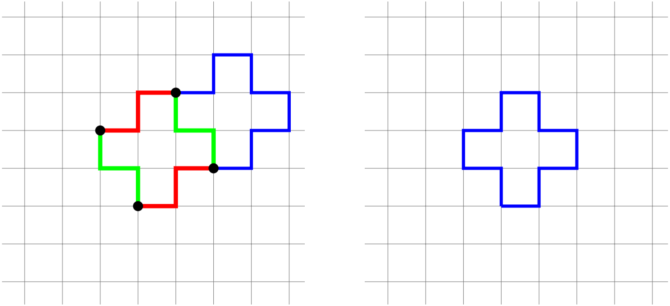
Square Tilings

The pentomino has two distinct square factorizations :



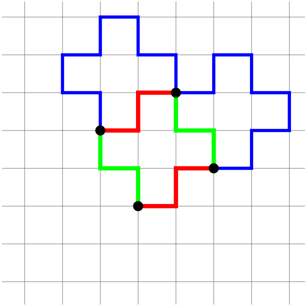
Square Tilings

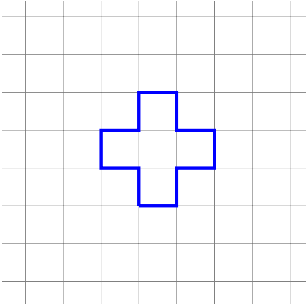
The pentomino has two distinct square factorizations :



Square Tilings

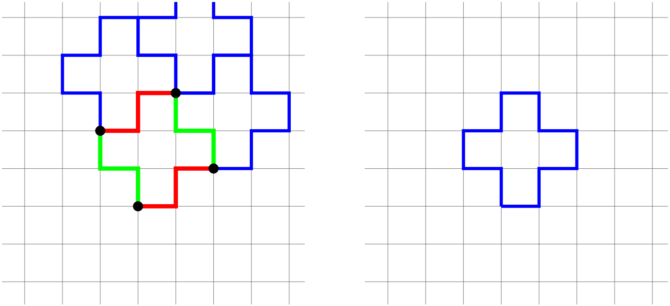
The pentomino has two distinct square factorizations :





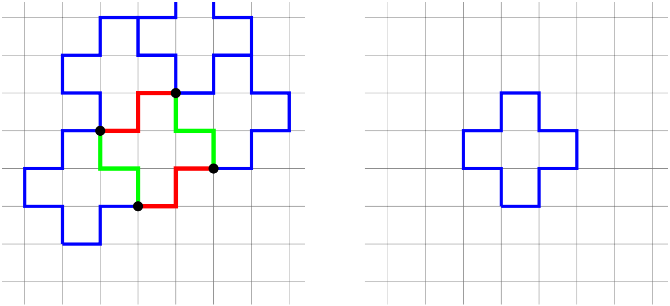
Square Tilings

The pentomino has two distinct square factorizations :



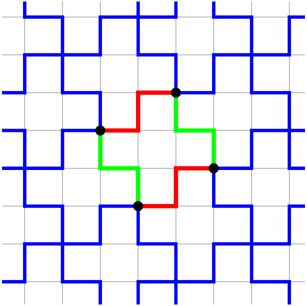
Square Tilings

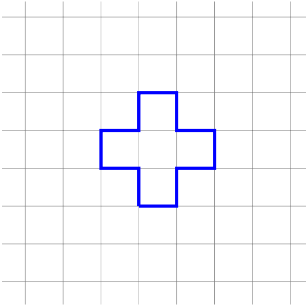
The pentomino has two distinct square factorizations :



Square Tilings

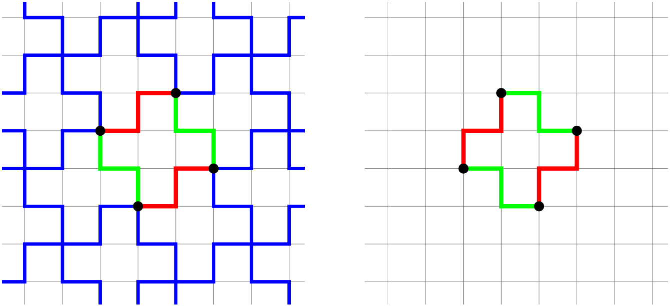
The pentomino has two distinct square factorizations :





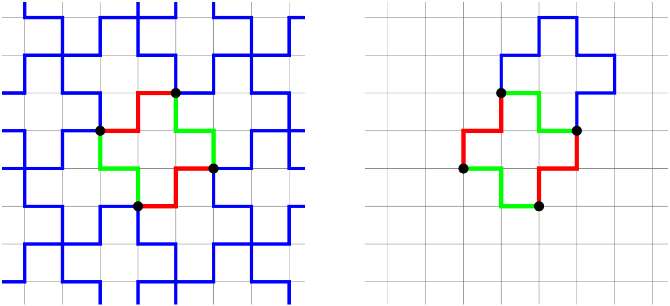
Square Tilings

The pentomino has two distinct square factorizations :

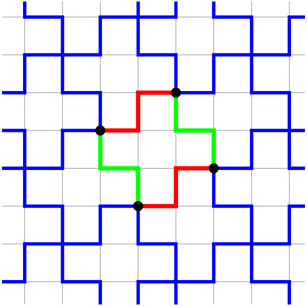


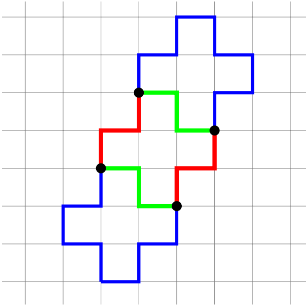
Square Tilings

The pentomino has two distinct square factorizations :



