Avancées récentes sur des questions issues des pavages du plan par translation d’une tuile

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Séminaire Structures Algébriques et Géométriques
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Travail en collaboration avec Alexandre Blondin Massé, Srečko Brlek et Ariane Garon
Outline

1 Introduction
   - Discrete Figures
   - Tilings
   - Beauquier and Nivat
   - Hexagonal and Square Tiles
   - A conjecture of Brlek, Dulucq, Fédou and Provençal, 2007
   - A conjecture of Provençal and Vuillon, 2008

2 (Idea of the) Proof of the first conjecture

3 (Idea of the) Proof of the second conjecture

4 Open problems
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4 Open problems
• Discrete plane: $\mathbb{Z}^2$

• **Definition**: A polyomino is a finite, 4-connected subset of the plane, without holes.
A set $S = \{P_1, P_2, \ldots, P_k\}$ of polyominoes tiles the plane if there exists a partition of $\mathbb{Z}^2$ into translated copies of $P_i$. 
Results on tilings

Theorem (Berger, 1966)

Il existe un ensemble de polyominos qui pavent le plan uniquement de manière apériodique.

Corollary (Berger, 1966)

Le problème de pavage du plan par un ensemble fini de polyominos est indécidable.

Theorem (Wijshoff et van Leuven, 1984)

- Si un polyomino pave le plan par translation, alors il peut également le faire de manière régulière.
- Donc, le problème du pavage du plan par un polyomino est décidable.
A polyomino $P$ is called a tile if it tiles the plane.
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**Problem**

*Does a given polyomino $P$ tile the plane?*
Freeman Chain Code

\[ \Sigma = \mathbb{Z}_4 = \{0, 1, 2, 3\} \]
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\[ w = 0103301103301111232211233233 \]
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Any conjugate \( w' \) of \( w \) codes the same polyomino.

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\( w \) and \( w' \) are conjugate if there exist \( u, v \in \Sigma^* \) such that \( w = uv \) and \( w' = vu \). We write \( w \equiv_{|u|} w' \).
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\( w \) and \( w' \) are conjugate if there exist \( u, v \in \Sigma^* \) such that \( w = uv \) and \( w' = vu \). We write \( w \equiv |u| \ w' \).
Characterization: A polyomino $P$ tiles the plane if and only if there exist $X, Y, Z \in \Sigma^*$ such that

$$[w] \equiv XYZ\hat{X}\hat{Y}\hat{Z}.$$ 

$X = 0 \, 0 \, 1 \, 0 \, 3 \, 0 \, 1$ 

$\hat{X} = 3 \, 2 \, 1 \, 2 \, 3 \, 2 \, 2$
There are polyominoes admitting many hexagon tilings:

A $1 \times n$ rectangle tiles the plane \textit{as an hexagon in $n - 1$ ways and as a square in only 1 way}.
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The pentomino has two distinct square factorizations:
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The \textit{pentomino} has two distinct square factorizations:

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{pentomino.png}
\end{figure}
The pentomino has two distinct square factorizations:
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\[ \text{Diagram of two square factorizations} \]
The **pentomino** has **two** distinct square factorizations:

![Square Tilings Diagram](image-url)
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Conjecture (Brlek, Dulucq, Fédou, Provençal 2007)

A tile has \textit{at most 2} square factorizations.
Composition of tiles

The factorization $AB\hat{A}\hat{B}$ of a square $S$ allows to define the substitution

$$\varphi_S : 0 \mapsto A, 1 \mapsto B, 2 \mapsto \hat{A}, 3 \mapsto \hat{B}.$$  

For any polyomino $P$ having boundary $w$ we define the composition

$$S \circ P := \varphi_S(w).$$

Note: This is not commutative.
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Note: This is not commutative.

Definition

A polyomino $Q$ is prime if $Q = S \circ P$ implies that $S$ or $P$ is the unit square.
Conjecture (X. Provençal and L. Vuillon, 2008)

If $XY\hat{X}\hat{Y} \equiv WZ\hat{W}\hat{Z}$ are distinct Beauquier-Nivat factorizations of a prime double square tile, then $X$, $Y$, $W$ and $Z$ are palindromes.

Note: a palindrome is a word that reads the same forward as it does backward.
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4. Open problems
Idea of the proof: at most 2 square factorizations

Lemma (Brlek, Fédou, Provençal, 2008)

The factorizations $U \hat{V} \hat{U} V \equiv_{d_1} X Y \hat{X} \hat{Y}$ of a double square tile must alternate, that is $0 < d_1 < |U| < d_1 + |X|$.

Suppose that there is a triple square tile having the following boundary:

$$U \hat{V} \hat{U} V \equiv_{d_1} X Y \hat{X} \hat{Y} \equiv_{d_2} W Z \hat{W} \hat{Z}.$$
Examples

Suppose that $|U| = |V| = |X| = |Y| = |W| = |Z| = 3$.