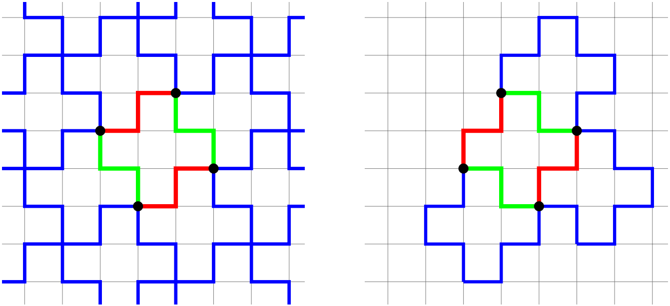
The pentomino has two distinct square factorizations :





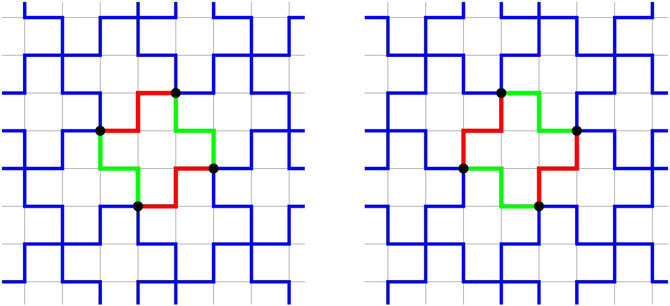
Square Tilings

The pentomino has two distinct square factorizations :

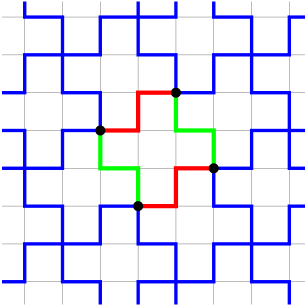


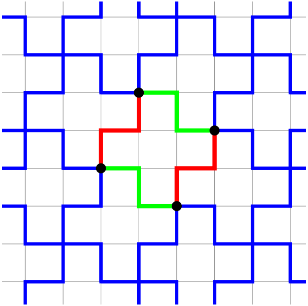
Square Tilings

The pentomino has two distinct square factorizations :



The pentomino has two distinct square factorizations :





Conjecture (Brlek, Dulucq, Fédou, Provençal 2007)

A tile has at most 2 square factorizations.

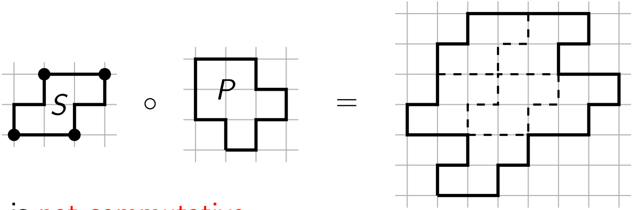
Composition of tiles

The factorization $AB\widehat{A}\widehat{B}$ of a square S allows to define the substitution

$$\varphi_S : \mathbf{0} \mapsto A, \mathbf{1} \mapsto B, \mathbf{2} \mapsto \widehat{A}, \mathbf{3} \mapsto \widehat{B}.$$

For any polyomino P having boundary w we define the composition

$$S \circ P := \varphi_S(w).$$



Note : This is not commutative.

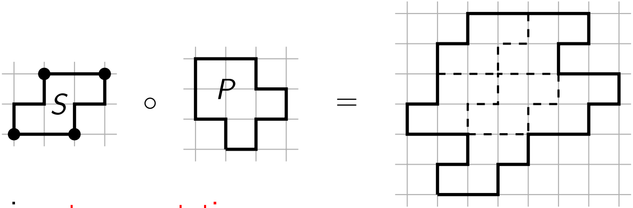
Composition of tiles

The factorization $AB\widehat{A}\widehat{B}$ of a square S allows to define the substitution

$$\varphi_S : \mathbf{0} \mapsto A, \mathbf{1} \mapsto B, \mathbf{2} \mapsto \widehat{A}, \mathbf{3} \mapsto \widehat{B}.$$

For any polyomino P having boundary w we define the composition

$$S \circ P := \varphi_S(w).$$



Note : This is not commutative.

Definition

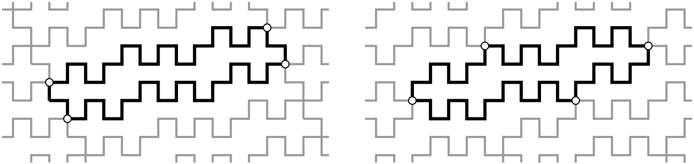
A polyomino Q is prime if $Q = S \circ P$ implies that S or P is the unit square.

Palindromes in Prime Double Square Tiles

Conjecture (X. Provençal and L. Vuillon, 2008)

If $XY\widehat{X}\widehat{Y} \equiv WZ\widehat{W}\widehat{Z}$ are distinct Beauquier-Nivat factorizations of a prime double square tile, then X, Y, W and Z are palindromes.

Note : a palindrome is a word that reads the same forward as it does backward.



Outline

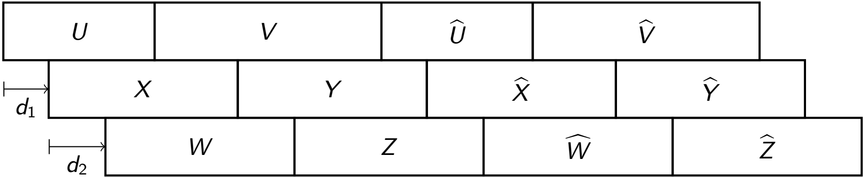
Idea of the proof : at most 2 square factorizations

Lemma (Brlek, Fédou, Provençal, 2008)

The factorizations $UV\widehat{U}\widehat{V} \equiv_{d_1} XY\widehat{X}\widehat{Y}$ *of a double square tile must alternate, that is* $0 < d_1 < |U| < d_1 + |X|$.

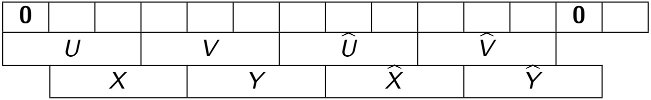
Suppose that there is a triple square tile having the following boundary :

$$UV\widehat{U}\widehat{V} \equiv_{d_1} XY\widehat{X}\widehat{Y} \equiv_{d_2} WZ\widehat{W}\widehat{Z}.$$



Examples

Suppose that $|U| = |V| = |X| = |Y| = |W| = |Z| = 3$.



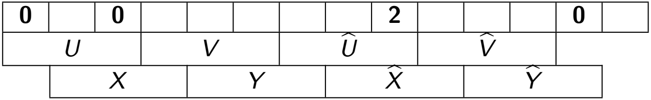
Examples

Suppose that $|U| = |V| = |X| = |Y| = |W| = |Z| = 3$.

<table>
<tr><td>0</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td>2</td><td></td><td></td><td></td><td>0</td><td></td></tr>
<tr><td colspan="3">U</td><td colspan="3">V</td><td colspan="3">$\widehat{U}$</td><td colspan="3">$\widehat{V}$</td><td colspan="2"></td></tr>
<tr><td></td><td colspan="3">X</td><td colspan="3">Y</td><td colspan="3">$\widehat{X}$</td><td colspan="3">$\widehat{Y}$</td><td></td></tr>
</table>

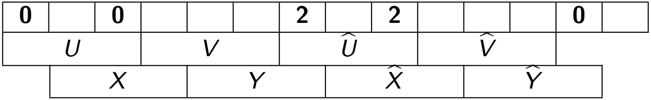
Examples

Suppose that $|U| = |V| = |X| = |Y| = |W| = |Z| = 3$.



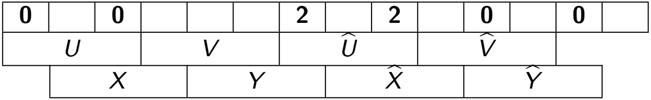
Examples

Suppose that $|U| = |V| = |X| = |Y| = |W| = |Z| = 3$.



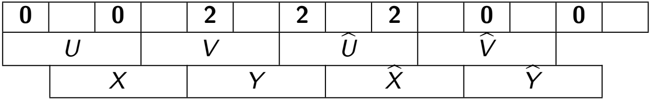
Examples

Suppose that $|U| = |V| = |X| = |Y| = |W| = |Z| = 3$.



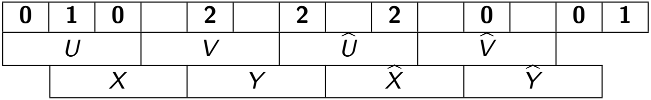
Examples

Suppose that $|U| = |V| = |X| = |Y| = |W| = |Z| = 3$.



Examples

Suppose that $|U| = |V| = |X| = |Y| = |W| = |Z| = 3$.



Examples

Suppose that $|U| = |V| = |X| = |Y| = |W| = |Z| = 3$.

0	**1**	**0**		**2**		**2**	**3**	**2**		**0**		**0**	**1**
	U			V			$\widehat{U}$			$\widehat{V}$			
		X			Y			$\widehat{X}$			$\widehat{Y}$		

Examples

Suppose that $|U| = |V| = |X| = |Y| = |W| = |Z| = 3$.

0	**1**	**0**	**1**	**2**		**2**	**3**	**2**		**0**		**0**	**1**
U			V			$\widehat{U}$			$\widehat{V}$				
	X			Y			$\widehat{X}$			$\widehat{Y}$			

Examples

Suppose that $|U| = |V| = |X| = |Y| = |W| = |Z| = 3$.

0	**1**	**0**	**1**	**2**		**2**	**3**	**2**		**0**	**3**	**0**	**1**
	U			V			$\widehat{U}$			$\widehat{V}$			
		X			Y			$\widehat{X}$			$\widehat{Y}$		

Examples

Suppose that $|U| = |V| = |X| = |Y| = |W| = |Z| = 3$.

<table>
<tr><td>0</td><td>1</td><td>0</td><td>1</td><td>2</td><td>1</td><td>2</td><td>3</td><td>2</td><td></td><td>0</td><td>3</td><td>0</td><td>1</td></tr>
<tr><td colspan="3">U</td><td colspan="3">V</td><td colspan="3">$\widehat{U}$</td><td colspan="3">$\widehat{V}$</td><td colspan="2"></td></tr>
<tr><td></td><td colspan="3">X</td><td colspan="3">Y</td><td colspan="3">$\widehat{X}$</td><td colspan="3">$\widehat{Y}$</td><td></td></tr>
</table>

Examples

Suppose that $|U| = |V| = |X| = |Y| = |W| = |Z| = 3$.