If a third factorization $WZ \hat{W} \hat{Z}$ exists, then, $0 = 2$ and $1 = 3$ which is a contradiction. Hence, there is no triple square tile of perimeter 12.

Although, there are words having more than two square factorizations. An example of length 36 was provided by X. Proven¸cal.
Examples

Suppose that $|U| = |V| = |X| = |Y| = |W| = |Z| = 3$.

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\[
\begin{array}{cccc}
0 & 0 & & \\
U & V & U & V \\
X & Y & X & Y \\
\end{array}
\]
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$U$ $V$ $\hat{U}$ $\hat{V}$

$X$ $Y$ $\hat{X}$ $\hat{Y}$

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Note that the factor $221003$ is a closed path...
Suppose that $|U| = |V| = |X| = |Y| = |W| = |Z| = 3$.

\[
\begin{array}{cccccc}
0 & 0 & 2 & 2 & 2 & 0 & 0 \\
U & V & \hat{U} & \hat{V} & \\
X & Y & \hat{X} & \hat{Y} \\
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\[
\begin{array}{cccccccc}
0 & 1 & 0 & 1 & 2 & 1 & 2 & 3 & 2 & 3 & 0 & 3 & 0 & 1 \\
\hline
U & V & \hat{U} & \hat{V} & \\
X & Y & \hat{X} & \hat{Y} & \\
W & Z & \hat{W} & \hat{Z} & \\
\end{array}
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0 & 1 & 0 & 1 & 2 & 1 & 2 & 3 & 2 & 3 & 0 & 3 & 0 & 1 \\
U & V & \hat{U} & \hat{V} & X & Y & \hat{X} & \hat{Y} & W & Z & \hat{W} & \hat{Z}
\end{array}
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```
0 0 1 1 1 1 0 0 1 1 1 1 0 0 1 1 1 1 0 0 1 1 1 1 0 0 1 1 1 1 0 0 1 1 1