<table>
<tr><td>0</td><td>1</td><td>0</td><td>1</td><td>2</td><td>1</td><td>2</td><td>3</td><td>2</td><td>3</td><td>0</td><td>3</td><td>0</td><td>1</td></tr>
<tr><td colspan="3">U</td><td colspan="3">V</td><td colspan="3">$\widehat{U}$</td><td colspan="3">$\widehat{V}$</td><td colspan="2"></td></tr>
<tr><td></td><td colspan="3">X</td><td colspan="3">Y</td><td colspan="3">$\widehat{X}$</td><td colspan="3">$\widehat{Y}$</td><td></td></tr>
</table>

Examples

Suppose that $|U| = |V| = |X| = |Y| = |W| = |Z| = 3$.

<table>
<tr><td>0</td><td>1</td><td>0</td><td>1</td><td>2</td><td>1</td><td>2</td><td>3</td><td>2</td><td>3</td><td>0</td><td>3</td><td>0</td><td>1</td></tr>
<tr><td colspan="3">U</td><td colspan="3">V</td><td colspan="3">$\widehat{U}$</td><td colspan="3">$\widehat{V}$</td><td colspan="2"></td></tr>
<tr><td></td><td colspan="3">X</td><td colspan="3">Y</td><td colspan="3">$\widehat{X}$</td><td colspan="3">$\widehat{Y}$</td><td></td></tr>
<tr><td colspan="2"></td><td colspan="3">W</td><td colspan="3">Z</td><td colspan="3">$\widehat{W}$</td><td colspan="3">$\widehat{Z}$</td></tr>
</table>

Examples

Suppose that $|U| = |V| = |X| = |Y| = |W| = |Z| = 3$.

<table>
<tr><td>0</td><td>1</td><td>0</td><td>1</td><td>2</td><td>1</td><td>2</td><td>3</td><td>2</td><td>3</td><td>0</td><td>3</td><td>0</td><td>1</td></tr>
<tr><td colspan="3">U</td><td colspan="3">V</td><td colspan="3">$\widehat{U}$</td><td colspan="3">$\widehat{V}$</td><td colspan="2"></td></tr>
<tr><td></td><td colspan="3">X</td><td colspan="3">Y</td><td colspan="3">$\widehat{X}$</td><td colspan="3">$\widehat{Y}$</td><td></td></tr>
<tr><td colspan="2"></td><td colspan="3">W</td><td colspan="3">Z</td><td colspan="3">$\widehat{W}$</td><td colspan="3">$\widehat{Z}$</td></tr>
</table>

If a third factorization $WZ\widehat{W}\widehat{Z}$ exists, then, $\mathbf{0} = \mathbf{2}$ and $\mathbf{1} = \mathbf{3}$ which is a contradiction. Hence, there is no triple square tile of perimeter 12.

Examples

Suppose that $|U| = |V| = |X| = |Y| = |W| = |Z| = 3$.

0	**1**	**0**	**1**	**2**	**1**	**2**	**3**	**2**	**3**	**0**	**3**	**0**	**1**
U			V			$\widehat{U}$			$\widehat{V}$				
	X			Y			$\widehat{X}$			$\widehat{Y}$			
		W			Z			$\widehat{W}$			$\widehat{Z}$		

If a third factorization $WZ\widehat{W}\widehat{Z}$ exists, then, $\mathbf{0} = \mathbf{2}$ and $\mathbf{1} = \mathbf{3}$ which is a contradiction. Hence, there is no triple square tile of perimeter 12.

Although, there are words having more than two square factorizations. An example of length 36 was provided by X. Provençal :

0	**0**	**122**	**10012**	**21001**	**221**	**0**	**0**	**322**	**30032**	**23003**	**223**	
U			V			$\widehat{U}$			$\widehat{V}$			
	X			Y				$\widehat{X}$			$\widehat{Y}$	
		W			Z			$\widehat{W}$			$\widehat{Z}$	

Examples

Suppose that $|U| = |V| = |X| = |Y| = |W| = |Z| = 3$.

<table>
<tr><td>0</td><td>1</td><td>0</td><td>1</td><td>2</td><td>1</td><td>2</td><td>3</td><td>2</td><td>3</td><td>0</td><td>3</td><td>0</td><td>1</td></tr>
<tr><td colspan="3">U</td><td colspan="3">V</td><td colspan="3">$\widehat{U}$</td><td colspan="3">$\widehat{V}$</td><td colspan="2"></td></tr>
<tr><td></td><td colspan="3">X</td><td colspan="3">Y</td><td colspan="3">$\widehat{X}$</td><td colspan="3">$\widehat{Y}$</td><td></td></tr>
<tr><td colspan="2"></td><td colspan="3">W</td><td colspan="3">Z</td><td colspan="3">$\widehat{W}$</td><td colspan="3">$\widehat{Z}$</td></tr>
</table>

If a third factorization $WZ\widehat{W}\widehat{Z}$ exists, then, $\mathbf{0} = \mathbf{2}$ and $\mathbf{1} = \mathbf{3}$ which is a contradiction. Hence, there is no triple square tile of perimeter 12.

Although, there are words having more than two square factorizations. An example of length 36 was provided by X. Provençal :

<table>
<tr><td>0</td><td>0</td><td>122</td><td>10012</td><td>21001</td><td>221</td><td>0</td><td>0</td><td>322</td><td>30032</td><td>23003</td><td>223</td><td></td></tr>
<tr><td colspan="3">U</td><td colspan="3">V</td><td colspan="3">$\widehat{U}$</td><td colspan="3">$\widehat{V}$</td><td></td></tr>
<tr><td></td><td colspan="3">X</td><td colspan="3">Y</td><td colspan="3">$\widehat{X}$</td><td colspan="3">$\widehat{Y}$</td></tr>
<tr><td colspan="2"></td><td colspan="3">W</td><td colspan="3">Z</td><td colspan="3">$\widehat{W}$</td><td colspan="2">$\widehat{Z}$</td></tr>
</table>

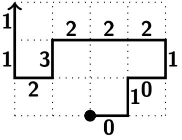
Note that the factor **221003** is a closed path...

First differences sequence

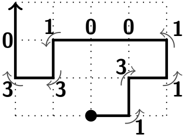
The first differences sequence of $w \in (\mathbb{Z}_4)^*$

$$\Delta w = (w_2 - w_1) \cdot (w_3 - w_2) \cdots (w_n - w_{n-1}).$$

represents the sequence of turns of the path.



$w = \mathbf{01012223211}$



$\Delta w = \mathbf{1311001330}$

First differences sequence

The first differences sequence of $w \in (\mathbb{Z}_4)^*$

$$\Delta w = (w_2 - w_1) \cdot (w_3 - w_2) \cdots (w_n - w_{n-1}).$$

represents the sequence of turns of the path.

$w = \mathbf{01012223211}$ $\Delta w = \mathbf{1311001330}$

We also consider $\Delta[w]$ well defined on the conjugacy classes :

$$\Delta[w] = (w_2 - w_1) \cdot (w_3 - w_2) \cdots (w_n - w_{n-1}) \cdot (w_1 - w_n) = \Delta w \cdot (w_1 - w_n).$$

Turning number

The turning number of a path w is $\mathcal{T}(w) = \frac{|\Delta w|_1 - |\Delta w|_3}{4}$ and corresponds to its total curvature divided by 2π (Wikipedia). We have that

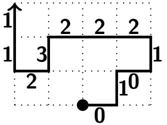
- $\mathcal{T}(w) = -\mathcal{T}(\widehat{w})$ for all path $w \in \Sigma^*$
- $\mathcal{T}([w]) = \pm 1$ for all simple and closed path w.

Turning number

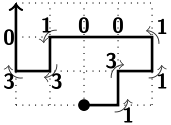
The turning number of a path w is $\mathcal{T}(w) = \frac{|\Delta w|_1 - |\Delta w|_3}{4}$ and corresponds to its total curvature divided by 2π (Wikipedia). We have that

- $\mathcal{T}(w) = -\mathcal{T}(\widehat{w})$ for all path $w \in \Sigma^*$
- $\mathcal{T}([w]) = \pm 1$ for all simple and closed path w.

For example,

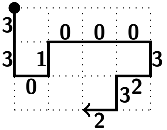


$w = \mathbf{01012223211}$

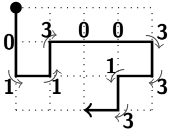


$\Delta w = \mathbf{1311001330}$

$\mathcal{T}(w) = 1/4$



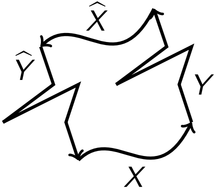
$\widehat{w} = \mathbf{33010003232}$



$\Delta\widehat{w} = \mathbf{0113003313}$

$\mathcal{T}(\widehat{w}) = -1/4$

Turning number



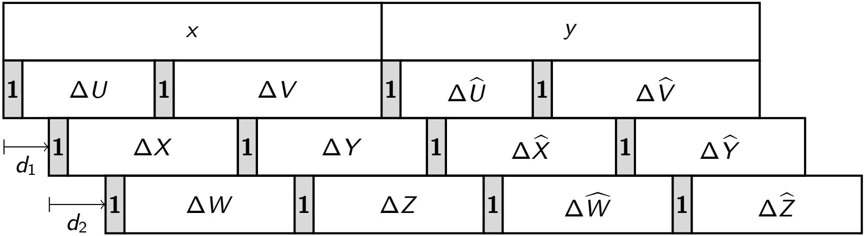
Then, for a square tile, the sum of the four angles between X, Y, $\widehat{X}$ and $\widehat{Y}$ must be 2π.

Lemma (Blondin-Massé, Brlek, Garon, L. 2010)

Si $XY\widehat{X}\widehat{Y}$ est la frontière orientée positivement d'une tuile carrée, alors

$$\Delta[XY\widehat{X}\widehat{Y}] = \Delta X \cdot \mathbf{1} \cdot \Delta Y \cdot \mathbf{1} \cdot \Delta\widehat{X} \cdot \mathbf{1} \cdot \Delta\widehat{Y} \cdot \mathbf{1}.$$

Idée de la preuve : au plus 2 factorisations carrées



On s'intéresse aux moitiés du contour

$$x = x_0 x_1 x_2 \cdots x_{n-1} = \mathbf{1} \cdot \Delta U \cdot \mathbf{1} \cdot \Delta V,$$
$$y = y_0 y_1 y_2 \cdots y_{n-1} = \mathbf{1} \cdot \Delta \widehat{U} \cdot \mathbf{1} \cdot \Delta \widehat{V}.$$

One has $x_i = y_i = \mathbf{1}$ for all $i \in I$ where
$I = \{0, d_1, d_1 + d_2, |U|, d_1 + |X|, d_1 + d_2 + |W|\} \subseteq \mathbb{Z}_n$.