<table>
<thead>
<tr>
<th>U</th>
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```
0 1 0 1 2 1 2 3 2 3 0 3 0 1
U V U V
X Y X Y
W Z W Z
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```
0 0 122 10012 21001 221 0 0 322 30032 23003 223
U V U V
X Y X Y
W Z W Z
```

Note that the factor $221003$ is a closed path...
The first differences sequence of $w \in (\mathbb{Z}_4)^*$

$$\Delta w = (w_2 - w_1) \cdot (w_3 - w_2) \cdots (w_n - w_{n-1}).$$

represents the sequence of turns of the path.

$w = 01012223211$

$\Delta w = 1311001330$
The first differences sequence of $w \in (\mathbb{Z}_4)^*$

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represents the sequence of turns of the path.

$$w = 01012223211 \quad \Delta w = 1311001330$$

We also consider $\Delta[w]$ well defined on the conjugacy classes:

$$\Delta[w] = (w_2 - w_1) \cdot (w_3 - w_2) \cdots (w_n - w_{n-1}) \cdot (w_1 - w_n) = \Delta w \cdot (w_1 - w_n).$$
The turning number of a path $w$ is $T(w) = \frac{|\Delta w|_1 - |\Delta w|_3}{4}$ and corresponds to its total curvature divided by $2\pi$ (Wikipedia). We have that

- $T(w) = -T(\hat{w})$ for all path $w \in \Sigma^*$
- $T([w]) = \pm 1$ for all simple and closed path $w$. 


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- $T([w]) = \pm 1$ for all simple and closed path $w$.

For example,

$w = 01012223211$

$\Delta w = 1311001330$

$\hat{w} = 33010003232$

$\Delta \hat{w} = 0113003313$

$T(w) = 1/4$

$T(\hat{w}) = -1/4$
Then, for a square tile, the sum of the four angles between $X$, $Y$, $\hat{X}$ and $\hat{Y}$ must be $2\pi$.

**Lemma (Blondin-Massé, Brlek, Garon, L. 2010)**

Si $XY\hat{X}\hat{Y}$ est la frontière orientée positivement d’une tuile carrée, alors

$$\Delta[XY\hat{X}\hat{Y}] = \Delta X \cdot 1 \cdot \Delta Y \cdot 1 \cdot \Delta \hat{X} \cdot 1 \cdot \Delta \hat{Y} \cdot 1.$$
Idée de la preuve : au plus 2 factorisations carrées

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On s’intéresse aux moitiés du contour

\[
x = x_0 x_1 x_2 \cdots x_{n-1} = 1 \cdot \Delta U \cdot 1 \cdot \Delta V,
\]

\[
y = y_0 y_1 y_2 \cdots y_{n-1} = 1 \cdot \Delta \hat{U} \cdot 1 \cdot \Delta \hat{V}.
\]

One has \(x_i = y_i = 1\) for all \(i \in I\) where

\(I = \{0, d_1, d_1 + d_2, |U|, d_1 + |X|, d_1 + d_2 + |W|\} \subseteq \mathbb{Z}_n.\)
Idée de la preuve : au plus 2 factorisations carrées

On définit trois réflexions sur $\mathbb{Z}_n$ :

\[
\begin{align*}
    s_1 : i & \mapsto |U| - i, \\
    s_2 : i & \mapsto |X| + 2d_1 - i, \\
    s_3 : i & \mapsto |W| + 2(d_1 + d_2) - i,
\end{align*}
\]

qui satisfont aux relations :

\[
s_1^2 = s_2^2 = s_3^2 = 1
\]

et pour tout $i, j, k \in \{1, 2, 3\}$

\[
(s_is_js_k)^2 = 1.
\]

Par exemple,

\[
s_2s_3s_1s_2s_3 = s_1.
\]
We say that the application $s_1$ is admissible on $i$ if $i \not\in \{0, |U|\}$ and similarly for the application $s_2$ if $i \not\in \{d_1, |X| + d_1\}$ and for $s_3$ if $i \not\in \{d_1 + d_2, |W| + d_1 + d_2\}$.

**Lemma**

*Soit $i \in \mathbb{Z}_n$ et $j \in \{1, 2, 3\}$ tels que $s_j$ est admissible sur $i$. Alors*

- $y_i = -x_{s_j(i)}$ et $x_i = -y_{s_j(i)}$ ou $x_i = -x_{s_j(i)}$ et $y_i = -y_{s_j(i)}$.
- Si $x_i = y_i$, alors $x_{s_j(i)} = y_{s_j(i)}$.

 où $-0 = 0$, $-1 = 3$, $-2 = 2$ et $-3 = 1$.

**Lemma**

*Let $i \in I$ and $S = s_{j_m} s_{j_{m-1}} \cdots s_2 s_1$ be an admissible product of reflections on $i$. Then $x_{S(i)} = y_{S(i)}$ and*

$$x_{S(i)} = \begin{cases} x_i & \text{if } m \text{ is even}, \\ -x_i & \text{if } m \text{ is odd}. \end{cases}$$
Idée de la preuve : au plus 2 factorisations carrées

Soit \( n = 30, \ d_1 = 3, \ d_2 = 5, \ |U| = 17, \ |X| = 17 \) et \( |W| = 15 \).

\( 1 = x_0 = -x_{s_3s_2s_1s_3s_2}(0) = -x_{17} = -1 = 3 \) une contradiction.

On a \( s_1 = s_3s_2s_1s_3s_2 \). Si \( s_3s_2s_1s_3s_2 \) est un produit admissible de réflexions sur 0, alors \( x_0 = -x_{17} \) ce qui est une contradiction. Autrement, des contradictions similaires sont obtenues.
Idée de la preuve : au plus 2 factorisations carrées

Theorem (Blondin Massé, Brlek, Garon, L. 2010)

A tile has at most 2 square factorizations.
1 Introduction
   • Discrete Figures
   • Tilings
   • Beauquier and Nivat
   • Hexagonal and Square Tiles
   • A conjecture of Brlek, Dulucq, Fédou and Provençal, 2007
   • A conjecture of Provençal and Vuillon, 2008

2 (Idea of the) Proof of the first conjecture

3 (Idea of the) Proof of the second conjecture

4 Open problems