Idée de la preuve : au plus 2 factorisations carrées

On définit trois réflexions sur $\mathbb{Z}_n$:

$$\begin{aligned} s_1 : i &\mapsto |U| - i, \\ s_2 : i &\mapsto |X| + 2d_1 - i, \\ s_3 : i &\mapsto |W| + 2(d_1 + d_2) - i, \end{aligned}$$

qui satisfont aux relations :

$$s_1^2 = s_2^2 = s_3^2 = 1$$

et pour tout $i, j, k \in \{1, 2, 3\}$

$$(s_i s_j s_k)^2 = 1.$$

Par exemple,

$$s_2 s_3 s_1 s_2 s_3 = s_1.$$

We say that the application s_1 is admissible on i if $i \notin \{0, |U|\}$ and similarly for the application s_2 if $i \notin \{d_1, |X| + d_1\}$ and for s_3 if $i \notin \{d_1 + d_2, |W| + d_1 + d_2\}$.

Lemma

Soit $i \in \mathbb{Z}_n$ et $j \in \{1, 2, 3\}$ tels que s_j est admissible sur i. Alors

- $y_i = -x_{s_j(i)}$ *et* $x_i = -y_{s_j(i)}$ *ou* $x_i = -x_{s_j(i)}$ *et* $y_i = -y_{s_j(i)}$.
- *Si* $x_i = y_i$, *alors* $x_{s_j(i)} = y_{s_j(i)}$.

où $-\mathbf{0} = \mathbf{0}$, $-\mathbf{1} = \mathbf{3}$, $-\mathbf{2} = \mathbf{2}$ *et* $-\mathbf{3} = \mathbf{1}$.

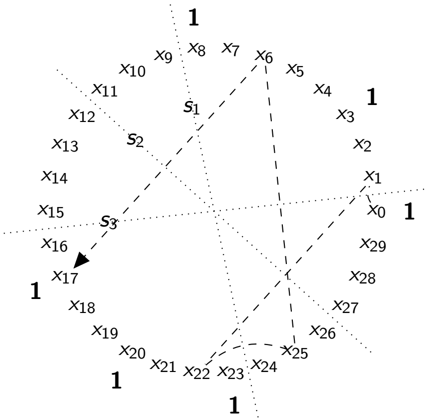
Lemma

Let $i \in I$ and $S = s_{j_m} s_{j_{m-1}} \cdots s_{j_2} s_{j_1}$ be an admissible product of reflections on i. Then $x_{S(i)} = y_{S(i)}$ and

$$x_{S(i)} = \begin{cases} x_i & \text{if } m \text{ is even,} \\ -x_i & \text{if } m \text{ is odd.} \end{cases}$$

Idée de la preuve : au plus 2 factorisations carrées

Soit $n = 30$, $d_1 = 3$, $d_2 = 5$, $|U| = 17$, $|X| = 17$ et $|W| = 15$.
$\mathbf{1} = x_0 = -x_{s_3 s_2 s_1 s_3 s_2(0)} = -x_{17} = -\mathbf{1} = \mathbf{3}$ une contradiction.



On a $s_1 = s_3 s_2 s_1 s_3 s_2$. Si $s_3 s_2 s_1 s_3 s_2$ est un produit admissible de réflexions sur 0, alors $x_0 = -x_{17}$ ce qui est une contradiction. Autrement, des contradictions similaires sont obtenues.

Idée de la preuve : au plus 2 factorisations carrées

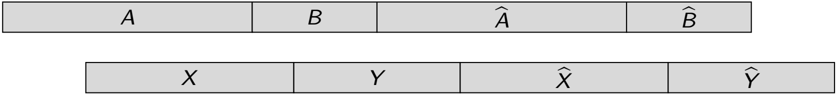
Theorem (Blondin Massé, Brlek, Garon, L. 2010)

A tile has at most 2 square factorizations.

Outline

Periods in the boundary of double square tiles

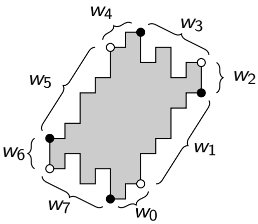
Let $AB\widehat{A}\widehat{B} \equiv XY\widehat{X}\widehat{Y}$ be the factorizations of a double square tile.



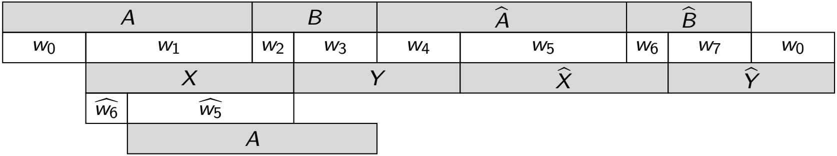
Periods in the boundary of double square tiles

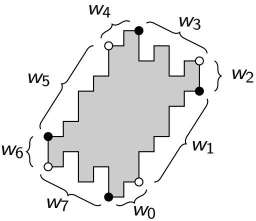
Let $AB\widehat{A}\widehat{B} \equiv XY\widehat{X}\widehat{Y}$ be the factorizations of a double square tile.

<table>
<tr><td colspan="2">A</td><td colspan="2">B</td><td colspan="2">$\widehat{A}$</td><td colspan="2">$\widehat{B}$</td><td></td></tr>
<tr><td>w_0</td><td>w_1</td><td>w_2</td><td>w_3</td><td>w_4</td><td>w_5</td><td>w_6</td><td>w_7</td><td>w_0</td></tr>
<tr><td></td><td colspan="2">X</td><td colspan="2">Y</td><td colspan="2">$\widehat{X}$</td><td colspan="2">$\widehat{Y}$</td></tr>
</table>



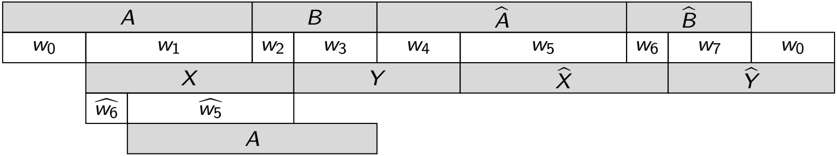
Let $AB\widehat{A}\widehat{B} \equiv XY\widehat{X}\widehat{Y}$ be the factorizations of a double square tile.





Periods in the boundary of double square tiles

Let $AB\widehat{A}\widehat{B} \equiv XY\widehat{X}\widehat{Y}$ be the factorizations of a double square tile.

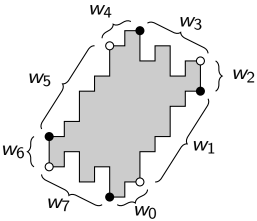


In general

- $|w_{i-1}| + |w_{i+1}|$ is a period of $w_{i-1}w_iw_{i+1}$.

Hence we write

- $w_i = (u_iv_i)^{n_i}u_i$ where $|u_iv_i| = |w_{i-1}| + |w_{i+1}|$.



Transformations

Let $S = (w_0, w_1, w_2, w_3, w_4, w_5, w_6, w_7)$. We define

$$\text{SHRINK}_0(S) = ((u_0 v_0)^{n_i - 1} u_0, w_1, w_2, w_3, (u_4 v_4)^{n_4 - 1} u_4, w_5, w_6, w_7),$$

$$\text{EXTEND}_0(S) = ((u_0 v_0)^{n_i + 1} u_0, w_1, w_2, w_3, (u_4 v_4)^{n_4 + 1} u_4, w_5, w_6, w_7)$$

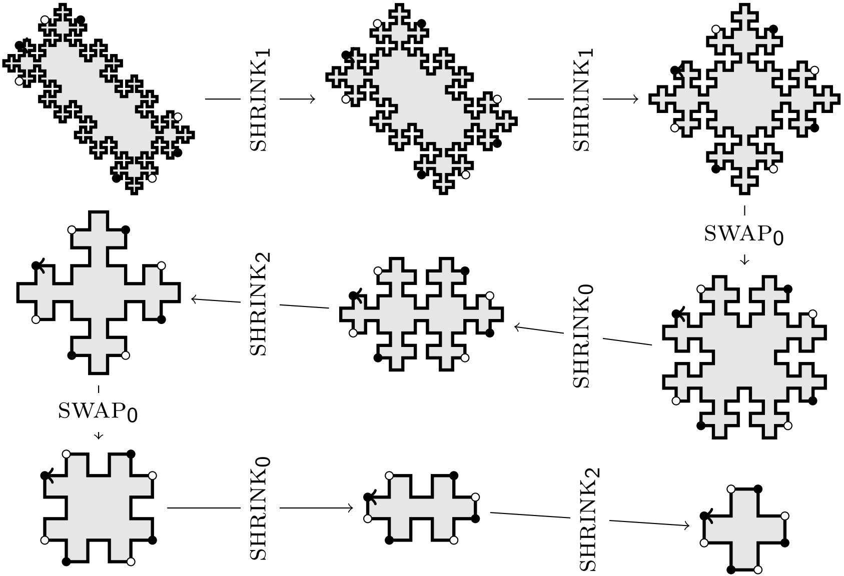
and

$$\text{SWAP}_0(S) = (\widehat{w_4}, (v_1 u_1)^{n_1} v_1, \widehat{w_6}, (v_3 u_3)^{n_3} v_3, \widehat{w_0}, (v_5 u_5)^{n_5} v_5, \widehat{w_2}, (v_7 u_7)^{n_7} v_7)$$

and their conjugates

- $\text{SHIFT}(S) = (w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_0)$,
- $\text{SHRINK}_i(S) = \text{SHIFT}^{-i} \circ \text{SHRINK}_0 \circ \text{SHIFT}^i(S)$,
- $\text{SWAP}_i(S) = \text{SHIFT}^{-i} \circ \text{SWAP}_0 \circ \text{SHIFT}^i(S)$,
- $\text{EXTEND}_i(S) = \text{SHIFT}^{-i} \circ \text{EXTEND}_0 \circ \text{SHIFT}^i(S)$.