Let $A\hat{B}\hat{A}\hat{B} \equiv X\hat{Y}\hat{X}\hat{Y}$ be the factorizations of a double square tile.

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$X$ $\hat{X}$ $Y$ $\hat{Y}$
$\hat{w}_6$ $\hat{w}_5$

$A$

In general

- $|w_{i-1}| + |w_{i+1}|$ is a period of $w_{i-1}w_iw_{i+1}$.

Hence we write

- $w_i = (u_iv_i)^n_i u_i$ where $|u_iv_i| = |w_{i-1}| + |w_{i+1}|$. 
Transformations

Let $S = (w_0, w_1, w_2, w_3, w_4, w_5, w_6, w_7)$. We define

$\text{SHRINK}_0(S) = ((u_0 v_0)^{n_i-1} u_0, w_1, w_2, w_3, (u_4 v_4)^{n_4-1} u_4, w_5, w_6, w_7)$,

$\text{EXTEND}_0(S) = ((u_0 v_0)^{n_i+1} u_0, w_1, w_2, w_3, (u_4 v_4)^{n_4+1} u_4, w_5, w_6, w_7)$

and

$\text{SWAP}_0(S) = (\widehat{w_4}, (v_1 u_1)^{n_1} v_1, \widehat{w_6}, (v_3 u_3)^{n_3} v_3, \widehat{w_0}, (v_5 u_5)^{n_5} v_5, \widehat{w_2}, (v_7 u_7)^{n_7} v_7)$

and their conjugates

- $\text{SHIFT}(S) = (w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_0)$,
- $\text{SHRINK}_i(S) = \text{SHIFT}^{-i} \circ \text{SHRINK}_0 \circ \text{SHIFT}^i(S)$,
- $\text{SWAP}_i(S) = \text{SHIFT}^{-i} \circ \text{SWAP}_0 \circ \text{SHIFT}^i(S)$,
- $\text{EXTEND}_i(S) = \text{SHIFT}^{-i} \circ \text{EXTEND}_0 \circ \text{SHIFT}^i(S)$. 
Example of reduction
Theorem (Blondin Massé, Brlek, Garon, L. (GASCom 2010))

Every double square reduces to a composed cross pentomino.
Moreover,
- The transformations $T_i$ are invertible.
- The transformations $T_i^{-1}$ preserve palindromes.

**Proposition (Blondin Massé, Brlek, Garon, L. (GASCom 2010))**

Let $AB\hat{A}\hat{B} \equiv XY\hat{X}\hat{Y}$ be the boundary of a double square $D$. If $D$ reduces to the prime cross pentomino, then $A$, $B$, $X$ and $Y$ are palindromes.
Equivalent questions:

- Does every prime double square reduce to the prime cross pentomino?
- Does the reduction $T_i$ preserve prime tiles?
- Does the inverse $T_i^{-1}$ preserve composed tiles?
- Does the following diagram commute?
Idea of the proof

Let $H(w) = |w|_0 + |w|_2$ be the number of horizontal steps of the path $w$ and $V(w) = |w|_1 + |w|_3$ be its number of vertical steps.

**Lemma**

Let $S = (w_0, w_1, w_2, w_3, w_4, w_5, w_6, w_7)$ be the factorization of a double square. Then, $H(w_i) = H(w_{i+4})$ and $V(w_i) = V(w_{i+4})$. 

Idea of the proof

Let $H(w) = |w|_0 + |w|_2$ be the number of horizontal steps of the path $w$ and $V(w) = |w|_1 + |w|_3$ be its number of vertical steps.

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Let $S = (w_0, w_1, w_2, w_3, w_4, w_5, w_6, w_7)$ be the factorization of a double square. Then, $H(w_i) = H(w_{i+4})$ and $V(w_i) = V(w_{i+4})$.

Lemma

SWAP$_i$, SHRINK$_i$ and EXTEND$_i$ commute with the composition.
Idea of the proof

Let $H(w) = \|w\|_0 + \|w\|_2$ be the number of horizontal steps of the path $w$ and $V(w) = \|w\|_1 + \|w\|_3$ be its number of vertical steps.

Lemma

Let $S = (w_0, w_1, w_2, w_3, w_4, w_5, w_6, w_7)$ be the factorization of a double square. Then, $H(w_i) = H(w_{i+4})$ and $V(w_i) = V(w_{i+4})$.

Lemma

$\text{SWAP}_i$, $\text{SHRINK}_i$ and $\text{EXTEND}_i$ commute with the composition.

Proposition (Blondin-Massé, L., 2010)

Every prime double square reduces to the prime cross pentomino.

Corollary

If $XY\hat{X}\hat{Y} \equiv WZ\hat{W}\hat{Z}$ are distinct Beauquier-Nivat factorizations of a prime double square tile, then $X$, $Y$, $W$ and $Z$ are palindromes.
Outline

1 Introduction
   - Discrete Figures
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2 (Idea of the) Proof of the first conjecture

3 (Idea of the) Proof of the second conjecture

4 Open problems
Some problems are left open:

- Find an algorithm that decides whether a polyomino is prime.
- If $\alpha\alpha$ appears in the boundary word of a double square tile $D$, where $\alpha \in \{0, 1, 2, 3\}$, then $D$ is not prime.
- Prove that if $S \circ P$ is a square tile, then so is $P$.
- Describe the distribution and the proportion of prime square tiles of half-perimeter $n$ as $n$ goes to infinity.
- Show that SHRINK$_i$ and SWAP$_i$ are sufficient to reduce a double square tile: no need to use the more complicated L-SHRINK$_i$ and R-SHRINK$_i$ defined for limit cases.
- Extend the results to 8-connected polyominoes.
- Extend the results to continuous paths and tiles.
This research was driven by computer exploration using the open-source mathematical software Sage.

Les images de ce document ont été produites à l’aide de pgf/tikz.