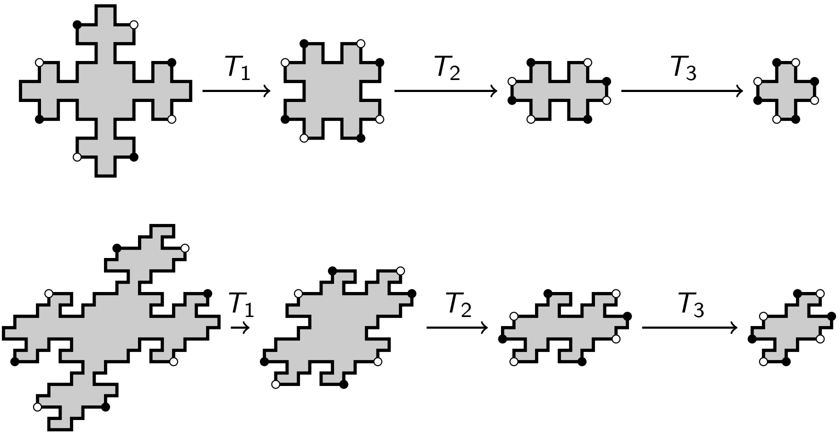
Example of reduction



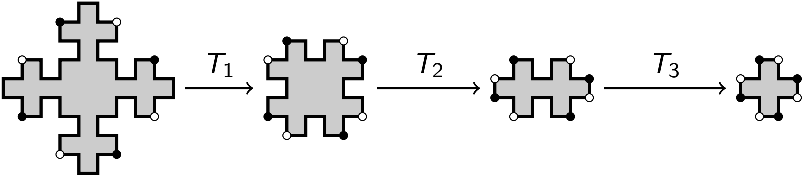
Reduction of double square tiles

Theorem (Blondin Massé, Brlek, Garon, L. (GASCom 2010))

Every double square reduces to a composed cross pentomino.



Reduction of double square tiles



Moreover,

- The transformations T_i are invertible.
- The transformations T_i^{-1} preserve palindromes.

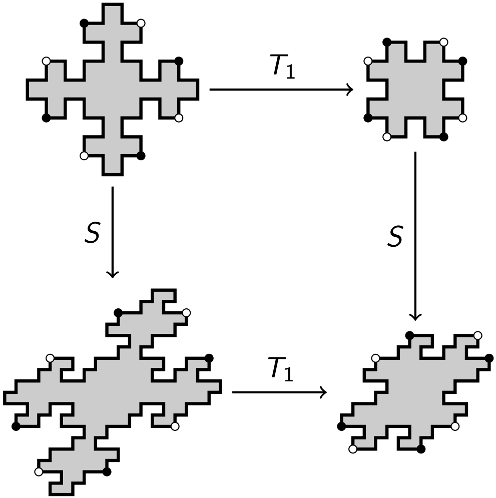
Proposition (Blondin Massé, Brlek, Garon, L. (GASCom 2010))

Let $AB\widehat{A}\widehat{B} \equiv XY\widehat{X}\widehat{Y}$ be the boundary of a double square D. If D reduces to the prime cross pentomino, then A, B, X and Y are palindromes.

Reduction vs Composition

Equivalent questions :

- Does every prime double square reduce to the prime cross pentomino ?
- Does the reduction T_i preserve prime tiles ?
- Does the inverse T_i^{-1} preserve composed tiles ?
- Does the following diagram commute ?



Idea of the proof

Let $H(w) = |w|_{\mathbf{0}} + |w|_{\mathbf{2}}$ be the number of horizontal steps of the path w and $V(w) = |w|_{\mathbf{1}} + |w|_{\mathbf{3}}$ be its number of vertical steps.

Lemma

Let $S = (w_0, w_1, w_2, w_3, w_4, w_5, w_6, w_7)$ be the factorization of a double square. Then, $H(w_i) = H(w_{i+4})$ and $V(w_i) = V(w_{i+4})$.

Idea of the proof

Let $H(w) = |w|_\mathbf{0} + |w|_\mathbf{2}$ be the number of horizontal steps of the path w and $V(w) = |w|_\mathbf{1} + |w|_\mathbf{3}$ be its number of vertical steps.

Lemma

Let $S = (w_0, w_1, w_2, w_3, w_4, w_5, w_6, w_7)$ be the factorization of a double square. Then, $H(w_i) = H(w_{i+4})$ and $V(w_i) = V(w_{i+4})$.

Lemma

SWAP_i, SHRINK_i *and* EXTEND_i *commute with the composition.*

Idea of the proof

Let $H(w) = |w|_{\mathbf{0}} + |w|_{\mathbf{2}}$ be the number of horizontal steps of the path w and $V(w) = |w|_{\mathbf{1}} + |w|_{\mathbf{3}}$ be its number of vertical steps.

Lemma

Let $S = (w_0, w_1, w_2, w_3, w_4, w_5, w_6, w_7)$ be the factorization of a double square. Then, $H(w_i) = H(w_{i+4})$ and $V(w_i) = V(w_{i+4})$.

Lemma

SWAP_i, SHRINK_i *and* EXTEND_i *commute with the composition.*

Proposition (Blondin-Massé, L., 2010)

Every prime double square reduces to the prime cross pentomino.

Corollary

If $XY\widehat{X}\widehat{Y} \equiv WZ\widehat{W}\widehat{Z}$ are distinct Beauquier-Nivat factorizations of a prime double square tile, then X, Y, W and Z are palindromes.

Outline

Open problems

Some problems are left open :

- Find an algorithm that decides whether a polyomino is prime.
- If $\alpha\alpha$ appears in the boundary word of a double square tile D, where $\alpha \in \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}\}$, then D is not prime.
- Prove that if $S \circ P$ is a square tile, then so is P.
- Describe the distribution and the proportion of prime square tiles of half-perimeter n as n goes to infinity.
- Show that SHRINK_i and SWAP_i are sufficient to reduce a double square tile : no need to use the more complicated L-SHRINK_i and R-SHRINK_i defined for limit cases.
- Extend the results to 8-connected polyominoes.
- Extend the results to continuous paths and tiles.

Credits

- This research was driven by computer exploration using the open-source mathematical software `Sage`.
- Les images de ce document ont été produites à l'aide de `pgf/tikz`.